Lec 5: Scaling Properties of Maxwell's Equaltions

Photonic Crystals:
Fundamentals and Applications

Lecture 05

Electromagnetism as an Eigenvalue Problem

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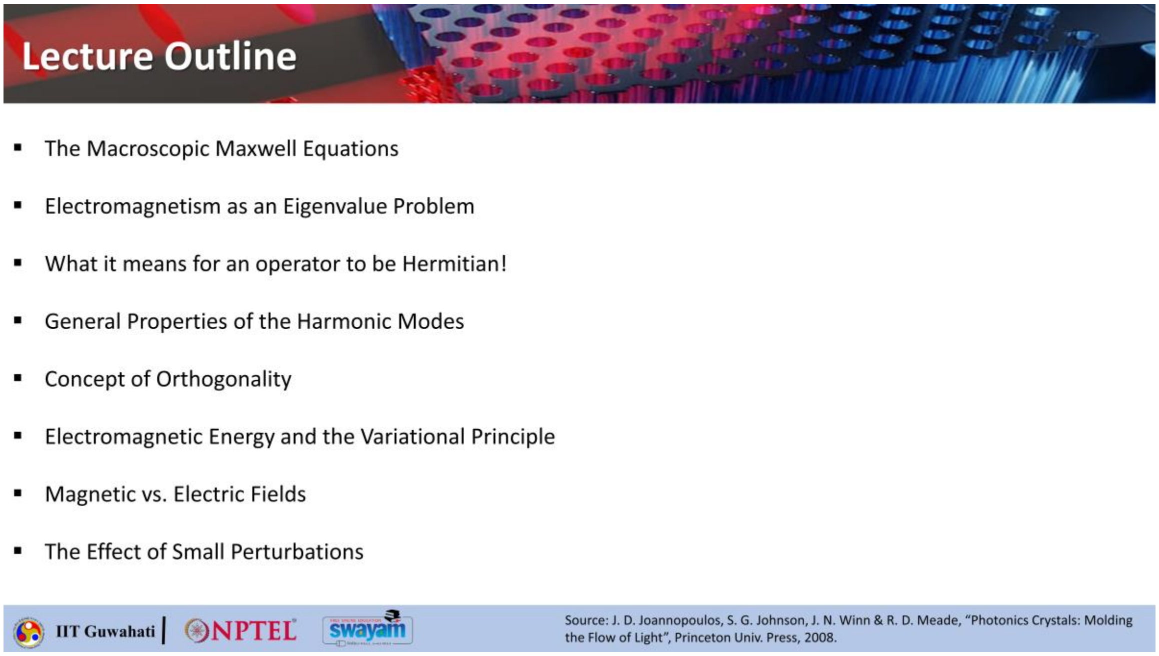
Hello students, welcome to lecture 5 of the online course on Photonic Crystals Fundamentals and Applications.

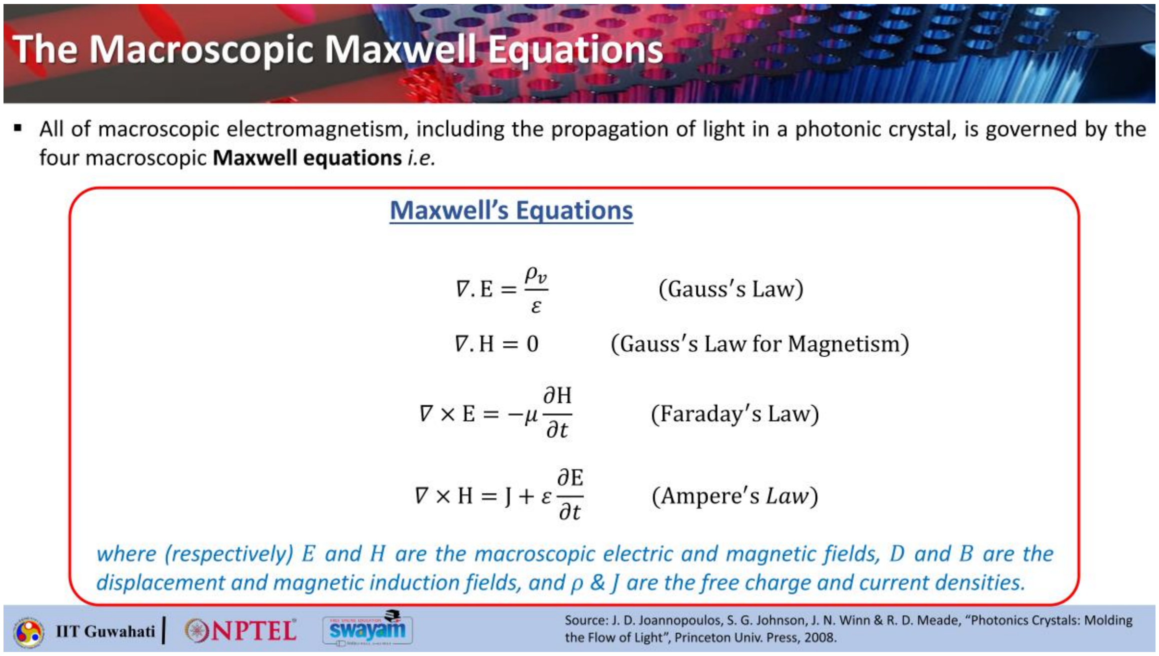


Lecture Outline

- The Macroscopic Maxwell Equations
- Electromagnetism as an Eigenvalue Problem
- What it means for an operator to be Hermitian!
- General Properties of the Harmonic Modes
- Concept of Orthogonality
- Electromagnetic Energy and the Variational Principle
- Magnetic vs. Electric Fields
- The Effect of Small Perturbations

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

Today's topic will be on electromagnetism as an eigenvalue problem. So, here is the lecture outline. So, we will look into the macroscopic Maxwell's equations, then we will introduce electromagnetism as an eigenvalue problem. What it means for an operator to be an Hermitian we'll discuss about the general properties of the harmonic modes. We'll introduce the concept of orthogonality and present electromagnetic energy, how to compute that and the variational principle and then we will discuss about why we prefer magnetic fields versus electric fields in these calculations.



The Macroscopic Maxwell Equations

- All of macroscopic electromagnetism, including the propagation of light in a photonic crystal, is governed by the four macroscopic **Maxwell equations** *i.e.*

Maxwell's Equations

$$\nabla . \mathrm{E} = \frac{\rho_v}{\varepsilon} \qquad \text{(Gauss's Law)}$$

$$\nabla . \mathrm{H} = 0 \qquad \text{(Gauss's Law for Magnetism)}$$

$$\nabla \times \mathrm{E} = -\mu \frac{\partial \mathrm{H}}{\partial t} \qquad \text{(Faraday's Law)}$$

$$\nabla \times \mathrm{H} = \mathrm{J} + \varepsilon \frac{\partial \mathrm{E}}{\partial t} \qquad \text{(Ampere's } Law\text{)}$$

where (respectively) E and H are the macroscopic electric and magnetic fields, D and B are the displacement and magnetic induction fields, and ρ & J are the free charge and current densities.

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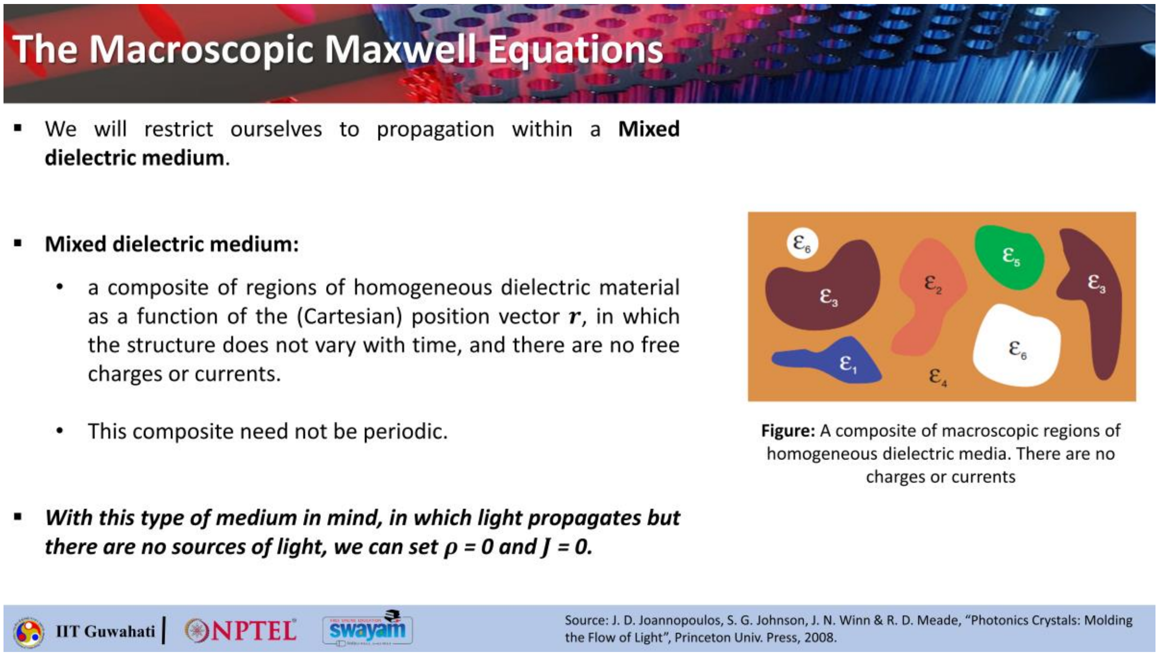
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, the first topic will be covering macroscopic Maxwell's equations. So, in order to study the propagation of light in photonic crystal, we begin with Maxwell's equation and this we have already seen in the previous lectures. So these are the four Maxwell's equations which are nothing but Gauss law, Gauss law for magnetism, Faraday's law and Ampere's Maxwell's law. So these are basically two divergence equation and two curl equations that correlate electric and magnetic field with electric displacement field and magnetic induction field along with ρ and J which are basically free charge and current densities.



The Macroscopic Maxwell Equations

- We will restrict ourselves to propagation within a **Mixed dielectric medium**.

- **Mixed dielectric medium:**
  - a composite of regions of homogeneous dielectric material as a function of the (Cartesian) position vector $\boldsymbol{r}$, in which the structure does not vary with time, and there are no free charges or currents.
  - This composite need not be periodic.

- ***With this type of medium in mind, in which light propagates but there are no sources of light, we can set ρ = 0 and J = 0.***

Figure: A composite of macroscopic regions of homogeneous dielectric media. There are no charges or currents

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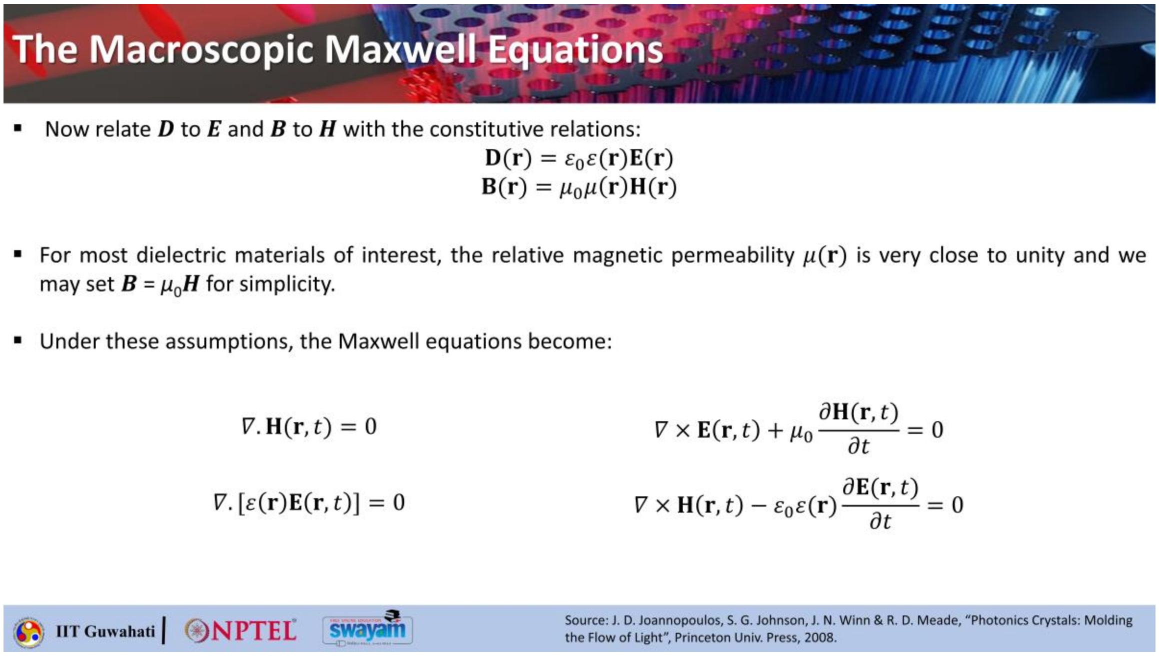
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, in this discussion we restrict ourselves to propagation within a mixed dielectric medium. So, when we say mixed dielectric medium it is basically a composite of regions of homogeneous dielectric material. as a function of the Cartesian position vector which is r, okay, in which the structure does not vary with time and there are no free charges or current. So, you can actually see small islands of different permittivity but each of this region are basically homogeneous. So, this is a overall a composite of the macroscopic regions, okay.

So, what is important here to note that there are no charges or currents and this composite need not be periodic. So, with this type of medium in mind in which the light propagates, but there are no sources of light. So, we can set ρ equals 0 and J equals 0. So, now we relate the electric displacement field to electric field and magnetic induction field or magnetic flux density to the magnetization field or magnetic field, okay. and that is done via the constitutive relations.



The Macroscopic Maxwell Equations

- Now relate $\boldsymbol{D}$ to $\boldsymbol{E}$ and $\boldsymbol{B}$ to $\boldsymbol{H}$ with the constitutive relations:

$$\mathbf{D}(\mathbf{r}) = \varepsilon_0 \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$$
$$\mathbf{B}(\mathbf{r}) = \mu_0 \mu(\mathbf{r})\mathbf{H}(\mathbf{r})$$

- For most dielectric materials of interest, the relative magnetic permeability $\mu(\mathbf{r})$ is very close to unity and we may set $\boldsymbol{B} = \mu_0 \boldsymbol{H}$ for simplicity.

- Under these assumptions, the Maxwell equations become:

$$\nabla . \mathbf{H}(\mathbf{r}, t) = 0 \qquad \nabla \times \mathbf{E}(\mathbf{r}, t) + \mu_0 \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t} = 0$$

$$\nabla . [\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}, t)] = 0 \qquad \nabla \times \mathbf{H}(\mathbf{r}, t) - \varepsilon_0 \varepsilon(\mathbf{r}) \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} = 0$$

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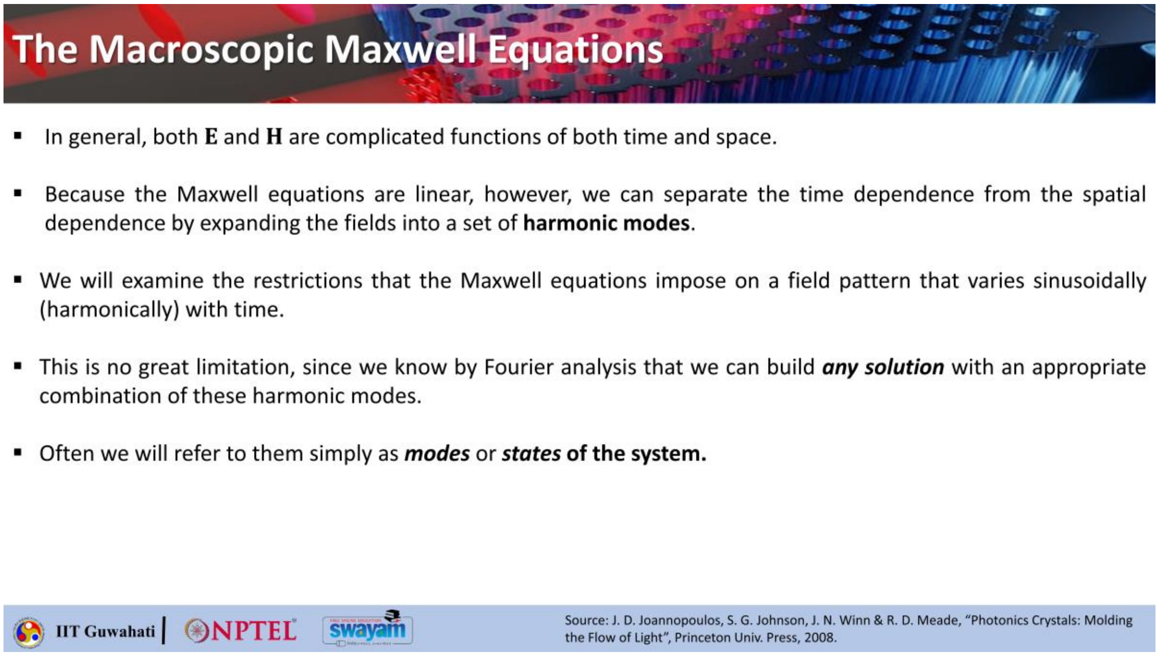
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, these are the two constitutive relations where D is a function of r can be written as ε_0 ε_r. So, here the permittivity is basically function of position and also electric field which is also function of position. Similarly, B(r) can be written as μ naught μr H(r). So, for most dielectric materials of interest the relative magnetic permeability which is μr is found to be very close to 1. So, you can simply put this as 1 that simplifies your this equation and you can write it as B equals μ naught H for simplicity.

So under this kind of assumption the Maxwell's equation will look like this. So you can write $\nabla.\mathbf{H}(\mathbf{r},t)$ which is 0 okay and ∇.D equals 0 and D can be written as $\varepsilon(\mathbf{r})$ okay. E(t) or E (r, t) and these are the two curl equations.

So, curl of E is nothing but -∂B/∂t. So, if you take the term on the left hand side and B you can write as $\mu 0\mathbf{H}$ you get this particular equation. Similarly, the fourth equation where J equals 0 you can write it as $\nabla\times\mathbf{H}$ equals ∂D/∂t. So, that can be taken on the left side and you can write $\nabla\times\mathbf{H}(\mathbf{r},t)-\varepsilon_0\ \varepsilon(\mathbf{r})(\partial\mathbf{E}(\mathbf{r},t))/\partial t$ equals 0. So, in general both E and H fields are complicated functions of both time and space.



The Macroscopic Maxwell Equations

- In general, both **E** and **H** are complicated functions of both time and space.
- Because the Maxwell equations are linear, however, we can separate the time dependence from the spatial dependence by expanding the fields into a set of **harmonic modes**.
- We will examine the restrictions that the Maxwell equations impose on a field pattern that varies sinusoidally (harmonically) with time.
- This is no great limitation, since we know by Fourier analysis that we can build ***any solution*** with an appropriate combination of these harmonic modes.
- Often we will refer to them simply as ***modes*** or ***states*** **of the system.**

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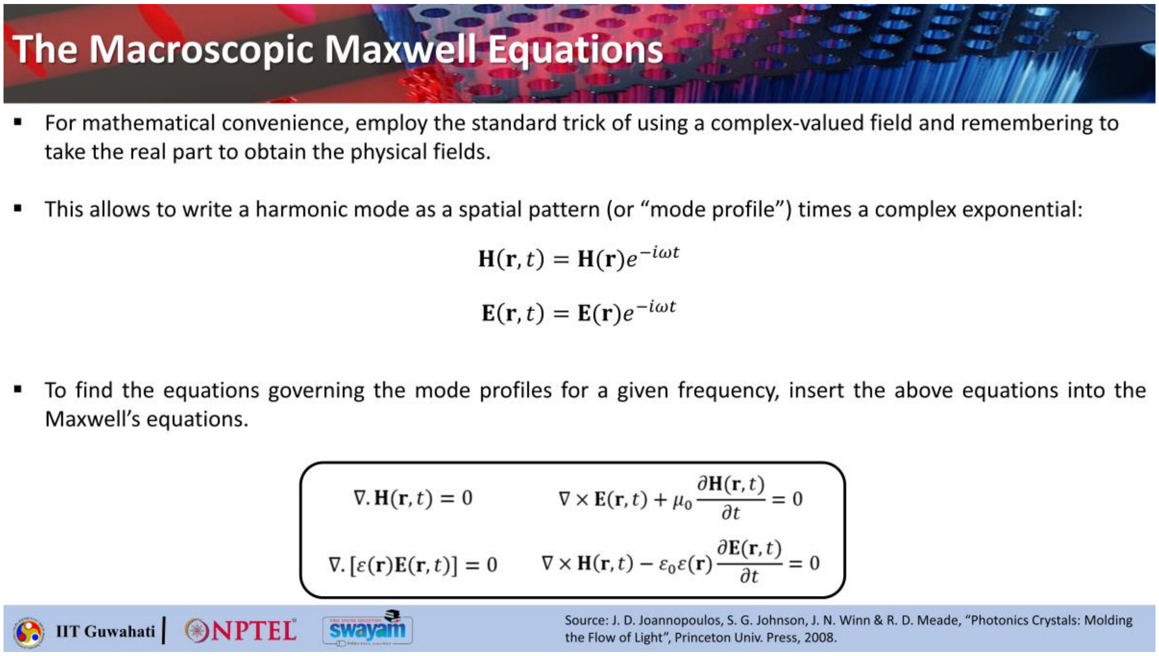
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, “Photonics Crystals: Molding the Flow of Light”, Princeton Univ. Press, 2008.

because the Maxwell's equations are linear. However, we can separate the time dependence from the spatial dependence by expanding the fields into a set of harmonic modes. So, we will examine these restrictions that Maxwell's equation impose on the field pattern that varies sinusoidally or you can say harmonically with time. There is no great limitation since we know that by Fourier analysis we can build any solution with an appropriate combination of this harmonic modes. So often, we will refer to them simply as modes or states of the system.

So we don't call them harmonic modes all the time, but we can simply call them modes. For mathematical convenience, employ the standard trick of using a complex valued field and remembering to take the real part of it to obtain the physical field. And this will allow us to write the harmonic field as a spatial pattern which is also known as the mode profile times a complex exponential,



The Macroscopic Maxwell Equations

- For mathematical convenience, employ the standard trick of using a complex-valued field and remembering to take the real part to obtain the physical fields.
- This allows to write a harmonic mode as a spatial pattern (or "mode profile") times a complex exponential:

$$\mathbf{H}(\mathbf{r},t) = \mathbf{H}(\mathbf{r})e^{-i\omega t}$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})e^{-i\omega t}$$

- To find the equations governing the mode profiles for a given frequency, insert the above equations into the Maxwell's equations.

$$\boxed{\begin{array}{ll} \nabla.\mathbf{H}(\mathbf{r},t) = 0 & \nabla\times\mathbf{E}(\mathbf{r},t) + \mu_0\dfrac{\partial\mathbf{H}(\mathbf{r},t)}{\partial t} = 0 \\ \nabla.[\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r},t)] = 0 & \nabla\times\mathbf{H}(\mathbf{r},t) - \varepsilon_0\varepsilon(\mathbf{r})\dfrac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t} = 0 \end{array}}$$

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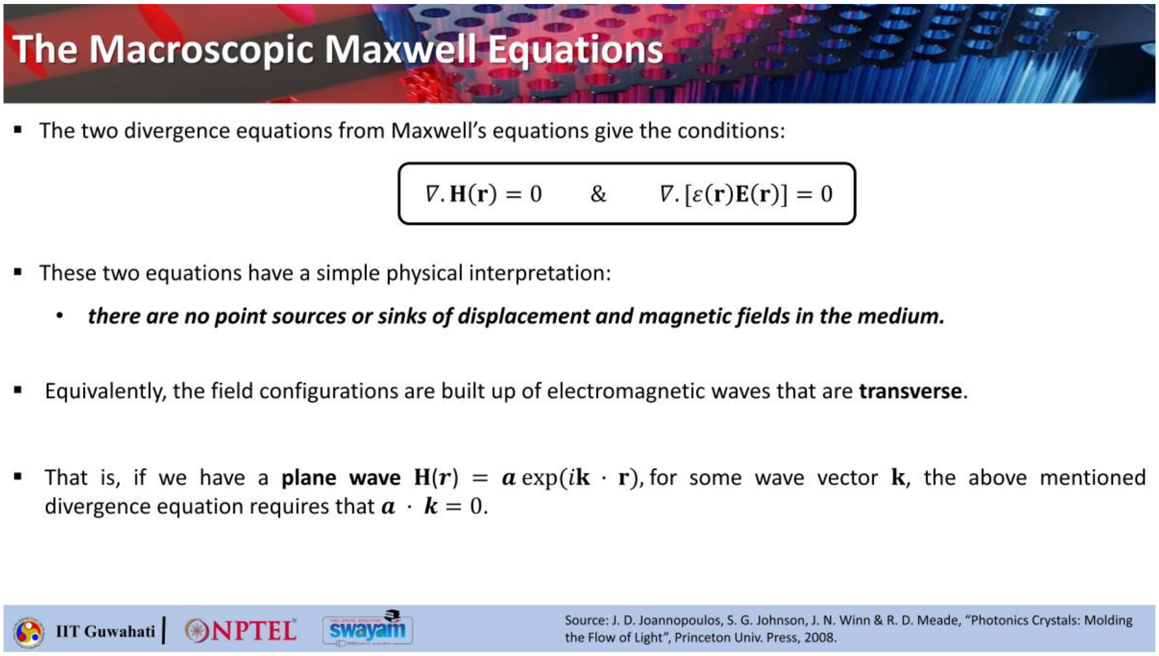
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, mathematically that field can be written as this. So, H which is a function of position and time can be written as H(r) which is only having spatial dependence and then you have a time exponential which is $e\hat{}(-i\omega t)$.

So, this is the complex exponential. So, similarly the form for magnetic field can also be written for electric field and you get this particular equation. So, to find the equations governing the mode profiles for a given frequency what you can do you can insert the above equations into the Maxwell's equation. So, when you put them into the Maxwell's equation, so these are the Maxwell's equation that you have seen in the previous slide. You can obtain the two divergence equation which are simply this ∇.

$\mathbf{H}(\mathbf{r})$ equals 0 and you will find $\nabla.[\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})]$ equals 0. Now these two equations have very simple physical interpretation okay. So there are no point sources or sink of displacement and magnetic field in the medium. So this is the physical interpretation of these two particular equations.



The Macroscopic Maxwell Equations

- The two divergence equations from Maxwell's equations give the conditions:

$$\boxed{\nabla.\mathbf{H}(\mathbf{r}) = 0 \qquad \& \qquad \nabla.[\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})] = 0}$$

- These two equations have a simple physical interpretation:
  - ***there are no point sources or sinks of displacement and magnetic fields in the medium.***

- Equivalently, the field configurations are built up of electromagnetic waves that are **transverse**.

- That is, if we have a **plane wave** $\mathbf{H}(\boldsymbol{r}) = \boldsymbol{a}\exp(i\mathbf{k}\cdot\mathbf{r})$, for some wave vector $\mathbf{k}$, the above mentioned divergence equation requires that $\boldsymbol{a}\cdot\boldsymbol{k} = 0$.

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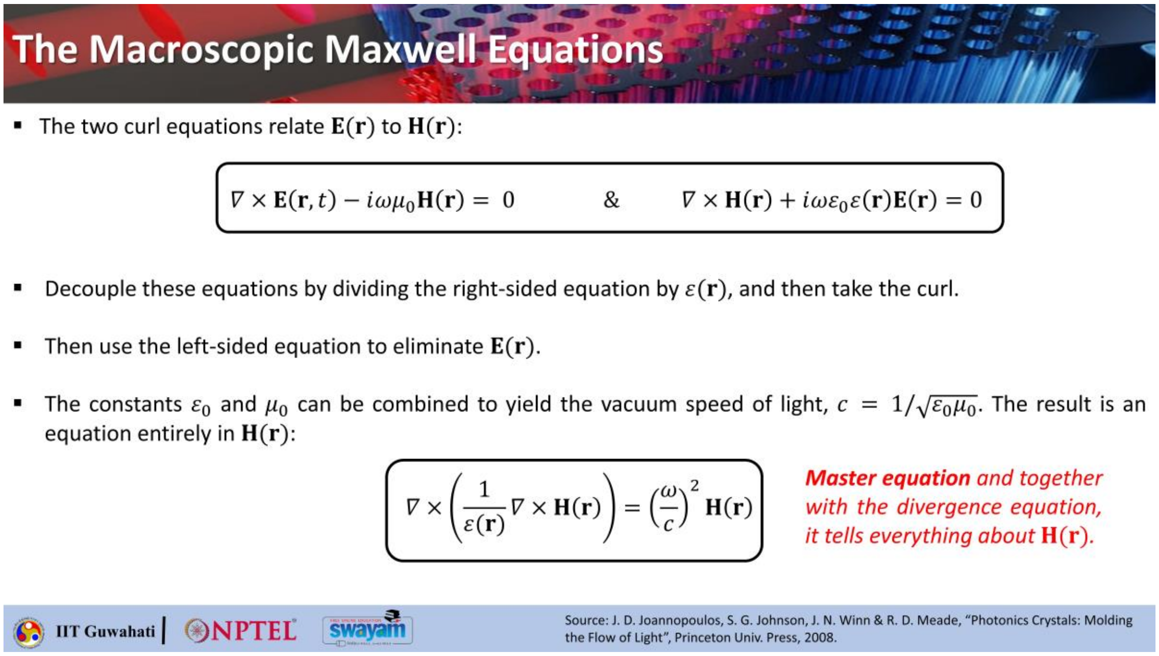
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

Now equivalently the field configuration are built up of electromagnetic waves that are transverse. if we have a plane wave that is expressed as $\mathbf{H}(\boldsymbol{r}) = \boldsymbol{a}\exp(i\mathbf{k}\cdot\mathbf{r})$. So k is the wave vector and r is the position vector okay. So in this case the ever mentioned divergence equation okay of this one will require that $\boldsymbol{a}\cdot\boldsymbol{k}$ the dot product of these two should be 0. Now, you can use the two curl equations which basically correlate electric field with magnetic field.



The Macroscopic Maxwell Equations

- The two curl equations relate $\mathbf{E}(\mathbf{r})$ to $\mathbf{H}(\mathbf{r})$:

$$\boxed{\nabla \times \mathbf{E}(\mathbf{r},t) - i\omega\mu_0\mathbf{H}(\mathbf{r}) = 0 \qquad \& \qquad \nabla \times \mathbf{H}(\mathbf{r}) + i\omega\varepsilon_0\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}) = 0}$$

- Decouple these equations by dividing the right-sided equation by $\varepsilon(\mathbf{r})$, and then take the curl.
- Then use the left-sided equation to eliminate $\mathbf{E}(\mathbf{r})$.
- The constants ε_0 and μ_0 can be combined to yield the vacuum speed of light, $c = 1/\sqrt{\varepsilon_0\mu_0}$. The result is an equation entirely in $\mathbf{H}(\mathbf{r})$:

$$\boxed{\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})}\nabla \times \mathbf{H}(\mathbf{r})\right) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})}$$

***Master equation** and together with the divergence equation, it tells everything about $\mathbf{H}(\mathbf{r})$.*

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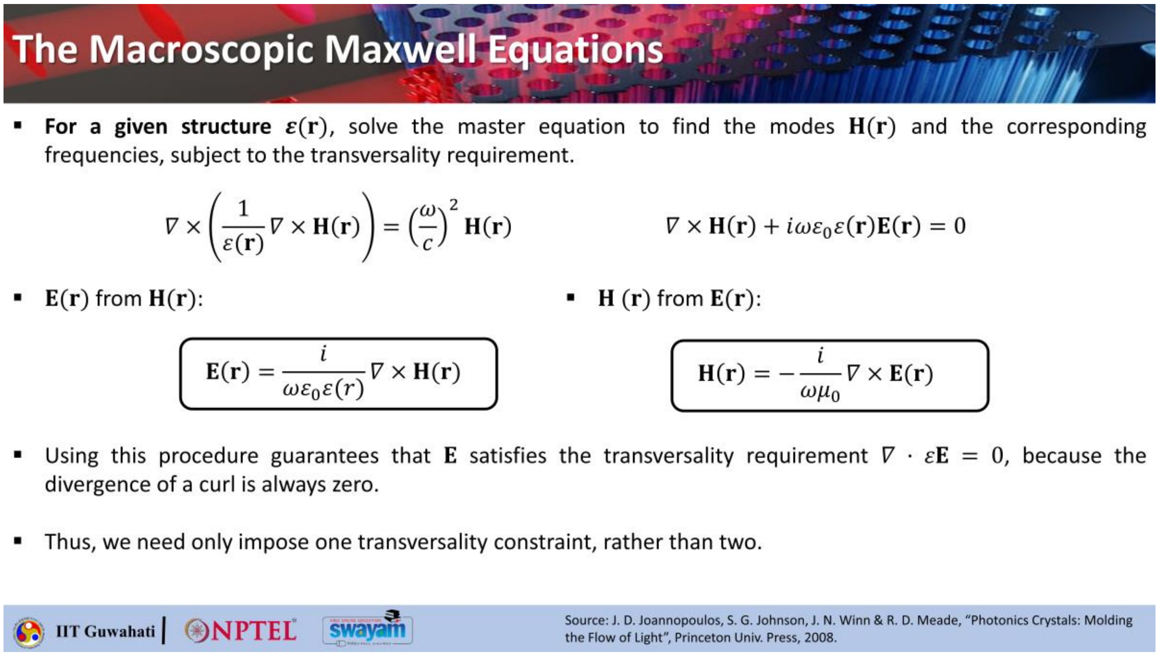
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, “Photonics Crystals: Molding the Flow of Light”, Princeton Univ. Press, 2008.

So, you can write this from the third Maxwell's equation. So, $\nabla\times\mathbf{E}(\mathbf{r},t)$ equals you know $-i\omega\mu_0\,\mathbf{H}(\mathbf{r})$. So, you have actually you know taken that time dependence and try to write this down okay. So, if you remember the time dependence that you have seen here, so you can take this okay. So, $e^{(-i\omega t)}$ you can take the time derivative of it that will give you that $i\omega$ term that comes in the that comes over here right.

So, this is how you can write that. So, similarly the fourth equation can be written like this So, decouple these equations by dividing the right side equation by epsilon r. So, if you divide this by $\varepsilon(\mathbf{r})$ you will only have the expression for E(r) okay and then you take it on the other side you take a curl of that okay and that curl can be equated to this one because this is also curl of E(t). So, the mathematics is pretty simple here I urge all of you who were not able to you know visualize this you can do it on pen and paper and you will see that you can actually eliminate E(r) from the expression by using these two expression and you can have everything in terms of H ok. And the constant is ε_0 and μ_0 can be combined to yield the vacuum speed of light that is c equals 1 over square root of ε_0 μ_0 okay and the result is an equation that entirely depends on H(r) okay.

So, you can finally have the expression looking like this. So, which is curl of this parameter, which is 1 by ε_0 and then curl of H(r), okay? And that is equal to omega by c whole square H(r). So, this is the master equation and together with the divergence equation that is $\nabla.\mathbf{H}(\mathbf{r})$ equals 0, that tells us everything about the magnetic field. And the beauty of the system is that once you know everything about the magnetic field, you can find everything about the electric field also because they are correlated.



The Macroscopic Maxwell Equations

- **For a given structure $\boldsymbol{\varepsilon}(\mathbf{r})$**, solve the master equation to find the modes $\mathbf{H}(\mathbf{r})$ and the corresponding frequencies, subject to the transversality requirement.

$$\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r}) \right) = \left(\frac{\omega}{c} \right)^2 \mathbf{H}(\mathbf{r}) \qquad \nabla \times \mathbf{H}(\mathbf{r}) + i\omega\varepsilon_0\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}) = 0$$

- $\mathbf{E}(\mathbf{r})$ from $\mathbf{H}(\mathbf{r})$:

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{i}{\omega\varepsilon_0\varepsilon(r)} \nabla \times \mathbf{H}(\mathbf{r})}$$

- $\mathbf{H}\,(\mathbf{r})$ from $\mathbf{E}(\mathbf{r})$:

$$\boxed{\mathbf{H}(\mathbf{r}) = -\frac{i}{\omega\mu_0} \nabla \times \mathbf{E}(\mathbf{r})}$$

- Using this procedure guarantees that $\mathbf{E}$ satisfies the transversality requirement $\nabla \cdot \varepsilon\mathbf{E} = 0$, because the divergence of a curl is always zero.

- Thus, we need only impose one transversality constraint, rather than two.

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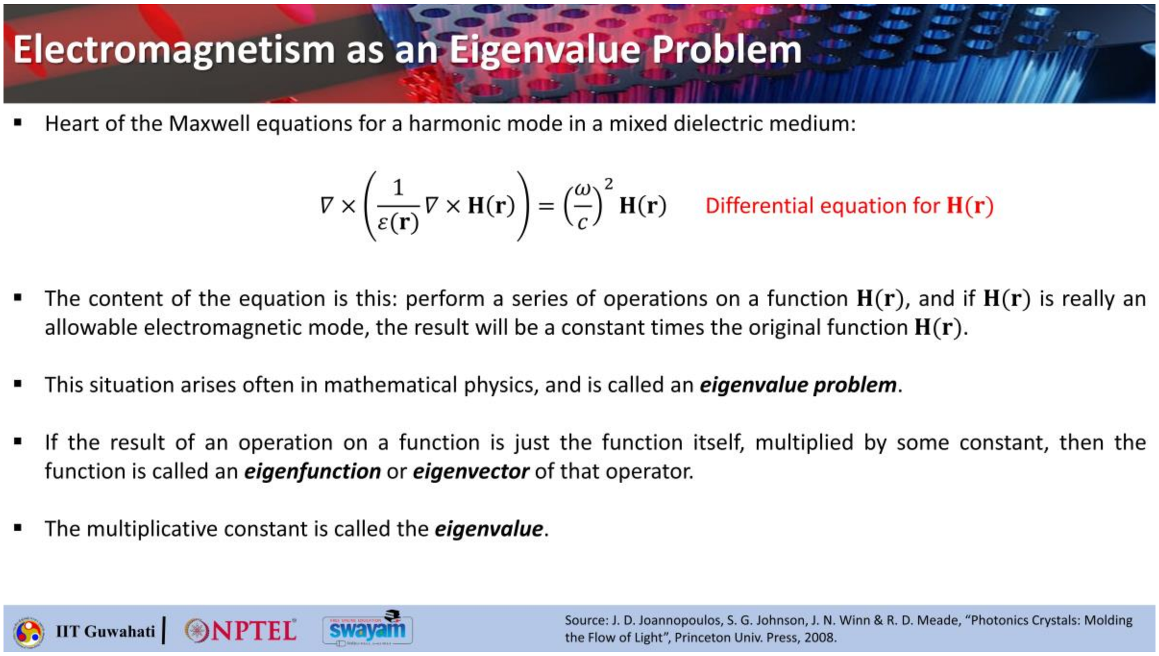
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, “Photonics Crystals: Molding the Flow of Light”, Princeton Univ. Press, 2008.

So, for a given structure εr we solve for the master equation to find out the modes of H(r) and its corresponding frequencies subject to the transversality requirement. So this is the equation, the master equation and this is the fourth Maxwell's equation that correlates the curl of H with E(r). So you can also find out from here what is E(r) in terms of curl of H and you can also find H(r) in terms of E(r). So this is how you can obtain one and then get the other. Now, using this process guarantees that electric field satisfied the transversatility requirement that is divergence of D is 0 because the divergence of a curl is always 0 right.



Electromagnetism as an Eigenvalue Problem

- Heart of the Maxwell equations for a harmonic mode in a mixed dielectric medium:

$$\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r}) \right) = \left(\frac{\omega}{c} \right)^2 \mathbf{H}(\mathbf{r})$$

Differential equation for $\mathbf{H}(\mathbf{r})$

- The content of the equation is this: perform a series of operations on a function $\mathbf{H}(\mathbf{r})$, and if $\mathbf{H}(\mathbf{r})$ is really an allowable electromagnetic mode, the result will be a constant times the original function $\mathbf{H}(\mathbf{r})$.
- This situation arises often in mathematical physics, and is called an ***eigenvalue problem***.
- If the result of an operation on a function is just the function itself, multiplied by some constant, then the function is called an ***eigenfunction*** or ***eigenvector*** of that operator.
- The multiplicative constant is called the ***eigenvalue***.

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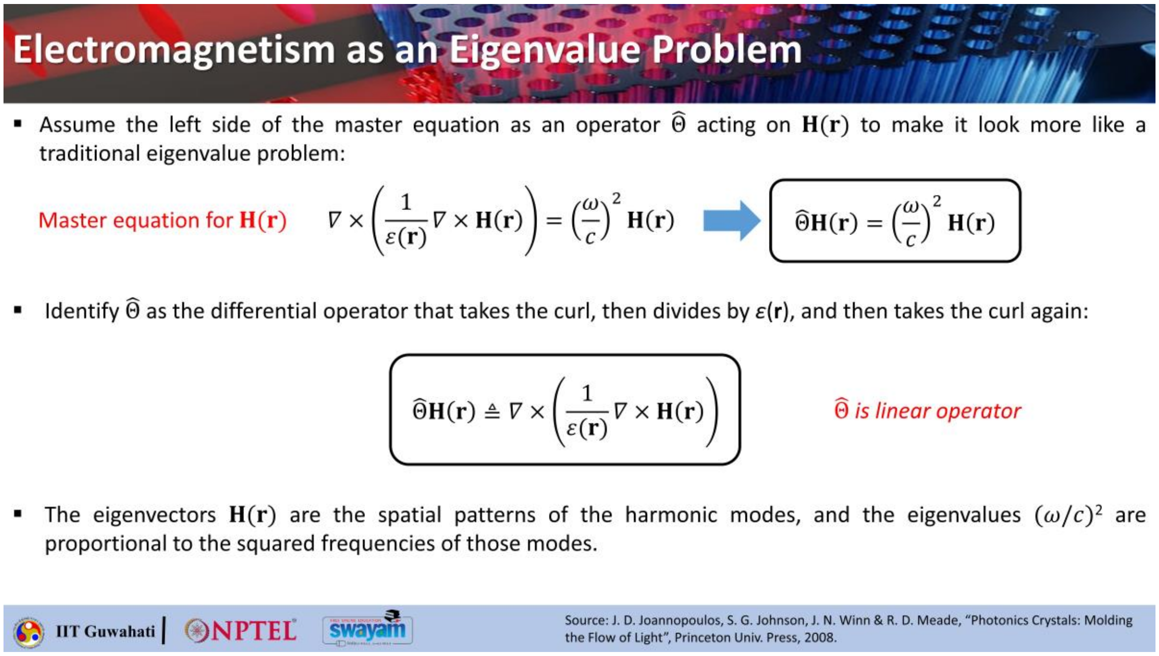
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, in this case what is understood that we do not require two transversatility constraint rather we can only impose one and that will give us one particular field and from that you can obtain the other field. So, now we will move on to you know formulate electromagnetism as an eigenvalue problem. So, the heart of the Maxwell's equation for harmonic mode in a dielectric medium can be written as this okay. So, this is nothing but a differential equation of H(r) okay and the content of this equation if you see that you are basically performing a series of operations on the magnetic field H(r) okay. And if H(r) is really and allowable electromagnetic mode, okay, the result that you are obtained on the right side is nothing but a constant times the original function H(r), okay.

So, what is the function here? You first take a curl, then you divide that curl by εr and then again you take the curl of that entire quantity. So, you are actually basing a lot of operations on the magnetic field. And at the end, what do you get? You are again getting that magnetic field function times a constant. So this kind of situation arises often in mathematical physics and these are called eigenvalue problem. So if the result of an operation on a function is just the function itself multiplied by some constant, then that function is called an eigenfunction or eigenvector for that particular operator.

And the multiplicative constant that you see here is called an eigenvalue, right?



Electromagnetism as an Eigenvalue Problem

- Assume the left side of the master equation as an operator $\widehat{\Theta}$ acting on $\mathbf{H}(\mathbf{r})$ to make it look more like a traditional eigenvalue problem:

Master equation for $\mathbf{H}(\mathbf{r})$

$$\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})}\nabla \times \mathbf{H}(\mathbf{r})\right) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r}) \Rightarrow \boxed{\widehat{\Theta}\mathbf{H}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})}$$

- Identify $\widehat{\Theta}$ as the differential operator that takes the curl, then divides by $\varepsilon(\mathbf{r})$, and then takes the curl again:

$$\boxed{\widehat{\Theta}\mathbf{H}(\mathbf{r}) \triangleq \nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})}\nabla \times \mathbf{H}(\mathbf{r})\right)}$$

$\widehat{\Theta}$ is linear operator

- The eigenvectors $\mathbf{H}(\mathbf{r})$ are the spatial patterns of the harmonic modes, and the eigenvalues $(\omega/c)^2$ are proportional to the squared frequencies of those modes.

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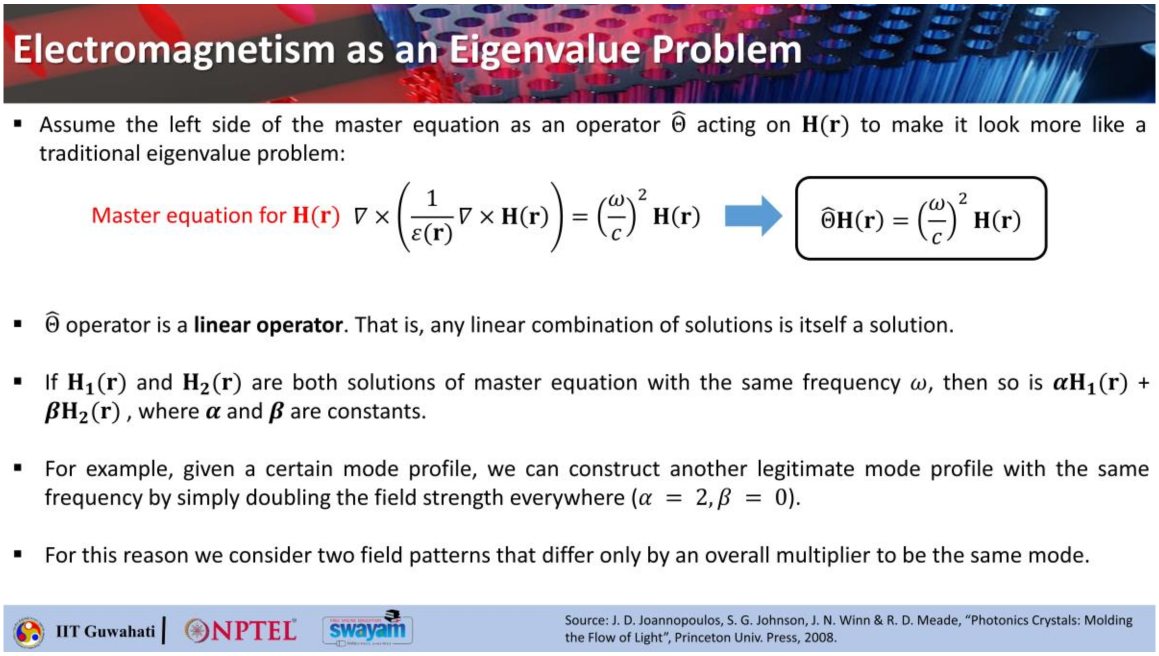
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So assume that the left side of the equation. So, this is the master equation. So, all these things if you if you you know assume that as an operator Θ cap which is acting on magnetic field okay then this master equation can be rewritten like this that you have a Maxwell operator that operates on magnetic field and gives you a constant times that same function. So, it looks pretty much like the eigenvalue problem. So, what is this Θ cap? This is basically the differential operator that takes the curl of magnetic field divides with you know εr and then again takes another curl okay.

So, this is this operator. So, curl of H is basically this sorry Θ cap H(r) is basically this. So, Θ cap is a linear operator. Now, the eigenvector, so H(r) is eigenvectors, okay, so which are basically the spatial patterns of the harmonic modes and $(\omega/c)^2$ that you see here are basically proportional to the squared frequencies of those modes. So, ω is basically the frequency of the particular eigenmode.



Electromagnetism as an Eigenvalue Problem

- Assume the left side of the master equation as an operator $\hat{\Theta}$ acting on $\mathbf{H}(\mathbf{r})$ to make it look more like a traditional eigenvalue problem:

Master equation for $\mathbf{H}(\mathbf{r})$ $$\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})}\nabla \times \mathbf{H}(\mathbf{r})\right) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r}) \Rightarrow \boxed{\hat{\Theta}\mathbf{H}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})}$$

- $\hat{\Theta}$ operator is a **linear operator**. That is, any linear combination of solutions is itself a solution.
- If $\mathbf{H_1}(\mathbf{r})$ and $\mathbf{H_2}(\mathbf{r})$ are both solutions of master equation with the same frequency ω, then so is $\boldsymbol{\alpha}\mathbf{H_1}(\mathbf{r}) + \boldsymbol{\beta}\mathbf{H_2}(\mathbf{r})$, where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are constants.
- For example, given a certain mode profile, we can construct another legitimate mode profile with the same frequency by simply doubling the field strength everywhere ($\alpha = 2, \beta = 0$).
- For this reason we consider two field patterns that differ only by an overall multiplier to be the same mode.

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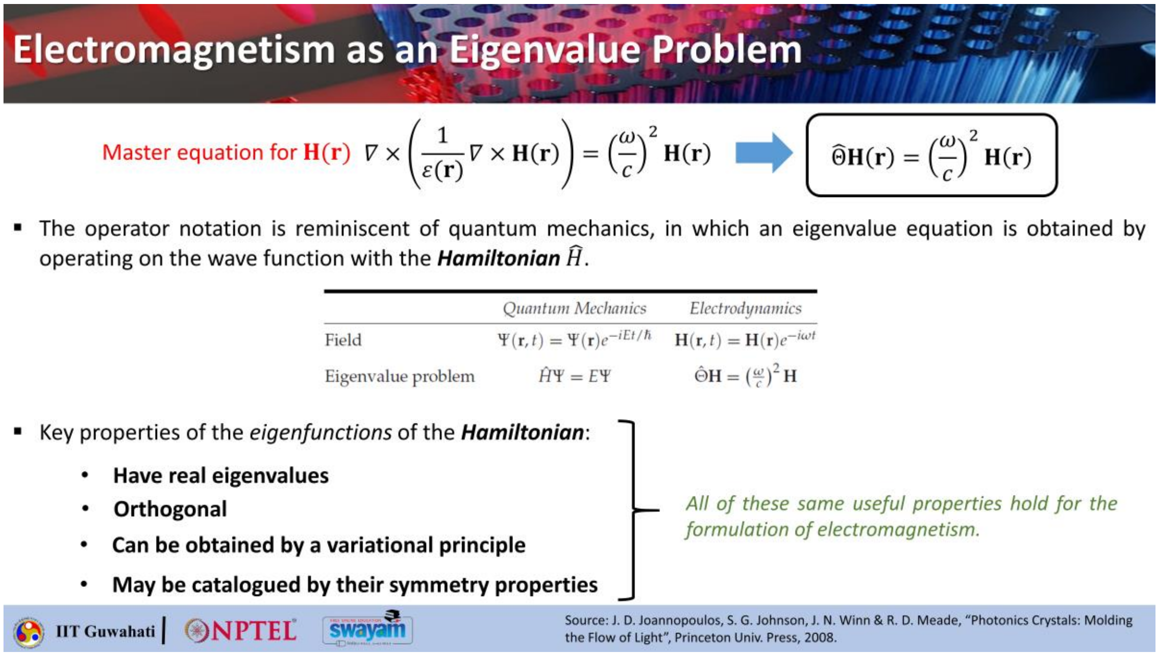
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, we can assume that the left side okay, so this we have already seen that this is typically how the you know Maxwell's equation or the master equation looks like more of a eigenvalue problem and we have also understood that this operator Maxwell operator theta cap is a linear operator and for any linear combination of the solution will again be a solution right.

So, we can actually assume that if $\mathbf{H_1}$ $(\mathbf{r})$ and $\mathbf{H_2}$ $(\mathbf{r})$ both are the solutions of this master equation with same frequency ω then a combination of that say $\boldsymbol{\alpha}\mathbf{H}_1(\mathbf{r})$ + $\boldsymbol{\beta}\mathbf{H}_2$ $(\mathbf{r})$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are constant that is also a solution of the master equation. For example, given a certain mode profile we can construct another legitimate mode profile with the same frequency by simply doubling the you know field strength everywhere. So, you can choose $\boldsymbol{\alpha}$ equals 2. So, that will double up the magnetic field strength everywhere and that will also be a solution. So, for this region we consider two field patterns that differ only by an overall multiplier to be the same mode.



Electromagnetism as an Eigenvalue Problem

Master equation for $\mathbf{H}(\mathbf{r})$ $$\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})}\nabla \times \mathbf{H}(\mathbf{r})\right) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r}) \Rightarrow \boxed{\hat{\Theta}\mathbf{H}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})}$$

- The operator notation is reminiscent of quantum mechanics, in which an eigenvalue equation is obtained by operating on the wave function with the ***Hamiltonian*** $\widehat{H}$.

	Quantum Mechanics	*Electrodynamics*
Field	$\Psi(\mathbf{r},t) = \Psi(\mathbf{r})e^{-iEt/\hbar}$	$\mathbf{H}(\mathbf{r},t) = \mathbf{H}(\mathbf{r})e^{-i\omega t}$
Eigenvalue problem	$\hat{H}\Psi = E\Psi$	$\hat{\Theta}\mathbf{H} = \left(\frac{\omega}{c}\right)^2\mathbf{H}$

- Key properties of the *eigenfunctions* of the ***Hamiltonian***:
  - **Have real eigenvalues**
  - **Orthogonal**
  - **Can be obtained by a variational principle**
  - **May be catalogued by their symmetry properties**

All of these same useful properties hold for the formulation of electromagnetism.

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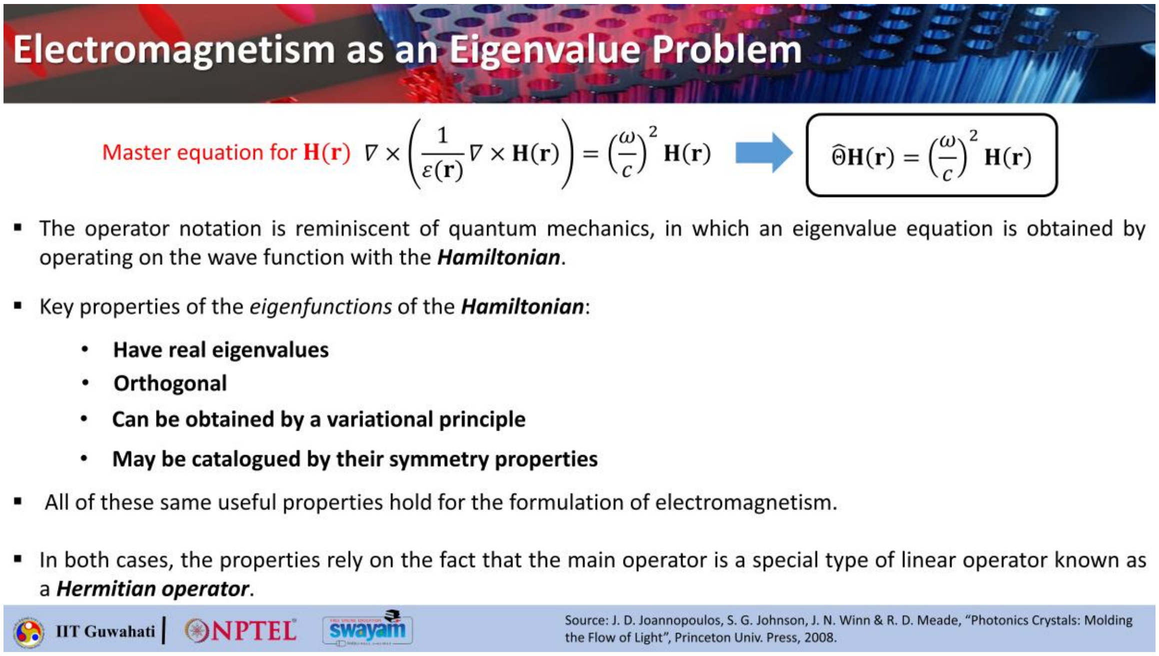
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, we actually now consider those as same mode because they just only differ by a multiplying factor ok. So, the operator notation that you see here is pretty similar to that of quantum mechanics in which an eigenvalue function is basically obtained by operating on wave function with the Hamiltonian H cap. okay. So, you can see that in quantum mechanics and electrodynamics if you put them side by side this is the wave function okay or wave scalar potential okay. So, if you compare quantum mechanics and electrodynamics side by side you can see the field is expressed as this in quantum mechanics whether in electrodynamics we represent it by this which we have seen earlier.



Electromagnetism as an Eigenvalue Problem

Master equation for $\mathbf{H}(\mathbf{r})$ $$\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})}\nabla \times \mathbf{H}(\mathbf{r})\right) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r}) \Rightarrow \boxed{\hat{\Theta}\mathbf{H}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})}$$

- The operator notation is reminiscent of quantum mechanics, in which an eigenvalue equation is obtained by operating on the wave function with the ***Hamiltonian***.
- Key properties of the *eigenfunctions* of the ***Hamiltonian***:
  - **Have real eigenvalues**
  - **Orthogonal**
  - **Can be obtained by a variational principle**
  - **May be catalogued by their symmetry properties**
- All of these same useful properties hold for the formulation of electromagnetism.
- In both cases, the properties rely on the fact that the main operator is a special type of linear operator known as a ***Hermitian operator***.

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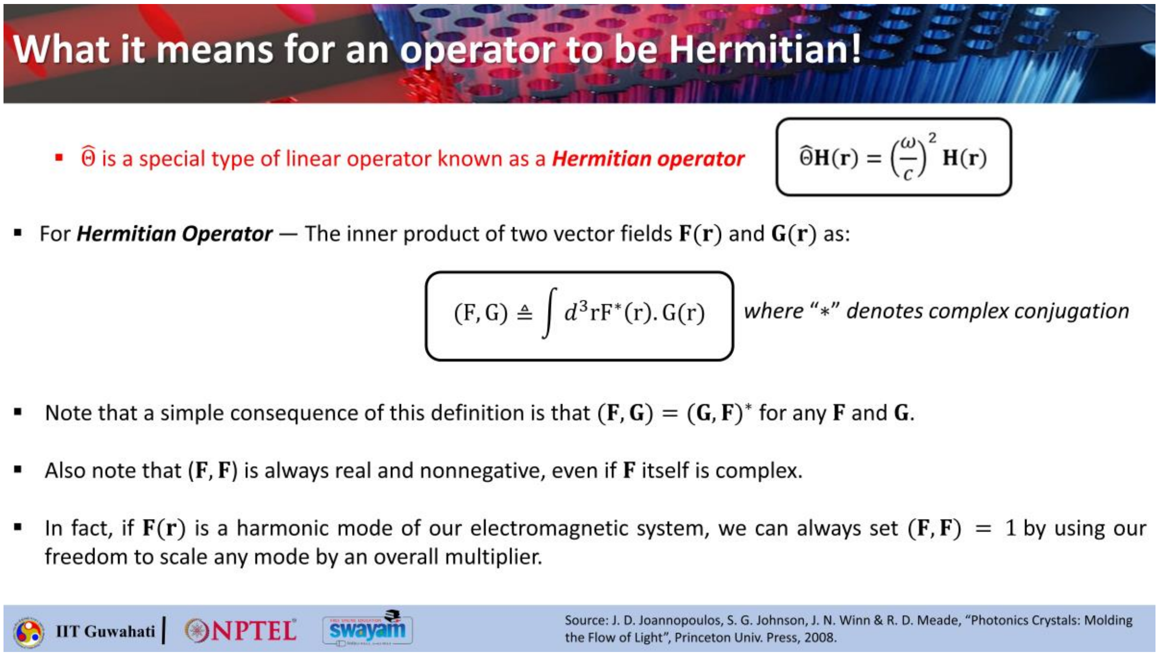
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, the eigenvalue here eigenvalue problem here. looks like H cap $\mathbf{\Psi}$ equals E $\mathbf{\Psi}$ whereas in this case this is the eigenvalue problem. So, you have Θ cap H equals $(\omega/c)^2$ into H. So, this kind of operator notation is very similar to that of quantum mechanics. So, we are basically an eigenvalue equation is obtained by operating on the wave function with the Hamiltonian here.



What it means for an operator to be Hermitian!

- $\hat{\Theta}$ is a special type of linear operator known as a ***Hermitian operator***

$$\boxed{\hat{\Theta}\mathbf{H}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})}$$

- For ***Hermitian Operator*** — The inner product of two vector fields $\mathbf{F}(\mathbf{r})$ and $\mathbf{G}(\mathbf{r})$ as:

$$\boxed{(\mathrm{F},\mathrm{G}) \triangleq \int d^3\mathrm{r}\,\mathrm{F}^*(\mathrm{r}).\,\mathrm{G}(\mathrm{r})}$$

where "$*$" denotes complex conjugation

- Note that a simple consequence of this definition is that $(\mathbf{F},\mathbf{G}) = (\mathbf{G},\mathbf{F})^*$ for any $\mathbf{F}$ and $\mathbf{G}$.
- Also note that $(\mathbf{F},\mathbf{F})$ is always real and nonnegative, even if $\mathbf{F}$ itself is complex.
- In fact, if $\mathbf{F}(\mathbf{r})$ is a harmonic mode of our electromagnetic system, we can always set $(\mathbf{F},\mathbf{F}) = 1$ by using our freedom to scale any mode by an overall multiplier.

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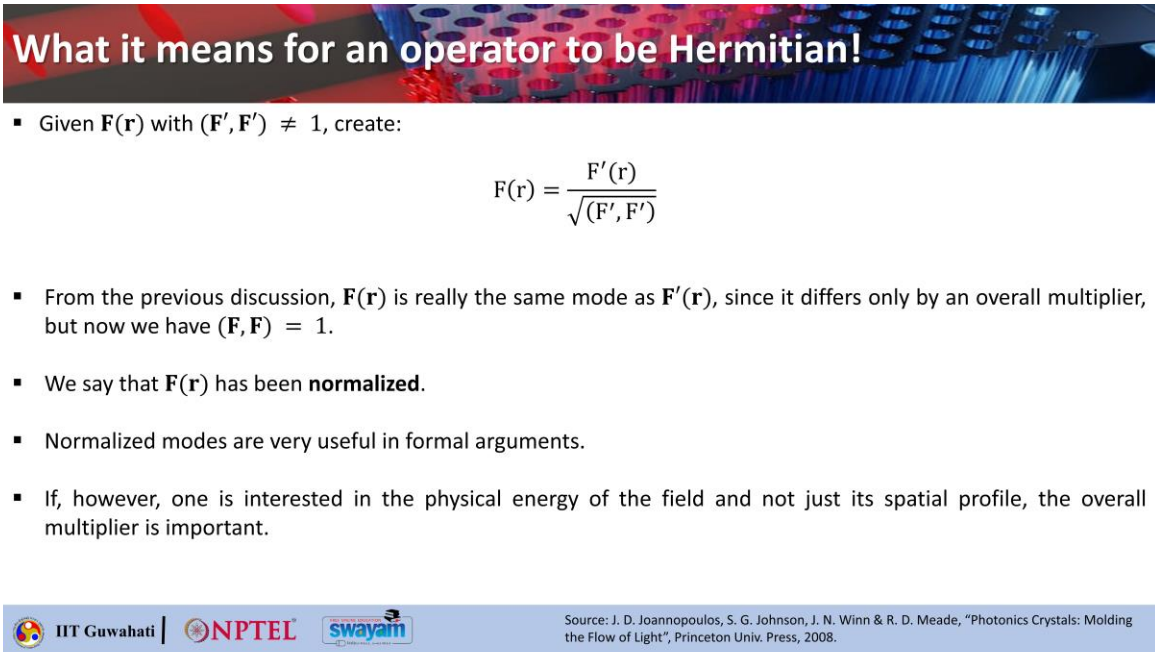
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, here The key properties of the eigenfunctions of the Hamiltonian will be to have real eigenvalues okay, they will be orthogonal, they can be obtained by variational principle and they can be categorized by their symmetric properties. And all these properties also you know hold for the formulation of electrode magnetism right.So, now let us look into what it means for an operator to be an Hermitian. So, Θ cap which is the Maxwell's operator is a special type of linear operator which is known as a Hermitian operator. So, what is that Hermitian operator and what does it take for an operator to be Hermitian? So, this is the master equation.

So, for an Hermitian operator the inner product of two vector fields, so if you take $\mathbf{F}(\mathbf{r})$ and $\mathbf{G}(\mathbf{r})$, their inner product can be written as this. So, it is a volume integral of one like F conjugate (r) times you know or dot G(r). So, now that a simple consequence of this definition will be that the inner product of (F,G) is nothing but (G,F) conjugate and that is true for any F and G. Now, if you take the inner product of a vector field with itself, so you can write $(\mathbf{F}, \mathbf{F})$ which is always real and non-negative even if F is complex. So, in fact if F(r) is a harmonic mode of our electromagnetic system, we can always set that the inner product of F with itself is 1 and by using our freedom to scale any mode by you know an overall multiplier.

So, this actually if you set this you can get that freedom of scaling any mode by a multiplier.



What it means for an operator to be Hermitian!

- Given $\mathbf{F}(\mathbf{r})$ with $(\mathbf{F}',\mathbf{F}') \neq 1$, create:

$$\mathrm{F}(\mathrm{r}) = \frac{\mathrm{F}'(\mathrm{r})}{\sqrt{(\mathrm{F}',\mathrm{F}')}}$$

- From the previous discussion, $\mathbf{F}(\mathbf{r})$ is really the same mode as $\mathbf{F}'(\mathbf{r})$, since it differs only by an overall multiplier, but now we have $(\mathbf{F},\mathbf{F}) = 1$.
- We say that $\mathbf{F}(\mathbf{r})$ has been **normalized**.
- Normalized modes are very useful in formal arguments.
- If, however, one is interested in the physical energy of the field and not just its spatial profile, the overall multiplier is important.

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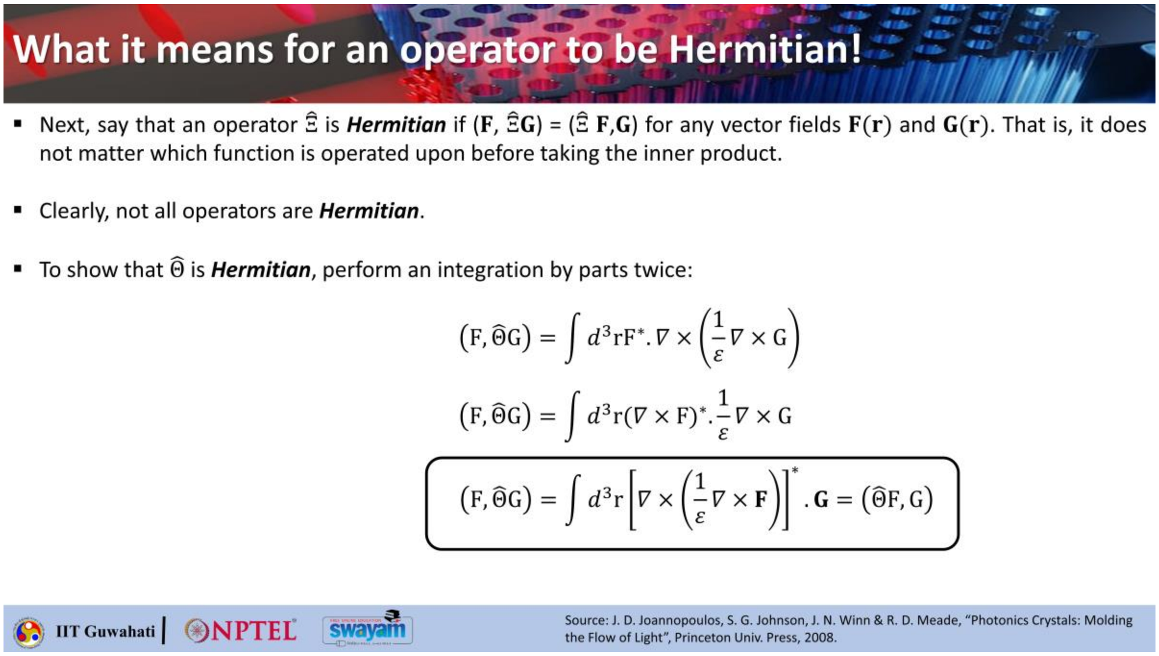
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, given $\mathrm{F}'(\mathrm{r})$ where you know the inner product of $\mathbf{F}'$ with itself is not 1 you can write F as $\mathrm{F}'(\mathrm{r})$ divided by whatever is that value right. So, from the previous discussion we understand that F(r) is basically the same mode as $\mathrm{F}'(\mathrm{r})$ just that it differs only by overall multiplier. but now we have this inner product of F with itself is 1 that means F(r) is basically normalized. And normalized modes are very useful in formal arguments if however if one is interested in the physical energy of the mode and not just its spatial profile, then the overall multiplier is also important.



What it means for an operator to be Hermitian!

- Next, say that an operator $\hat{\Xi}$ is ***Hermitian*** if $(\mathbf{F}, \hat{\Xi}\mathbf{G}) = (\hat{\Xi}\,\mathbf{F},\mathbf{G})$ for any vector fields $\mathbf{F}(\mathbf{r})$ and $\mathbf{G}(\mathbf{r})$. That is, it does not matter which function is operated upon before taking the inner product.
- Clearly, not all operators are ***Hermitian***.
- To show that $\hat{\Theta}$ is ***Hermitian***, perform an integration by parts twice:

$$(\mathrm{F}, \hat{\Theta}\mathrm{G}) = \int d^3\mathrm{r}\mathrm{F}^*.\nabla \times \left(\frac{1}{\varepsilon}\nabla \times \mathrm{G}\right)$$

$$(\mathrm{F}, \hat{\Theta}\mathrm{G}) = \int d^3\mathrm{r}(\nabla \times \mathrm{F})^*.\frac{1}{\varepsilon}\nabla \times \mathrm{G}$$

$$\boxed{(\mathrm{F}, \hat{\Theta}\mathrm{G}) = \int d^3\mathrm{r}\left[\nabla \times \left(\frac{1}{\varepsilon}\nabla \times \mathbf{F}\right)\right]^* .\mathbf{G} = (\hat{\Theta}\mathrm{F}, \mathrm{G})}$$

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Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, “Photonics Crystals: Molding the Flow of Light”, Princeton Univ. Press, 2008.

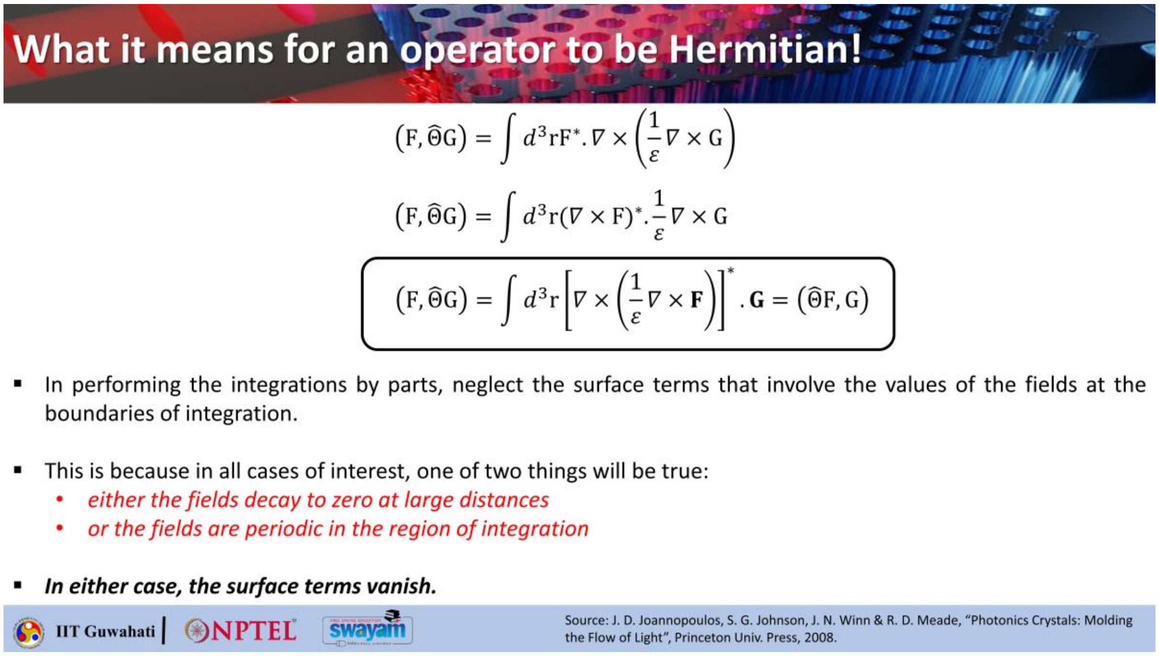
Because normalized will not tell you exactly about the actual physical energy of that field, it will only show you the spatial profile. So, you need to know the overall multiplier to know the exact value of the physical fields. Next let us assume that an operator Ξ cap, capital Ξ cap is Hermitian. If the inner product of F with this ΞG is same as you know (ΞF,G), the inner product of this with this and that is true for any vector field F and G. That is it does not matter which function is operated upon before taking the inner product.

So, if you either like take the you operate the function on G. and then you take the inner product or you operate the function on F and you take the inner product, the inner product should come out to be same in that case you know you can say that function is Hermitian. So, obviously you can understand that not all operators are Hermitian. Now we have to show that the Maxwell's operator theta cap is Hermitian. So, in that case we perform an integration by parts twice.

So, you take this kind of you know F, Θ cap G. So, you write down the inner product ok. So, this is your Θ cap G right. And here you can actually use two identities, two vector identities. One is the divergence of this $\nabla.(F\times G)$ can be written as you know curl of F dot product with G minus F dot product with curl of G okay.

So, here you will see that you are getting this kind of situation where you are getting curl of this parameter. So, you can actually use this vector identity and find out what it is. So, this can be written as curl of F times okay So, you can actually obtain this from this one ok. And then integrating both sides and applying the divergence theorem you can finally figure out that integration of F dot curl of G will be same as you know integration of curl of F dot G ok plus a surface term. that term typically from the integral of curl of sorry from the integral of this term okay.

$\nabla.(F\times G)$ that surface term basically vanishes. So, you can actually write it in terms of this. So, which tells that you can operate (Θ cap on F and then G you take the inner product and that will result into the same. So, that establishes the fact that Θ cap the Maxwell operator is a Hermitian operator. So, just repeating this here telling or highlighting that we have performed integration by parts and neglect the surface term that involve the values of field at the boundary of integration. So, this is because in all cases of interest you can neglect the term because of two things.



What it means for an operator to be Hermitian!

$$(\mathrm{F},\hat{\Theta}\mathrm{G}) = \int d^3\mathrm{r}\mathrm{F}^*.\nabla\times\left(\frac{1}{\varepsilon}\nabla\times\mathrm{G}\right)$$

$$(\mathrm{F},\hat{\Theta}\mathrm{G}) = \int d^3\mathrm{r}(\nabla\times\mathrm{F})^*.\frac{1}{\varepsilon}\nabla\times\mathrm{G}$$

$$\boxed{(\mathrm{F},\hat{\Theta}\mathrm{G}) = \int d^3\mathrm{r}\left[\nabla\times\left(\frac{1}{\varepsilon}\nabla\times\mathbf{F}\right)\right]^*.\mathbf{G} = (\hat{\Theta}\mathrm{F},\mathrm{G})}$$

- In performing the integrations by parts, neglect the surface terms that involve the values of the fields at the boundaries of integration.
- This is because in all cases of interest, one of two things will be true:
  - *either the fields decay to zero at large distances*
  - *or the fields are periodic in the region of integration*
- ***In either case, the surface terms vanish.***

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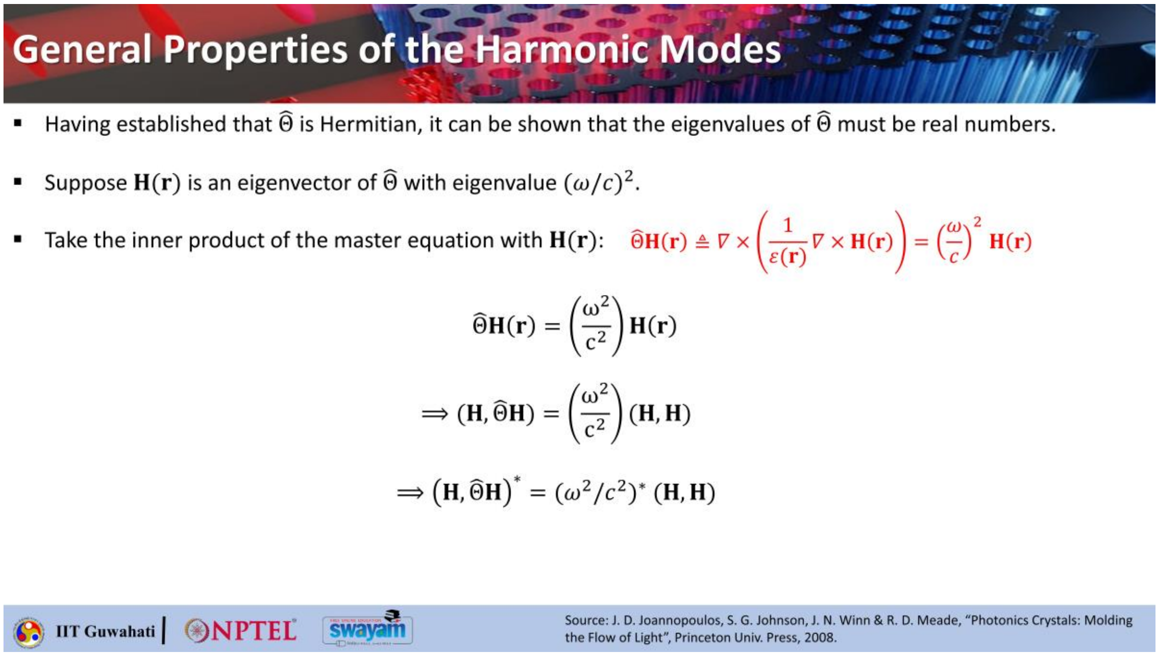
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, “Photonics Crystals: Molding the Flow of Light”, Princeton Univ. Press, 2008.

So, one of these things will be true either the field will decay to 0 at large distance or the field is basically periodic in the region of integration. So, in these two cases you can you know neglect the surface term and that involves the value of the field at the boundaries of the integration. So, that establishes that Θ cap or the Maxwell operator is a Hermitian operator. right. So, now we look into the general properties of the harmonic modes.



General Properties of the Harmonic Modes

- Having established that $\widehat{\Theta}$ is Hermitian, it can be shown that the eigenvalues of $\widehat{\Theta}$ must be real numbers.
- Suppose $\mathbf{H}(\mathbf{r})$ is an eigenvector of $\widehat{\Theta}$ with eigenvalue $(\omega/c)^2$.
- Take the inner product of the master equation with $\mathbf{H}(\mathbf{r})$: $\widehat{\Theta}\mathbf{H}(\mathbf{r}) \triangleq \nabla\times\left(\frac{1}{\varepsilon(\mathbf{r})}\nabla\times\mathbf{H}(\mathbf{r})\right) = \left(\frac{\omega}{c}\right)^2\mathbf{H}(\mathbf{r})$

$$\widehat{\Theta}\mathbf{H}(\mathbf{r}) = \left(\frac{\omega^2}{c^2}\right)\mathbf{H}(\mathbf{r})$$

$$\Rightarrow (\mathbf{H},\widehat{\Theta}\mathbf{H}) = \left(\frac{\omega^2}{c^2}\right)(\mathbf{H},\mathbf{H})$$

$$\Rightarrow (\mathbf{H},\widehat{\Theta}\mathbf{H})^* = (\omega^2/c^2)^*\,(\mathbf{H},\mathbf{H})$$

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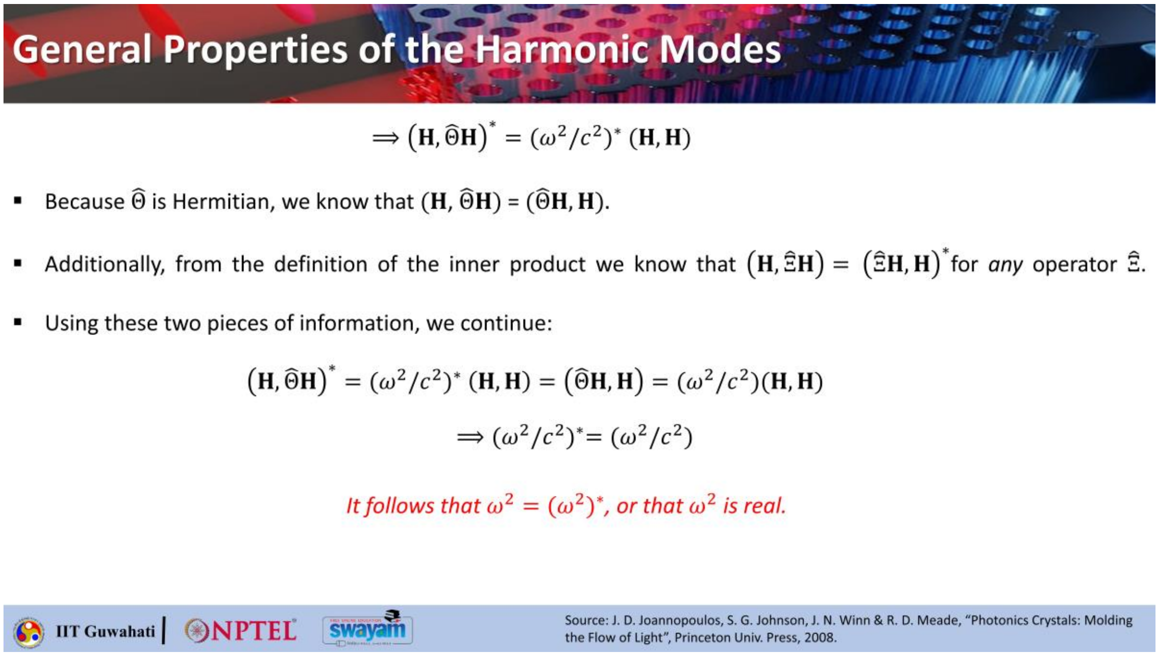
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, having established that Θ cap is the Hermitian operator, it can be shown that the eigenvalues of Θ cap must be real numbers okay. Suppose H(r) the magnetic field okay or the spatial profile of the magnetic field is an eigenvector of Θ cap with eigenvalue of ω/c whole square okay. It means okay you can take the inner product of the master equation you can write like this okay and you are getting this eigenvalue and again this particular eigenvector. In short you can write in terms of Θ cap operator this is how it will look like. Now if you take inner product of H and this thing what you get you will get you know ω^2/c^2.

and inner product of H with itself ok. If you take the conjugate of this you will get conjugate of ω^2/c^2 conjugate and then you have the inner product of H with itself ok. So, because Θ cap is a Hermitian operator, so this can be written as this. So, it does not matter where you are applying the operator, the inner product remains same and here the function is basically H, okay. Both vector fields are basically same. So, additionally from the definition of the inner product, we also know that H and then (**H**,Ξˆ cap **H**) will be equal to (Ξ cap **H**,**H**) conjugate for any operator Ξ cap ok.

So, if you use these two information together you can write that **H**,Ξˆ cap **H** conjugate will be equal to ω^2/c^2 conjugate and then the inner product of H with itself and this can be written as this ok. from here ok and that is simply ω^2/c^2 and inner product of H with itself. So, what we obtain from here is that ω^2/c^2 conjugate is basically ω^2/c^2 that means ω^2 is basically ω^2 conjugate that means ω^2 is basically real. okay and then this inner product okay is very easy that when you set F equals G equals H in this particular equation okay you basically get this. Inner product of H with H ω/c whole square that can be written as this turns out to be now a F and G are both equal and they are both H.



General Properties of the Harmonic Modes

$$\Rightarrow \left(\mathbf{H}, \widehat{\Theta}\mathbf{H}\right)^{*} = (\omega^2/c^2)^{*}\,(\mathbf{H},\mathbf{H})$$

- Because $\widehat{\Theta}$ is Hermitian, we know that $(\mathbf{H}, \widehat{\Theta}\mathbf{H}) = (\widehat{\Theta}\mathbf{H}, \mathbf{H})$.
- Additionally, from the definition of the inner product we know that $\left(\mathbf{H}, \widehat{\Xi}\mathbf{H}\right) = \left(\widehat{\Xi}\mathbf{H}, \mathbf{H}\right)^{*}$ for *any* operator $\widehat{\Xi}$.
- Using these two pieces of information, we continue:

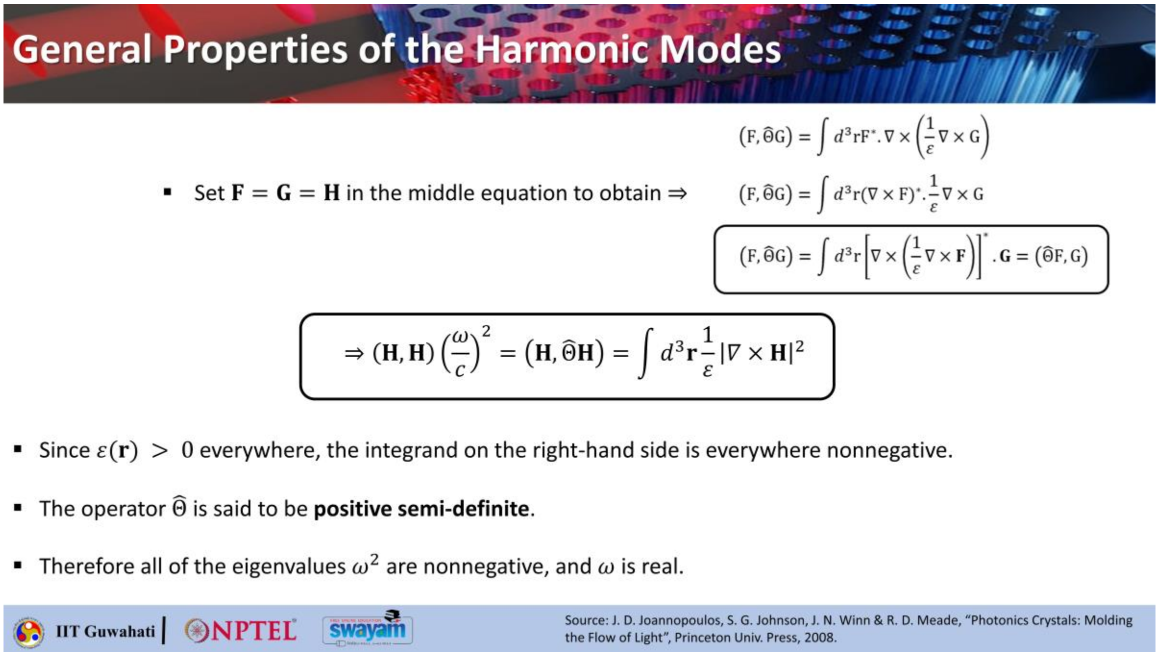
$$\left(\mathbf{H}, \widehat{\Theta}\mathbf{H}\right)^{*} = (\omega^2/c^2)^{*}\,(\mathbf{H},\mathbf{H}) = \left(\widehat{\Theta}\mathbf{H}, \mathbf{H}\right) = (\omega^2/c^2)(\mathbf{H},\mathbf{H})$$

$$\Rightarrow (\omega^2/c^2)^{*} = (\omega^2/c^2)$$

It follows that $\omega^2 = (\omega^2)^{*}$, or that ω^2 is real.

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, there are curl of H twice. So, you have curl of H whole square and this is how the equation looks like. So here since $\varepsilon(\mathbf{r})$ is positive everywhere the integrand on the right side is also everywhere non-negative. So that means you can say that this operator Θ cap the Maxwell operator is basically a positive semi-definite. Therefore, all the eigenvalues ω^2 are basically non-negative and ω is also real.



General Properties of the Harmonic Modes

$$(\mathrm{F}, \widehat{\Theta}\mathrm{G}) = \int d^3\mathrm{r}\,\mathrm{F}^{*}.\nabla \times \left(\frac{1}{\varepsilon}\nabla \times \mathrm{G}\right)$$

- Set $\mathbf{F} = \mathbf{G} = \mathbf{H}$ in the middle equation to obtain ⇒

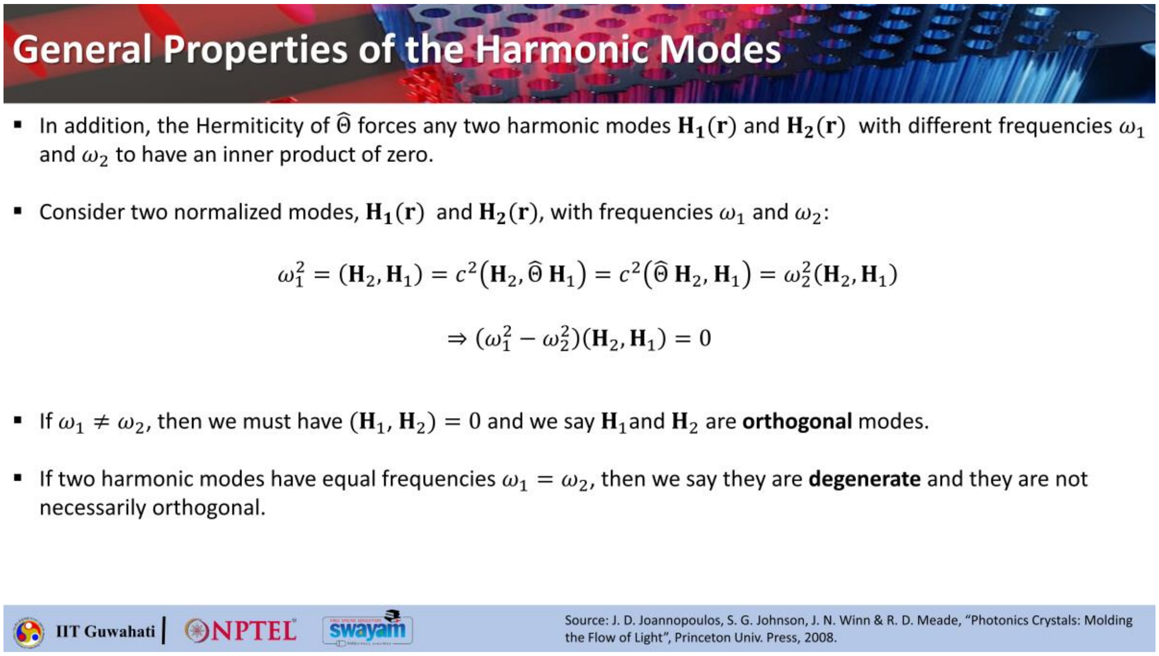
$$(\mathrm{F}, \widehat{\Theta}\mathrm{G}) = \int d^3\mathrm{r}(\nabla \times \mathrm{F})^{*}.\frac{1}{\varepsilon}\nabla \times \mathrm{G}$$

$$\boxed{(\mathrm{F}, \widehat{\Theta}\mathrm{G}) = \int d^3\mathrm{r}\left[\nabla \times \left(\frac{1}{\varepsilon}\nabla \times \mathbf{F}\right)\right]^{*}.\mathbf{G} = (\widehat{\Theta}\mathrm{F}, \mathrm{G})}$$

$$\boxed{\Rightarrow (\mathbf{H},\mathbf{H})\left(\frac{\omega}{c}\right)^2 = \left(\mathbf{H}, \widehat{\Theta}\mathbf{H}\right) = \int d^3\mathbf{r}\,\frac{1}{\varepsilon}|\nabla \times \mathbf{H}|^2}$$

- Since $\varepsilon(\mathbf{r}) > 0$ everywhere, the integrand on the right-hand side is everywhere nonnegative.
- The operator $\widehat{\Theta}$ is said to be **positive semi-definite**.
- Therefore all of the eigenvalues ω^2 are nonnegative, and ω is real.

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.



General Properties of the Harmonic Modes

- In addition, the Hermiticity of $\widehat{\Theta}$ forces any two harmonic modes $\mathbf{H_1(r)}$ and $\mathbf{H_2(r)}$ with different frequencies ω_1 and ω_2 to have an inner product of zero.
- Consider two normalized modes, $\mathbf{H_1(r)}$ and $\mathbf{H_2(r)}$, with frequencies ω_1 and ω_2:

$$\omega_1^2 = (\mathbf{H}_2, \mathbf{H}_1) = c^2\left(\mathbf{H}_2, \widehat{\Theta}\,\mathbf{H}_1\right) = c^2\left(\widehat{\Theta}\,\mathbf{H}_2, \mathbf{H}_1\right) = \omega_2^2(\mathbf{H}_2, \mathbf{H}_1)$$

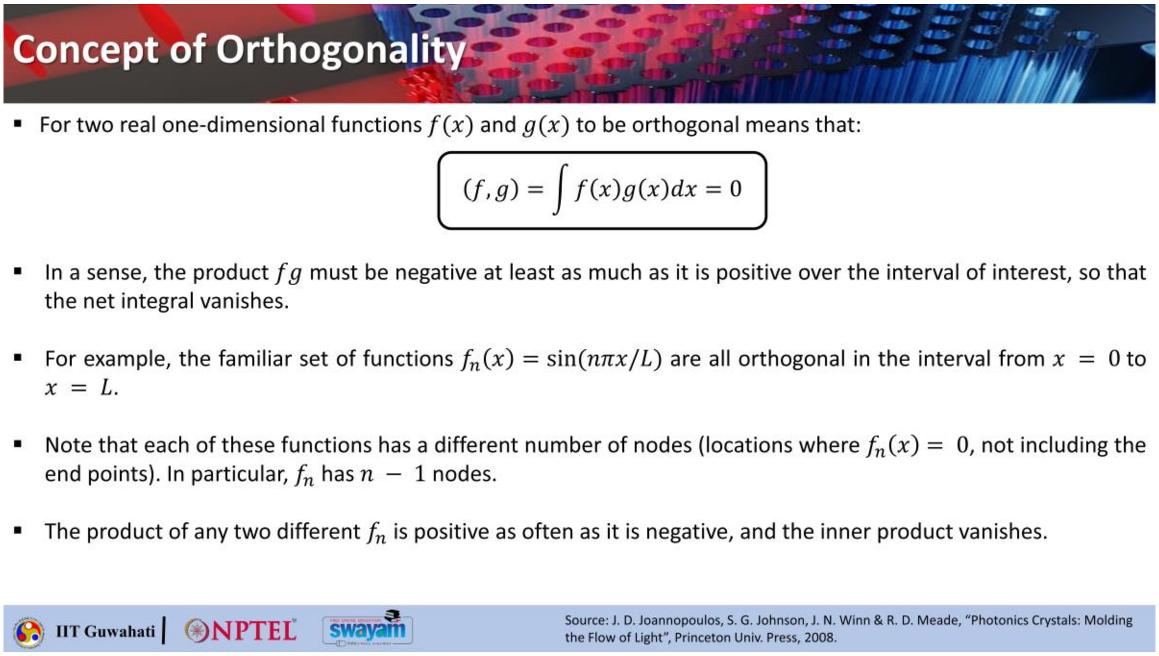
$$\Rightarrow (\omega_1^2 - \omega_2^2)(\mathbf{H}_2, \mathbf{H}_1) = 0$$

- If $\omega_1 \neq \omega_2$, then we must have $(\mathbf{H}_1, \mathbf{H}_2) = 0$ and we say $\mathbf{H}_1$ and $\mathbf{H}_2$ are **orthogonal** modes.
- If two harmonic modes have equal frequencies $\omega_1 = \omega_2$, then we say they are **degenerate** and they are not necessarily orthogonal.

Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

In addition, the hermicity of Θ cap forces any two harmonic modes $\mathbf{H_1(r)}$ and $\mathbf{H_1(r)}$ with different frequencies ω_1 and ω_2 to have an inner product of 0.

So, let us consider two normalized modes like $\mathbf{H_1(r)}$ and $\mathbf{H_1(r)}$ and they are having frequency of ω_1 and ω_2 respectively. So, you can write ω_1 square okay inner product of H_2 and H_1 equals c square H_2 and then Θ cap H_1 that will be same as c square okay Θ cap H_2 comma H_1 and that can be written as ω_2 square H_1. (H_2 H_1). So, if you take these two together okay you can write ω_1^2-ω_2^2 and the inner product of H_2 H_1. Now ω_1 and ω_2 are not 0 sorry ω_1 and ω_2 are not equal so this term will not be 0 that means this has to be 0 and this is 0 means you know H_1 and H_2 are basically orthogonal modes.



Concept of Orthogonality

- For two real one-dimensional functions $f(x)$ and $g(x)$ to be orthogonal means that:

$$\boxed{(f,g) = \int f(x)g(x)dx = 0}$$

- In a sense, the product fg must be negative at least as much as it is positive over the interval of interest, so that the net integral vanishes.
- For example, the familiar set of functions $f_n(x) = \sin(n\pi x/L)$ are all orthogonal in the interval from $x = 0$ to $x = L$.
- Note that each of these functions has a different number of nodes (locations where $f_n(x) = 0$, not including the end points). In particular, f_n has $n - 1$ nodes.
- The product of any two different f_n is positive as often as it is negative, and the inner product vanishes.

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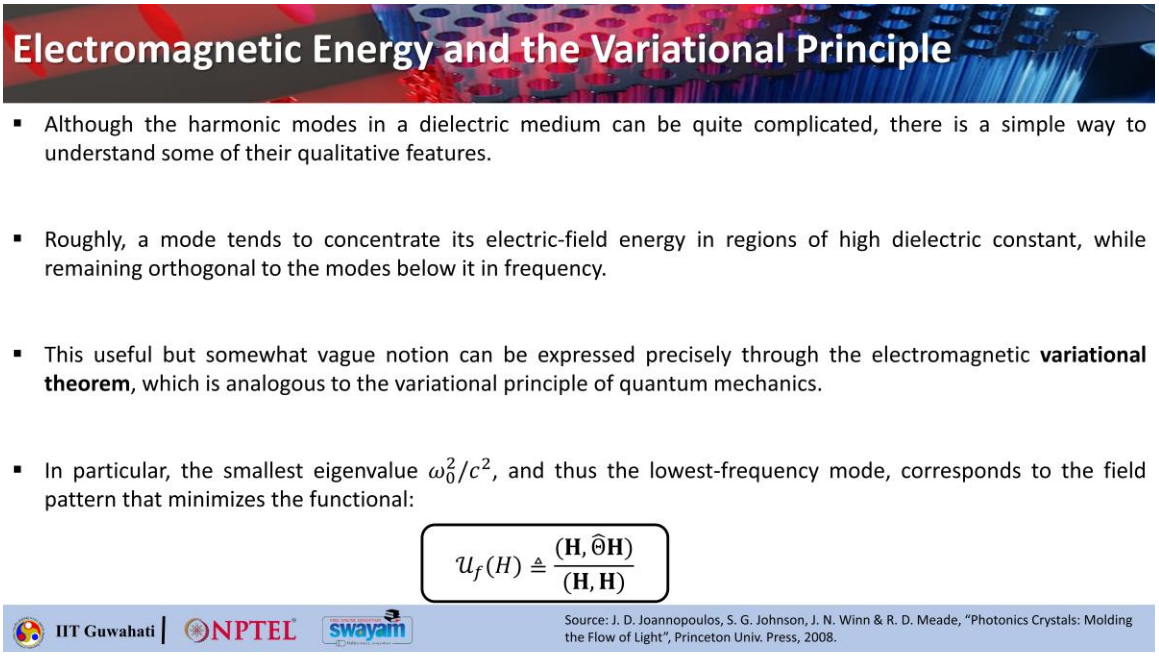
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, two harmonic modes with equal frequencies if they are equal then they need not be you know orthogonal. So, this term need not be orthogonal and we can say if the two modes are having same frequencies we can say that they are degenerate. So, now let us introduce the concept of orthogonality. For two real one dimensional functions if you take f(x) and g(x). they are orthogonal if you know you can write the inner product of f and g as integration f(x)dxg(x)dx equals 0.

So, in a sense the product fg must be negative at least as much as it is positive over the integral of interest and then only it can become 0 right. So, For example, if you take familiar set of functions like f and x and you can write it as $\sin(n\pi x/L)$, they are all orthogonal in the interval from x equals 0 to x equals L. So, you understand over the integral, some part of the function has to be positive, some part has to be negative and then only they can be, you know, the net integral can vanish. So note that each of this function has a different number of modes and the locations can be found from fn(x) equals 0 and remember not including the endpoints.

So in particular fn has $n - 1$ nodes. So the product of any two different fn is positive as often as it is negative. So, that you know the inner product vanishes. So, that is the whole idea of orthogonality fine.



Electromagnetic Energy and the Variational Principle

- Although the harmonic modes in a dielectric medium can be quite complicated, there is a simple way to understand some of their qualitative features.
- Roughly, a mode tends to concentrate its electric-field energy in regions of high dielectric constant, while remaining orthogonal to the modes below it in frequency.
- This useful but somewhat vague notion can be expressed precisely through the electromagnetic **variational theorem**, which is analogous to the variational principle of quantum mechanics.
- In particular, the smallest eigenvalue ω_0^2/c^2, and thus the lowest-frequency mode, corresponds to the field pattern that minimizes the functional:

$$\boxed{\mathcal{U}_f(H) \triangleq \frac{(\mathbf{H},\hat{\Theta}\mathbf{H})}{(\mathbf{H},\mathbf{H})}}$$

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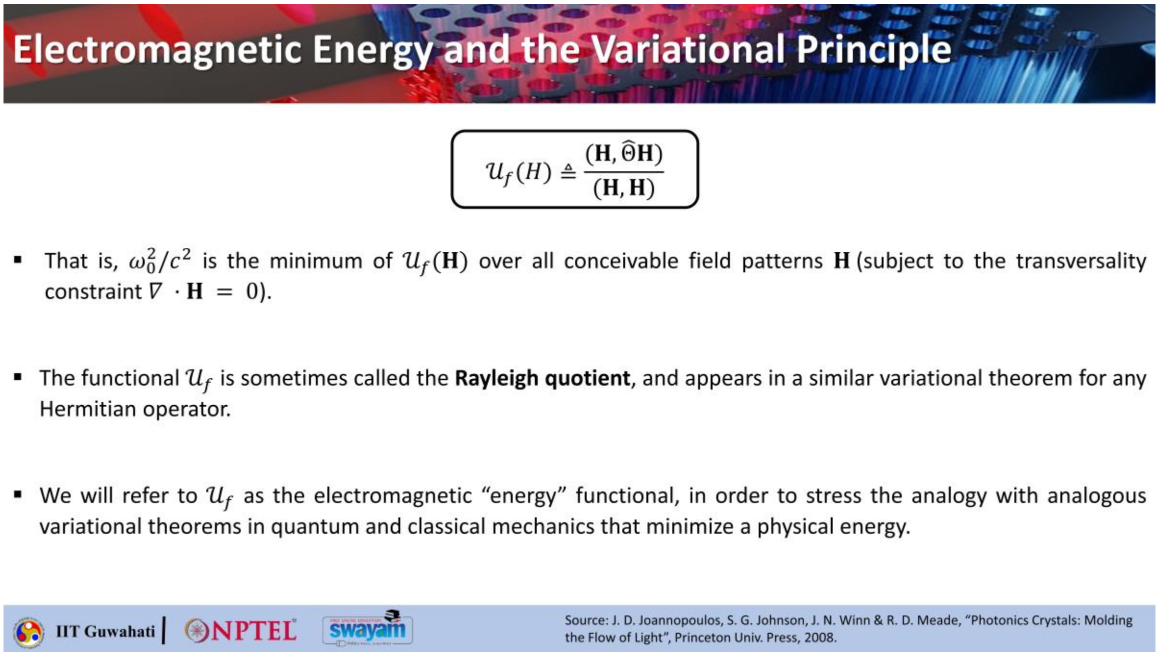
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

Now, let us look into how do we calculate electromagnetic energy and apply the variational principle. So, although the harmonic modes in a dielectric medium can be quite complicated, there is a simple way to understand some of their qualitative features.

Roughly a mode tends to concentrate its electrical energy or electric field energy in the region of high dielectric constant while remaining orthogonal to the modes below it in frequency. You understand the orthogonality right. So, f(x) integration f(x) g(x) will be 0 something like that ok. For mode it is slightly more complicated, but the idea is that though same. This useful, but somewhat vague notion can be expressed precisely through the electromagnetic variational theorem which is analogous to the variational principle of quantum mechanics.

In particular, the smallest eigenvalue which is $\omega_0{}^2/c^2$ and thus the lowest frequency mode corresponds to the field pattern that minimizes the functional. So, you can define a functional like this $\mathcal{U}_f\ (H)$ where it is inner product of $((\mathbf{H},\hat{\Theta}\mathbf{H}))$ and then inner product of divided by $(\mathbf{H},\mathbf{H})$ that is the inner product ok.



Electromagnetic Energy and the Variational Principle

$$\boxed{\mathcal{U}_f(H) \triangleq \frac{(\mathbf{H}, \widehat{\Theta}\mathbf{H})}{(\mathbf{H}, \mathbf{H})}}$$

- That is, ω_0^2/c^2 is the minimum of $\mathcal{U}_f(\mathbf{H})$ over all conceivable field patterns $\mathbf{H}$ (subject to the transversality constraint $\nabla \cdot \mathbf{H} = 0$).
- The functional $\mathcal{U}_f$ is sometimes called the **Rayleigh quotient**, and appears in a similar variational theorem for any Hermitian operator.
- We will refer to $\mathcal{U}_f$ as the electromagnetic “energy” functional, in order to stress the analogy with analogous variational theorems in quantum and classical mechanics that minimize a physical energy.

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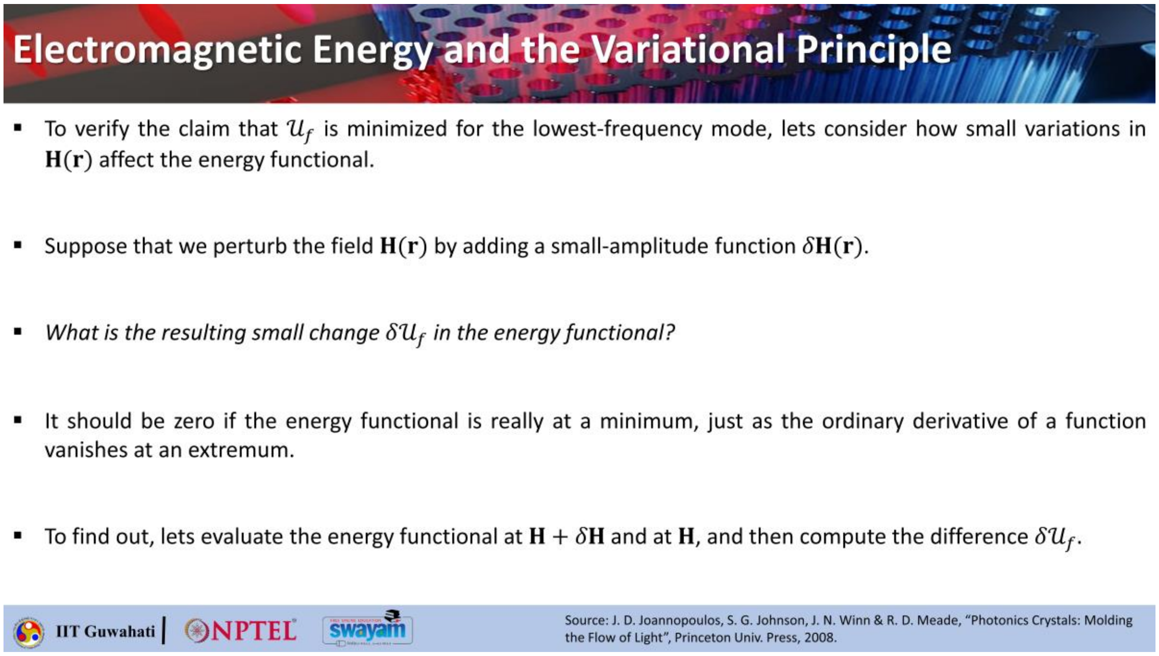
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, “Photonics Crystals: Molding the Flow of Light”, Princeton Univ. Press, 2008.

So, if you consider $\omega_0{}^2/c^2$ is the minimum of this functional over all conceivable field patterns H. okay which are subject to the transversality constraint that is $\nabla \cdot \mathbf{H}$ will be 0 okay. The functional $\mathcal{U}_f$(H)is also called Rayleigh quotient and it appears in a similar variational theorem for any Hermitian operator.

So, we will basically refer to this $\mathcal{U}_f$ as electromagnetic energy functional in order to stretch its analogy with the variational theorems in quantum and classical mechanics that minimizes a physical energy.



Electromagnetic Energy and the Variational Principle

- To verify the claim that $\mathcal{U}_f$ is minimized for the lowest-frequency mode, lets consider how small variations in $\mathbf{H}(\mathbf{r})$ affect the energy functional.
- Suppose that we perturb the field $\mathbf{H}(\mathbf{r})$ by adding a small-amplitude function $\delta\mathbf{H}(\mathbf{r})$.
- *What is the resulting small change $\delta\mathcal{U}_f$ in the energy functional?*
- It should be zero if the energy functional is really at a minimum, just as the ordinary derivative of a function vanishes at an extremum.
- To find out, lets evaluate the energy functional at $\mathbf{H} + \delta\mathbf{H}$ and at $\mathbf{H}$, and then compute the difference $\delta\mathcal{U}_f$.

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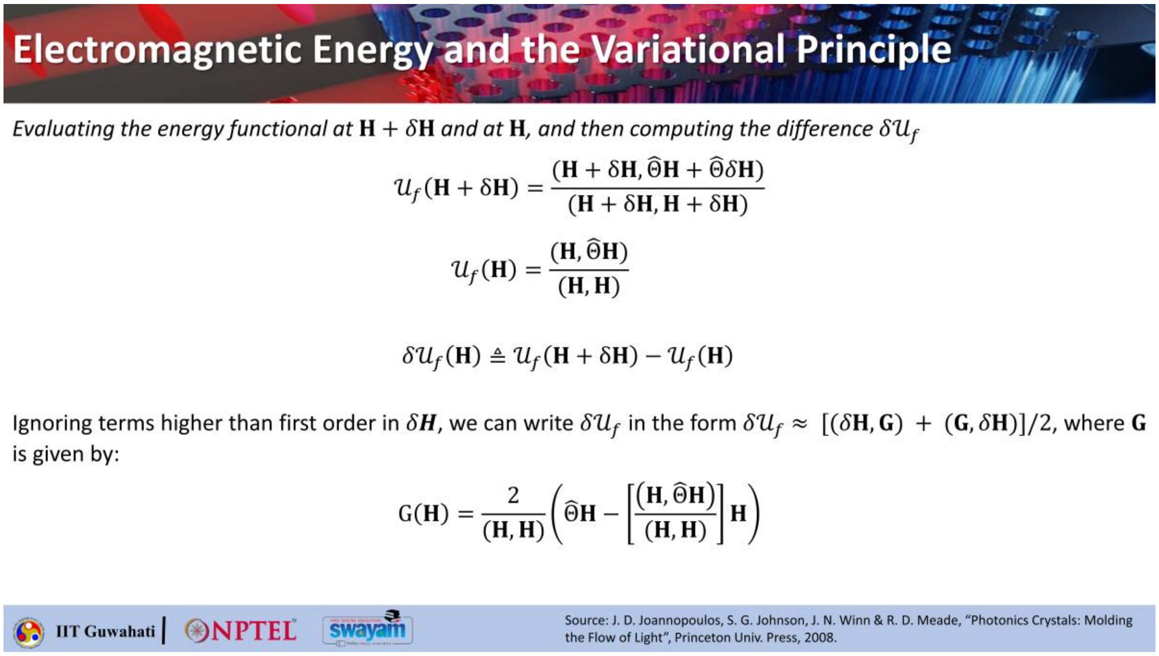
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, to verify the claim that $\mathcal{U}_f$ is basically minimized for the lowest frequency mode, let us consider how small variations in hr affect the energy functional. So, suppose that we perturb the magnetic field hr by adding a small amplitude function which is delta H(r) okay. what is the resulting small change $\delta\mathcal{U}_f$ in the energy functional? So, it should be 0 if the energy functional is really at minimum, thus an ordinary derivative of the function will vanish at the extremum. To find out, let us evaluate the energy functional at $\mathbf{H}+\delta\mathbf{H}$ and at H and then compute the difference which is $\delta\mathcal{U}f$.



Electromagnetic Energy and the Variational Principle

Evaluating the energy functional at $\mathbf{H}+\delta\mathbf{H}$ *and at* $\mathbf{H}$*, and then computing the difference* $\delta\mathcal{U}_f$

$$\mathcal{U}_f(\mathbf{H}+\delta\mathbf{H}) = \frac{(\mathbf{H}+\delta\mathbf{H}, \hat{\Theta}\mathbf{H}+\hat{\Theta}\delta\mathbf{H})}{(\mathbf{H}+\delta\mathbf{H}, \mathbf{H}+\delta\mathbf{H})}$$

$$\mathcal{U}_f(\mathbf{H}) = \frac{(\mathbf{H}, \hat{\Theta}\mathbf{H})}{(\mathbf{H}, \mathbf{H})}$$

$$\delta\mathcal{U}_f(\mathbf{H}) \triangleq \mathcal{U}_f(\mathbf{H}+\delta\mathbf{H}) - \mathcal{U}_f(\mathbf{H})$$

Ignoring terms higher than first order in $\delta\boldsymbol{H}$, we can write $\delta\mathcal{U}_f$ in the form $\delta\mathcal{U}_f \approx [(\delta\mathbf{H},\mathbf{G}) + (\mathbf{G},\delta\mathbf{H})]/2$, where $\mathbf{G}$ is given by:

$$\mathrm{G}(\mathbf{H}) = \frac{2}{(\mathbf{H},\mathbf{H})}\left(\hat{\Theta}\mathbf{H} - \left[\frac{(\mathbf{H},\hat{\Theta}\mathbf{H})}{(\mathbf{H},\mathbf{H})}\right]\mathbf{H}\right)$$

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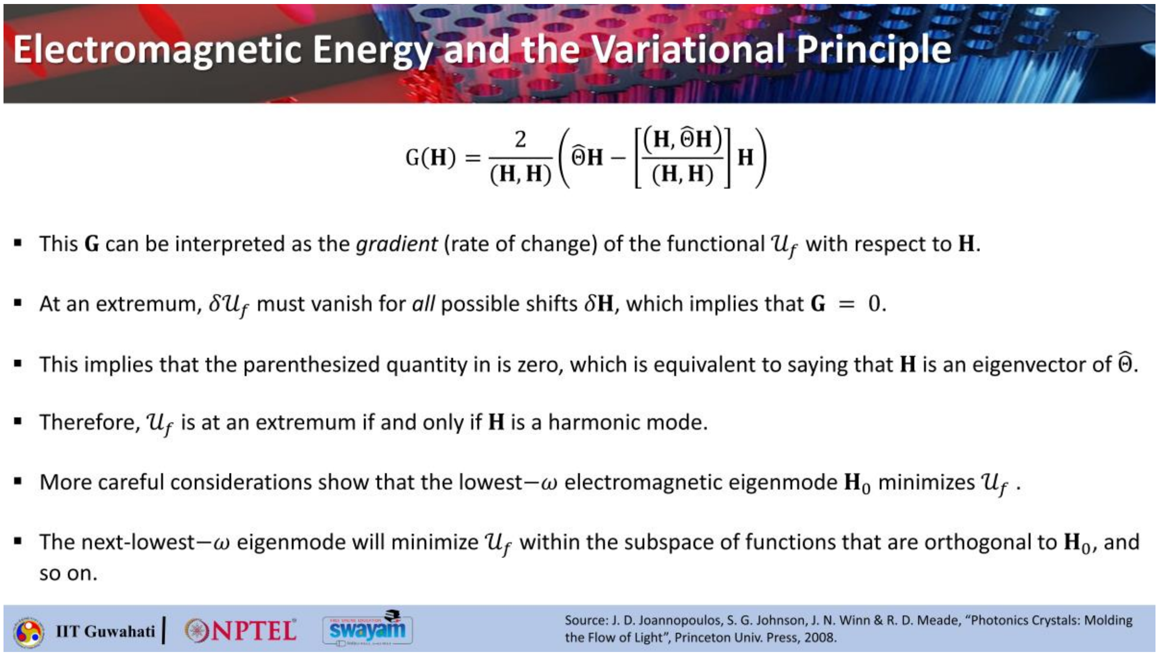
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, evaluating the energy functional at $\mathbf{H}+\delta\mathbf{H}$ this is how the equation looks like and you also have the energy functional at H ok. So, the difference between these two is basically $\delta\mathcal{U}^{f}$(H) ok which is this one. Now, if you ignore the terms higher than the first order in δ(H). okay. So, you can write $\delta\mathcal{U}_f$ in terms of this which is $\delta\mathbf{H}$,$\mathbf{G}$ the inner product of this plus **G,** $\delta\mathbf{H}$ the inner product of this by 2.

So, where **G**(H) can be defined as this one. So, it is basically 2 divided by the inner product of H and then you have that operator Θ cap with H and then the ratio of this times H.



Electromagnetic Energy and the Variational Principle

$$\mathrm{G}(\mathbf{H}) = \frac{2}{(\mathbf{H},\mathbf{H})}\left(\hat{\Theta}\mathbf{H} - \left[\frac{(\mathbf{H},\hat{\Theta}\mathbf{H})}{(\mathbf{H},\mathbf{H})}\right]\mathbf{H}\right)$$

- This $\mathbf{G}$ can be interpreted as the *gradient* (rate of change) of the functional $\mathcal{U}_f$ with respect to $\mathbf{H}$.
- At an extremum, $\delta\mathcal{U}_f$ must vanish for *all* possible shifts $\delta\mathbf{H}$, which implies that $\mathbf{G} = 0$.
- This implies that the parenthesized quantity in is zero, which is equivalent to saying that $\mathbf{H}$ is an eigenvector of $\hat{\Theta}$.
- Therefore, $\mathcal{U}_f$ is at an extremum if and only if $\mathbf{H}$ is a harmonic mode.
- More careful considerations show that the lowest$-\omega$ electromagnetic eigenmode $\mathbf{H}_0$ minimizes $\mathcal{U}_f$.
- The next-lowest$-\omega$ eigenmode will minimize $\mathcal{U}_f$ within the subspace of functions that are orthogonal to $\mathbf{H}_0$, and so on.

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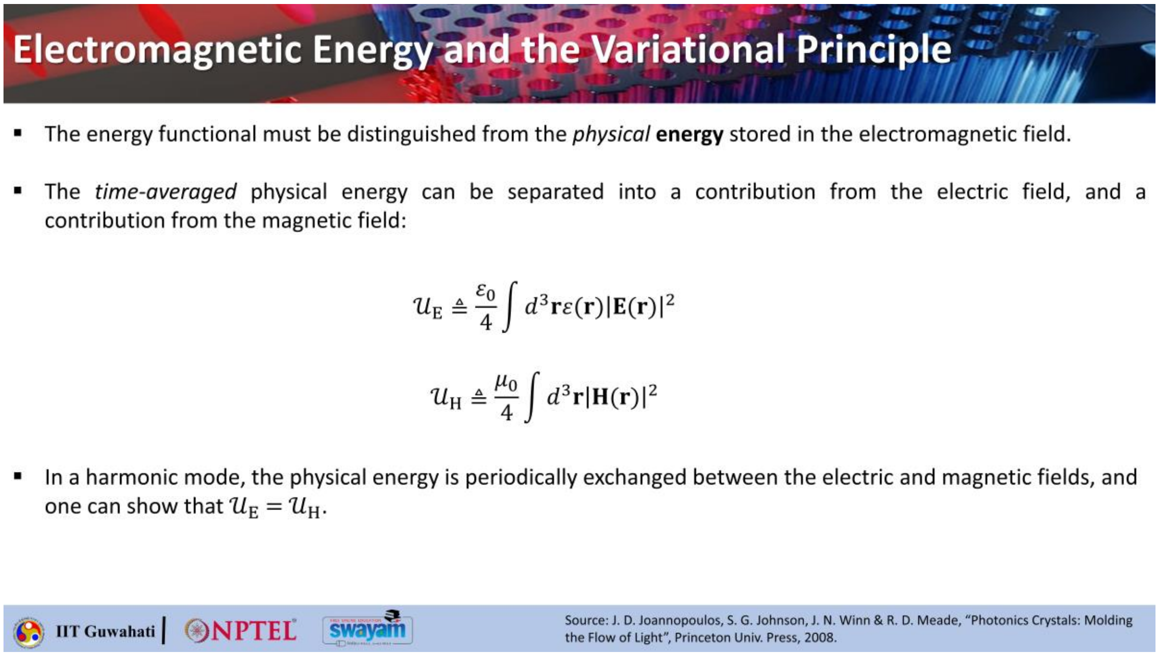
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, what is this? This G is basically interpreted as a gradient which is a rate of change of the functional $\mathcal{U}_f$ with respect to H. So, it is written as $\mathbf{G}(\mathrm{H})$ right, $\mathbf{G}$ as a function of H or you can say the gradient as a function of H. Now, at the at an at an extremum $\delta\mathcal{U}_f$ must vanish right that is how you find the minimum for all possible shifts of δH and that will show that the gradient is basically G equals 0. This implies that the parenthesis quantity here should be 0 ok which is equivalent to saying that this is 0.

which is again telling us that H is basically an eigenvector of Θ cap. So, which is true that means $\mathcal{U}_f$ is the functional is at an extremum if and only if H is an harmonic mode. So, more careful considerations will show us that the lowest frequency electromagnetic eigenmode H naught basically minimizes the energy functional $\mathcal{U}_f$. So, the next lowest energy okay ω eigen mode will minimize $\mathcal{U}_f$. within the subspace of the function that are orthogonal to H_0 and so on.



Electromagnetic Energy and the Variational Principle

- The energy functional must be distinguished from the *physical* **energy** stored in the electromagnetic field.
- The *time-averaged* physical energy can be separated into a contribution from the electric field, and a contribution from the magnetic field:

$$\mathcal{U}_{\mathrm{E}} \triangleq \frac{\varepsilon_0}{4}\int d^3\mathbf{r}\,\varepsilon(\mathbf{r})|\mathbf{E}(\mathbf{r})|^2$$

$$\mathcal{U}_{\mathrm{H}} \triangleq \frac{\mu_0}{4}\int d^3\mathbf{r}\,|\mathbf{H}(\mathbf{r})|^2$$

- In a harmonic mode, the physical energy is periodically exchanged between the electric and magnetic fields, and one can show that $\mathcal{U}_{\mathrm{E}} = \mathcal{U}_{\mathrm{H}}$.

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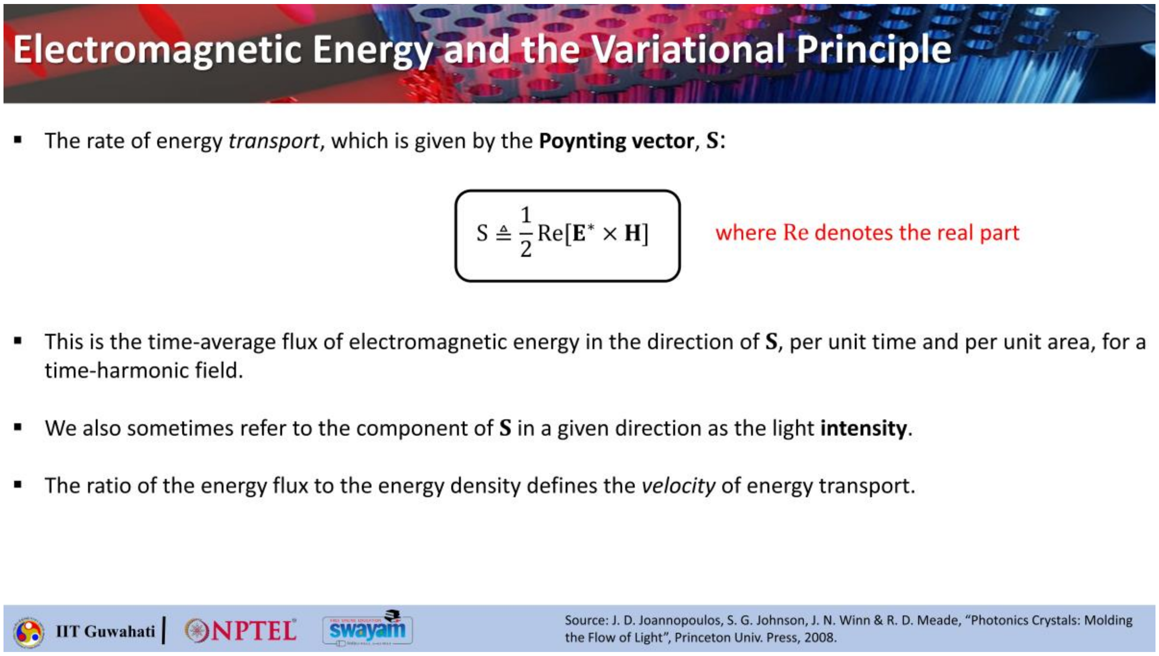
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, the next one will be orthogonal to this particular mode profile. So, the energy functional must be distinguished from the physical energy which is stored in the electromagnetic field. if you take the time averaged physical energy that can be separated into the contribution coming from electric field and also coming from the magnetic field and you can write $\mathcal{U}_{\mathrm{E}}$ that is basically the energy time averaged physical energy. So, you can write in terms of the electric field. So, this is the contribution from electric field, this is the contribution coming from magnetic field. So, in a harmonic mode the physical energy is basically periodically exchanged between the electric field and the magnetic field.

So, you can show that this energy can be totally converted into magnetic energy and back and forth. So, $\mathcal{U}_{\mathrm{E}}$ is same as $\mathcal{U}_{\mathrm{H}}$ right.



Electromagnetic Energy and the Variational Principle

- The rate of energy *transport*, which is given by the **Poynting vector**, $\mathbf{S}$:

$$\boxed{S \triangleq \frac{1}{2}\mathrm{Re}[\mathbf{E}^* \times \mathbf{H}]}$$

where Re denotes the real part

- This is the time-average flux of electromagnetic energy in the direction of **S**, per unit time and per unit area, for a time-harmonic field.
- We also sometimes refer to the component of **S** in a given direction as the light **intensity**.
- The ratio of the energy flux to the energy density defines the *velocity* of energy transport.

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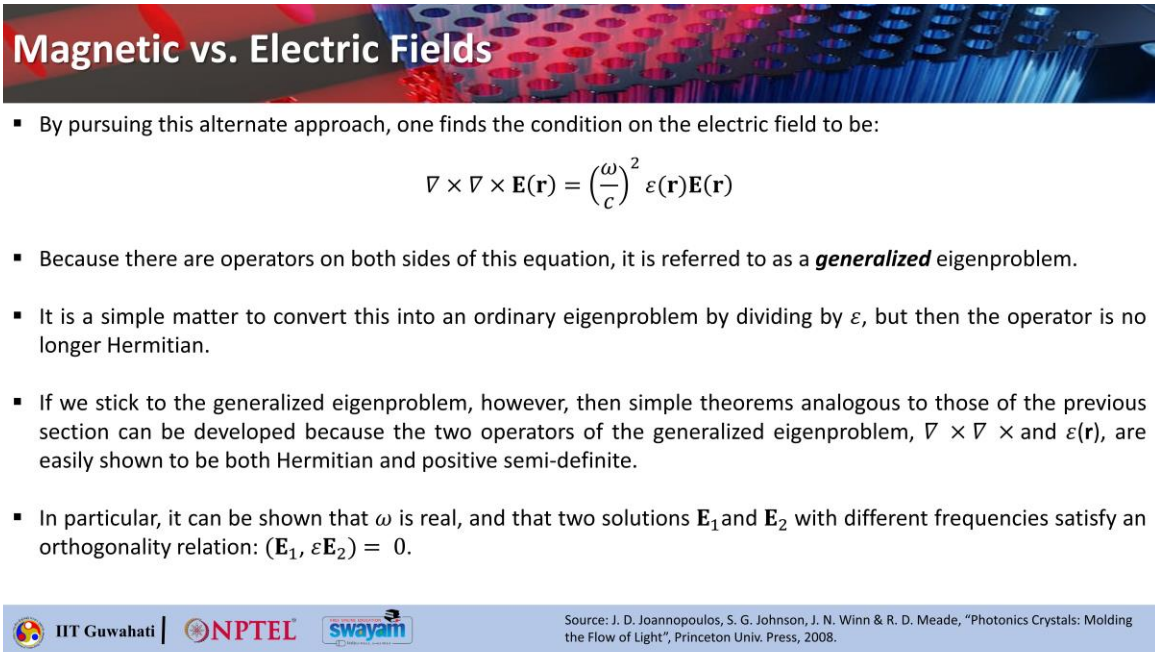
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, “Photonics Crystals: Molding the Flow of Light”, Princeton Univ. Press, 2008.

So, the physical energy and the energy functional are related, but remember there is an important difference. The energy functional has fields. in both the numerator and the denominator and is therefore, independent of the field strength.

So, the physical energy is basically proportional to the square of the field strength. So, in other words multiplying the fields by a constant affects the physical energy, but it will not affect the energy functional because energy functional has got filled in both numerator and denominator. So, if we are interested in the physical energy, we must pay attention to the amplitude of our modes. But if we are only interested in the mode profile, then we might as well normalize our modes. There you know the actual amplitude will not be seen, but the mode profile will be recorded. The rate of energy transport which is given by the pointing vector S can be written as S equals half real of you know E conjugate cross product with H.

So this is how you obtain the pointing vector. So that actually tells you the direction of energy transport or energy flow. So, this is the time average flux of the electromagnetic energy in the direction of s per unit time and per unit area for a time harmonic field. So, we also sometimes refer to the component of S in a given direction as the light intensity. The ratio of the energy flux to the energy density defines the velocity of the energy transport right. Now towards the end of this lecture, we will discuss why we discuss everything in terms of magnetic field instead of electric field.



Magnetic vs. Electric Fields

- By pursuing this alternate approach, one finds the condition on the electric field to be:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$$

- Because there are operators on both sides of this equation, it is referred to as a ***generalized*** eigenproblem.
- It is a simple matter to convert this into an ordinary eigenproblem by dividing by ε, but then the operator is no longer Hermitian.
- If we stick to the generalized eigenproblem, however, then simple theorems analogous to those of the previous section can be developed because the two operators of the generalized eigenproblem, $\nabla \times \nabla \times$ and $\varepsilon(\mathbf{r})$, are easily shown to be both Hermitian and positive semi-definite.
- In particular, it can be shown that ω is real, and that two solutions $\mathbf{E}_1$ and $\mathbf{E}_2$ with different frequencies satisfy an orthogonality relation: $(\mathbf{E}_1, \varepsilon\mathbf{E}_2) = 0$.

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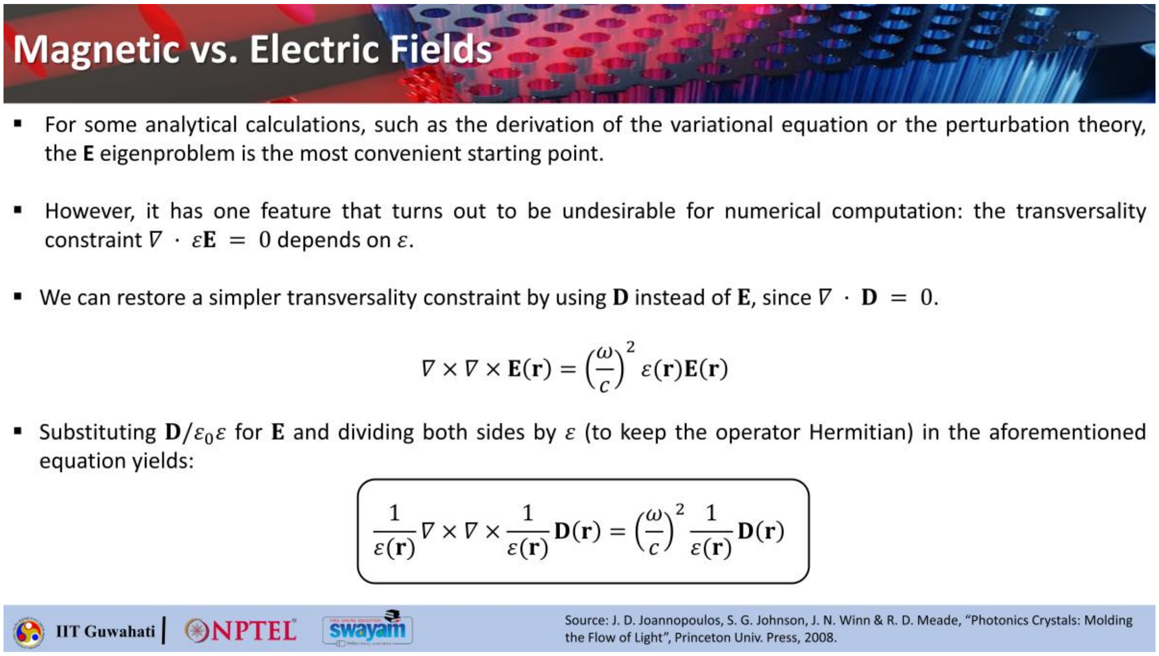
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, by now one should be asking this question that why we are always dealing with magnetic field instead of electric field. In the previous section, we reformulated the Maxwell's equation as an eigenvalue equation for the harmonic modes of magnetic field H(r). The idea was that for a given frequency, we could solve for H(r) and then determine what is E(r) by their correlation equations. But we could have equally well tried an alternative approach for the electric field and determine the magnetic field later on. Now, why did not we choose that particular route? So, we will see that there are certain difficulties and because of which working with magnetic field is preferred.

So, one can also find the condition on the electric field to be like this from the Maxwell's equation ok. Therefore, there are operators on the both sides of these equations and it is referred to as a generalized eigen problem. So, it is a simple matter to convert this into a ordinary eigen problem by dividing by ε_r, then the operator is no longer Hermitian. So, if we stick to the generalized eigen problem like here ok, where there are operators on both sides. However, then the simple theorem which are analogous to those of the previous sections can be developed because the two operators of the generalized eigenproblem one is curl of curl of this this one curl of curl and another one is ε_r.

They can be shown to be both Hermitian and positive semi-definite. So, in particular it can be shown that ω is real and that the two solutions E_1 and E_2 with different frequencies satisfy an orthogonality relation something like E_1 and then you have εE_2 okay which will be 0.



Magnetic vs. Electric Fields

- For some analytical calculations, such as the derivation of the variational equation or the perturbation theory, the **E** eigenproblem is the most convenient starting point.
- However, it has one feature that turns out to be undesirable for numerical computation: the transversality constraint $\nabla \cdot \varepsilon\mathbf{E} = 0$ depends on ε.
- We can restore a simpler transversality constraint by using $\mathbf{D}$ instead of $\mathbf{E}$, since $\nabla \cdot \mathbf{D} = 0$.

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$$

- Substituting $\mathbf{D}/\varepsilon_0\varepsilon$ for $\mathbf{E}$ and dividing both sides by ε (to keep the operator Hermitian) in the aforementioned equation yields:

$$\boxed{\frac{1}{\varepsilon(\mathbf{r})}\nabla \times \nabla \times \frac{1}{\varepsilon(\mathbf{r})}\mathbf{D}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \frac{1}{\varepsilon(\mathbf{r})}\mathbf{D}(\mathbf{r})}$$

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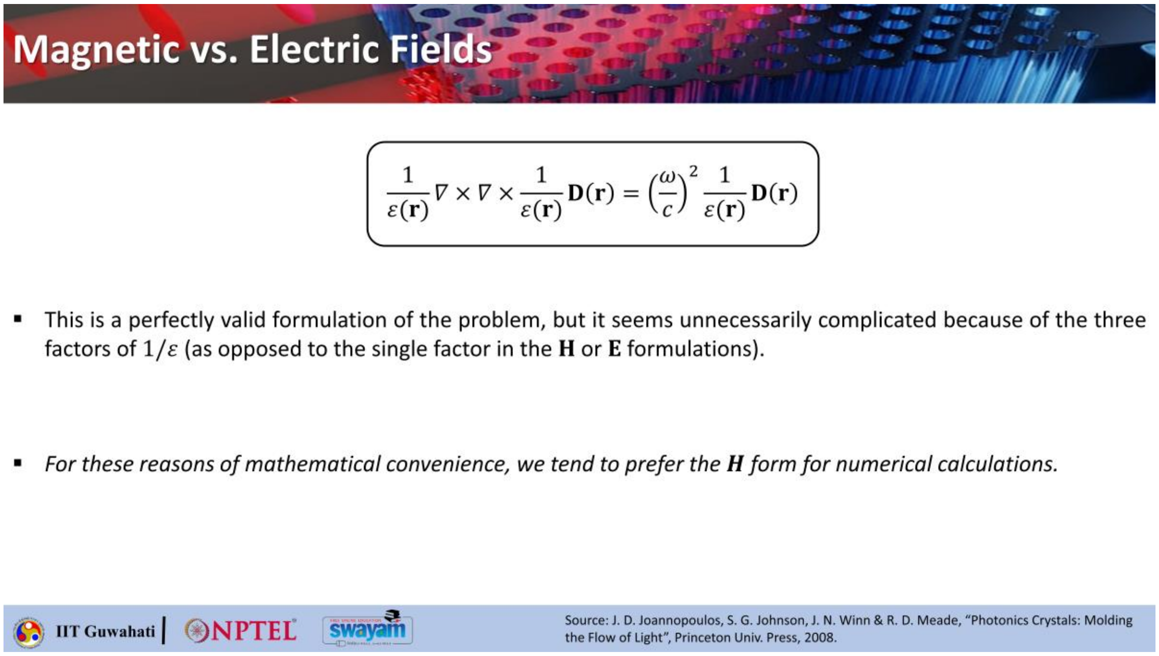
Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

So, for some analytical calculations such as you know derivation of the variational equation or the perturbation theory the electric field eigen problem is the most convenient starting point. However, it has one feature that turns out to be undesirable for numerical computation, which is the transversality constraint. So, if you look at the constraint here, it is $\nabla \cdot \varepsilon\mathbf{E}$ equals 0.

So, you see this transversality constraint depends on epsilon. So, we can restore a simpler transversality constraint by using D instead of E. So, it looks like $\nabla \cdot \mathbf{D}$ equals 0 that means you have to deal with you know the displacement field instead of the electric field. So, in this equation you can substitute you know $\mathbf{D}/\varepsilon_0\,\varepsilon$ this term can replace your E and then you divide both sides by ε to keep the Hermitian operator and then you can actually obtain this kind of a. So, in this equation you are substituting For E, you are putting $\mathbf{D}/\varepsilon_0\,\varepsilon$.



Magnetic vs. Electric Fields

$$\boxed{\frac{1}{\varepsilon(\mathbf{r})}\nabla \times \nabla \times \frac{1}{\varepsilon(\mathbf{r})}\mathbf{D}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \frac{1}{\varepsilon(\mathbf{r})}\mathbf{D}(\mathbf{r})}$$

- This is a perfectly valid formulation of the problem, but it seems unnecessarily complicated because of the three factors of $1/\varepsilon$ (as opposed to the single factor in the $\mathbf{H}$ or $\mathbf{E}$ formulations).
- *For these reasons of mathematical convenience, we tend to prefer the $\boldsymbol{H}$ form for numerical calculations.*

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Source: J. D. Joannopoulos, S. G. Johnson, J. N. Winn & R. D. Meade, "Photonics Crystals: Molding the Flow of Light", Princeton Univ. Press, 2008.

And then you are dividing both sides by ε to keep the operator Hermitian. And this above mentioned equation will yield this. So you can see this is perfectly valid formulation of the problem. But it is unnecessarily complicated because of the three factors of 1 by ε_r. that is getting into the picture as compared to a single factor which gets into the formulation of H or E alone. So, this is an extra mathematical burden and that is why for mathematical convenience people prefer to work on H form for the numerical calculations.

So, that is all for this lecture. We will start the discussion of scaling properties of Maxwell's equation in the next lecture. If you have any doubt or query regarding this lecture at any point of time, you can email me mentioning MOOC and photonic crystal on the email subject line and this is my email address. Thank you.