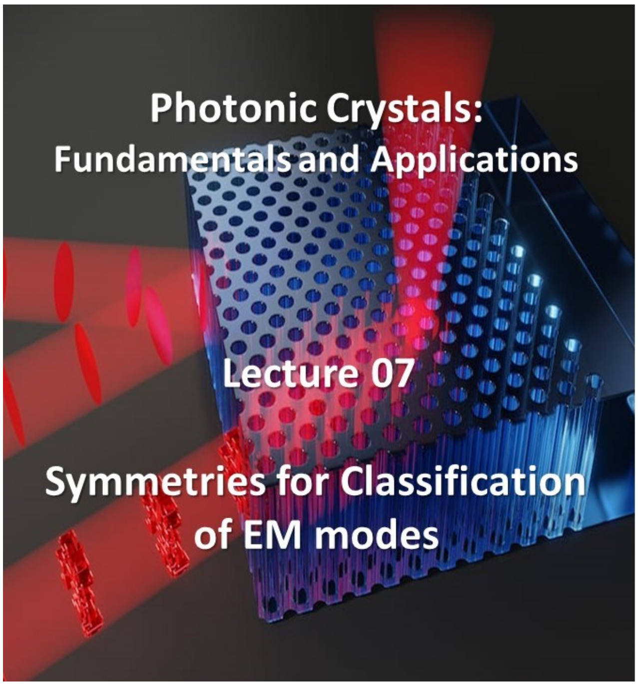


FREE ONLINE EDUCATION
swayam
शिक्षित भारत, उन्नत भारत

Dr. Debabrata Sikdar

Department of Electronics and Electrical Engineering
Indian Institute of Technology Guwahati

Web: https://www.iitg.ac.in/deb.sikdar
Email: deb.sikdar@iitg.ac.in

NPTEL

NPTEL ONLINE CERTIFICATION COURSE
AN INITIATIVE OF MoE, GOVT. OF INDIA

Lec 7: Symmetries for Classification of EM Modes

Hello students, welcome to lecture 7 of the online course on Photonic Crystals Fundamentals and Applications. Today's lecture will be on symmetries for classification of EM modes.

Lecture Outline

- **Continuous Translational Symmetry**
- **Discrete Translational Symmetry**
- **Rotational Symmetry**
- **Mirror Symmetry and the Separation of Modes**
- **Time-Reversal Symmetry**
- **Summary**

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

Here is the lecture outline. We will discuss about continuous translational symmetry, discrete translational symmetry, rotational symmetry, mirror symmetry and the separation of modes, time reversal symmetry, and finally, we will summarize all these different symmetries that occur in any electromagnetic system. So, why it is important? The symmetry in a dielectric structure serves as a convenient method for classifying electromagnetic modes within that system. So, we basically look for symmetry for mode classification.

Overview

- **Dielectric Symmetry and Mode Categorization:**
  - Symmetry in a dielectric structure serves as a convenient method for classifying electromagnetic modes within that system.

- **Focus on Translational Symmetries:**
  - Both discrete and continuous translational symmetries will be explored, with particular emphasis on their significance in the context of periodic dielectrics, such as photonic crystals.

- **Comprehensive Symmetry Exploration:**
  - Discussion will extend beyond translational symmetries which will include rotational, mirror, inversion, and time-reversal symmetries, offering a comprehensive examination of how various symmetries contribute to the understanding of electromagnetic modes in dielectric systems.

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

So, let us focus on the translational symmetries where both discrete and continuous translational symmetries will be explored with particular emphasis on their significance in the context of periodic dielectrics which is nothing but you know photonic crystals. We will then continue our discussion towards you know and beyond this translational symmetry and we will discuss about rotational, mirror, inversion and time reversal symmetries which will offer a comprehensive examination of how various symmetries could contribute towards the understanding of electromagnetic modes in any dielectric system. So, that way this particular lecture is very important. Now what is symmetry? So, symmetry refers to a balanced and harmonious arrangement of parts or elements.

What is symmetry?

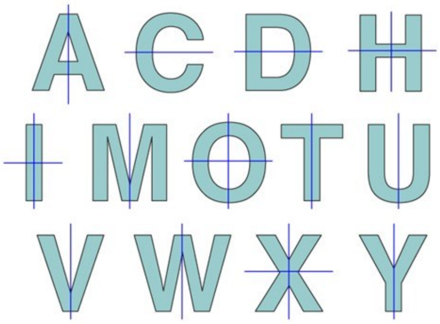


Figure 1 : Alphabet symmetries

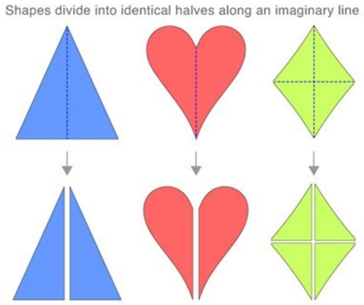


Figure 2 : Geometrical Shape symmetries

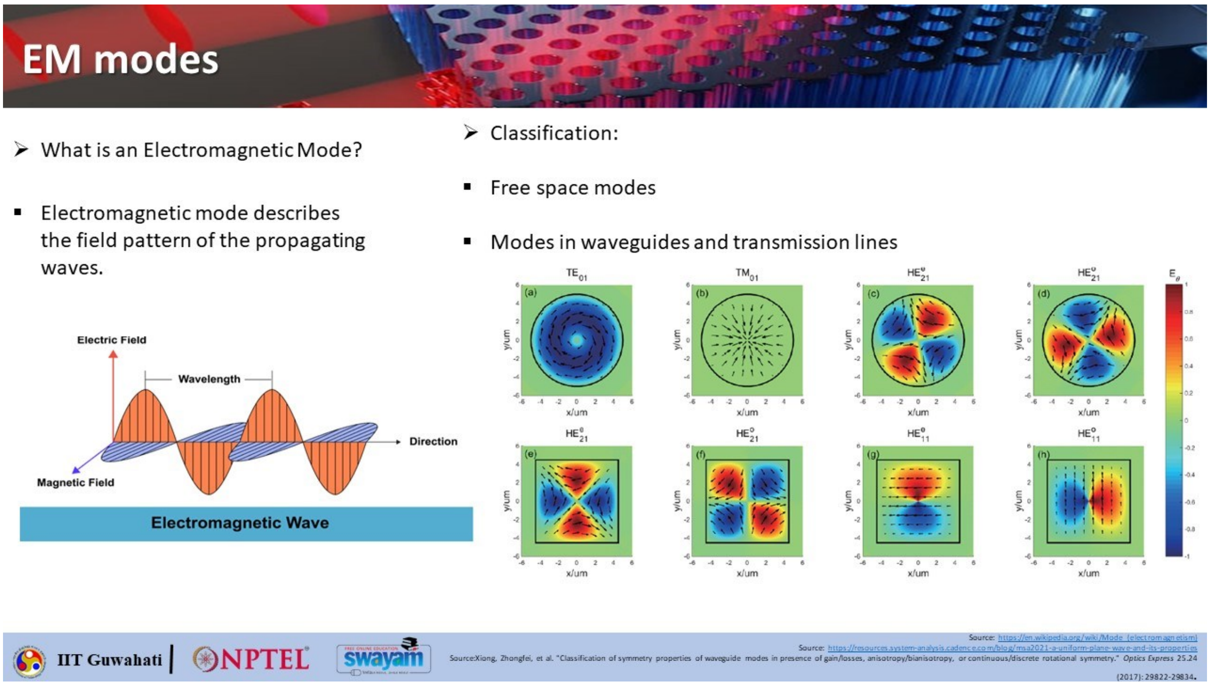
IIT Guwahati | NPTEL swayam

Source: https://mathmonks.com/symmetry

When something has symmetry, it means that one part mirrors or corresponds to the other part in a way that creates a pleasing or balanced whole structure. So, in our everyday life, you may notice symmetries in things such as butterfly wings, snowflake, even on human face, where one side is basically the mirror image of the other. In a broader context, if you discuss symmetry plays a critical role in various scientific and mathematical principles. that helps us to understand and describe patterns and relationships in the world around us, right? So here you can see some pictures that depicts the phenomena of symmetry. So, the lines that are drawn on these alphabets, they tell you the symmetry plane.

So on the two sides of the symmetry plane, you can see that they are basically similar kind of features. So, these are the two symmetry planes for the letter H, whereas for the letter U, you can see only one vertical symmetry plane. Similarly, if you divide different shapes, something like a triangle, you can actually see that this is the symmetry line. For the heart symbol also, this is the symmetry line. Whereas if you have a diamond kind of a symbol, these are the two symmetry lines.

It has got horizontal as well as vertical symmetry



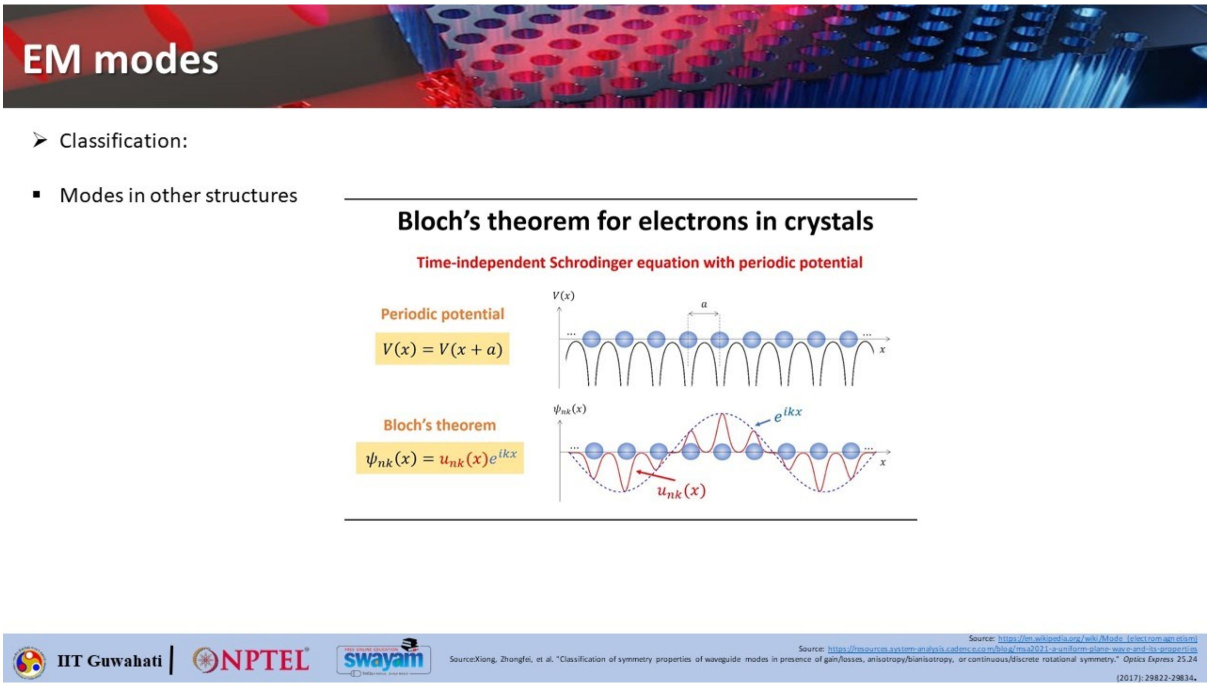
Now, in electromagnetic modes, we are discussing symmetries today to describe electromagnetic modes, EM modes. Now, what is an EM mode? So, EM mode basically describes the field pattern of the propagating wave. So, you can classify them as free space mode as you can see here.

okay. So, electromagnetic modes can be thought of analogous to the normal modes of vibration in other systems something like in mechanical systems okay. So, here you can think of free space modes and what you have shown here is plane waves that is propagating in vacuum or in any medium. So, here you can see that the electric and the magnetic field. So, this is the blue one is magnetic field, the electric field, okay, is this one. They are basically orthogonal to each other and also to the direction of propagation, right? So, these are the waves that exist in free space far from any antenna.

There are also modes which are EM modes, okay, seen in the waveguides and transmission line. So, the transverse modes okay is basically those where you know at least one of the electric or magnetic field entirely lies in the transverse direction right. So, you can also have TEM mode which are transverse electromagnetic modes where just like the plane wave both electric and magnetic fields are basically entirely transverse. So, when we discuss about transverse electric mode, it means that only electric field is entirely transverse. So those can be also notated as H modes because it indicates that there could be a longitudinal magnetic field component.

We can also think of TM modes, transverse magnetic modes. So, there only the magnetic field is entirely transverse and they can be notated as E mode indicating that there is a possible longitudinal electric component. Then there are hybrid modes. So, hybrid electromagnetic modes can be named as HEM or HE modes okay as you can see here these are different hybrid modes. So, here is what happens: both the electric and magnetic field components can lie along the longitudinal direction.

So, they can be analyzed as a superposition of their corresponding TE and TM modes, okay? So, HE modes are basically those where the electric, or you could say the TE component, dominates, and there could be EH modes. Those are the hybrid modes where TM components dominate. So, you can also see modes in other structures, such as block modes. Which are seen.   This sentence is a fragment and lacks context.



So, block modes are basically modes of block waves. So, you can see those occurring in periodically repeating structures. You can think of a periodic potential, which represents the Bloch theorem, where $u_{nk}(x)$ is basically this, and the envelope is given by e^{ikx}. So, this is Bloch's theorem, and the periodic potential can be written as $V(x) = V(x+a)$. So, a is essentially the period over which the potential is repeating.

Now, as you understand, EM modes are basically distributions of electromagnetic energy. So, you can actually take advantage of symmetries to classify different electromagnetic modes. So, in both classical mechanics and quantum mechanics, the study of symmetry provides a powerful tool for making general statements. The behavior of the system can be extended to our electromagnetic system as well. So, the mathematical analogy could highlight the role of symmetry in understanding the properties of electromagnetic systems.

Symmetries classify EM modes

- **Symmetry and System Behaviour:**
  - In both classical mechanics and quantum mechanics, the study of symmetries provides a powerful tool for making general statements about the behaviour of a system.
  - This principle is extended to electromagnetic systems.

- **Mathematical Analogy with Symmetry:**
  - The mathematical analogy highlights the role of symmetry in understanding the properties of electromagnetic systems.
  - Symmetry considerations become a valuable lens for interpreting and predicting system behaviour.

- **Concrete Example and Formal Discussion:**
  - The exploration begins with a specific example highlighting the impact of symmetry on electromagnetic systems.
  - This serves as a practical illustration before delving into a more formal discussion of symmetries in electromagnetism, offering a structured approach to comprehending their significance in this context.

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

Symmetries classify EM modes

- **Arbitrary Shape, Important Symmetry:**
  - The two-dimensional metal cavity in figure possesses an arbitrary shape, posing challenges for establishing precise boundary conditions and solving the problem analytically.
- **Central Symmetry Simplifies Analysis:**
  - Despite the complex shape, the cavity exhibits an important symmetry: inversion symmetry about its center.
  - If the cavity is inverted, the resulting shape remains identical.
  - This symmetry simplifies the analysis, offering a key insight into the behaviour of electromagnetic modes within the cavity.
- **Indistinguishability of Symmetric Modes:**
  - The inversion symmetry implies that if a particular mode pattern, denoted as H(r), is identified with a frequency ω, its symmetric counterpart, H(−r), must also be a mode with the same frequency ω.

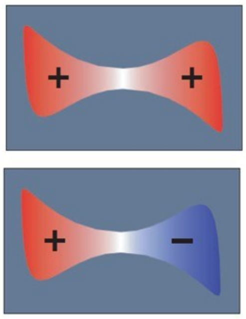


Figure: A two-dimensional metallic cavity with inversion symmetry. Red and blue suggest positive and negative fields. Top, an **even** mode occupies the cavity, for which **H**(**r**)= **H**(−**r**). Bottom, an **odd** mode occupies the cavity, for which **H**(**r**) = −**H**(−**r**).

IIT Guwahati | NPTEL swayam Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

Symmetry considerations become a valuable lens for interpreting and predicting system behavior. (The original sentence is already correct.) So, we will take specific examples as we go through each system, okay? So, we can see that you know the exploration begins with a very specific example highlighting the impact of symmetry on electromagnetic systems. And this could serve as a practical illustration before delving into a more formal discussion about symmetries in electromagnetism. This could offer a structured approach to understanding the importance of symmetry in this particular context.

So, the two-dimensional metal cavity that you can see here has an arbitrary shape. This shape is not a regular shape, and it can be challenging to establish the precise boundary conditions and solve this particular problem analytically. And that is where central symmetry can simplify your analysis. So, despite the complex shape, the cavity exhibits an important symmetry known as inversion symmetry about the center. So, if you think of this as the center line, appreciate the fact that this part and this part are basically similar; it's just that this one is inverted, right? So, if this particular cavity is inverted, you can actually obtain the remaining shape, which simplifies the analysis of this symmetry.

You know, offering a key insight into the behavior of the electromagnetic modes within the cavity, right? This is a two-dimensional metallic cavity with inversion symmetry. So, when you look at red and blue, they basically represent positive and negative fields. So, we are not discussing which field. You can assume that these are basically magnetic fields. The sentence is already correct as it is.

"Okay" is an acceptable expression. Positive and negative. (This phrase is already grammatically correct, but it may need context to convey a complete thought. If you have a specific context in mind, please provide it for further assistance.

) The sentence "Okay." is already grammatically correct. If you need a different kind of correction or context, please provide more details! And at the top, you can see that an even mode actually occupies the cavity because the field here is similar to the field there. You can write hr equals h minus r. Meanwhile, at the bottom, you can say that this is basically the odd mode. Because you can say that hr is basically minus h minus r, you can identify that both of these modes could have the same frequency omega.

Symmetries classify EM modes

- **Degeneracy and Mode Equivalence:**
  - Modes with the same frequency are termed degenerate.
  - If a mode H(r) is not part of a degenerate family, then its symmetric counterpart H(−r) with the same frequency must be identical.
  - This indicates that H(−r) is simply a multiple (α) of H(r).
- **Symmetry-Induced Constraints on α:**
  - Inverting the system twice should return to the original function, leading to the condition $\alpha^2 H(r)=H(r)$
- **Classification of Nondegenerate Modes:**
  - Nondegenerate modes are classified based on their response to inversion symmetry: even modes (H(−r)=H(r)), and odd modes (−H(−r)=H(r))
  - This classification provides insights into how these modes behave under the system's symmetry operations.

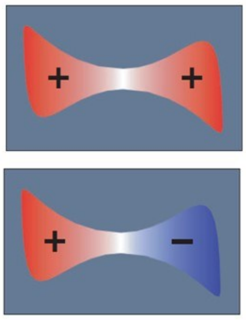


Figure: A two-dimensional metallic cavity with inversion symmetry. Red and blue suggest positive and negative fields. Top, an **even** mode occupies the cavity, for which **H**(**r**)= **H**(−**r**). Bottom, an **odd** mode occupies the cavity, for which **H**(**r**) = −**H**(−**r**).

IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

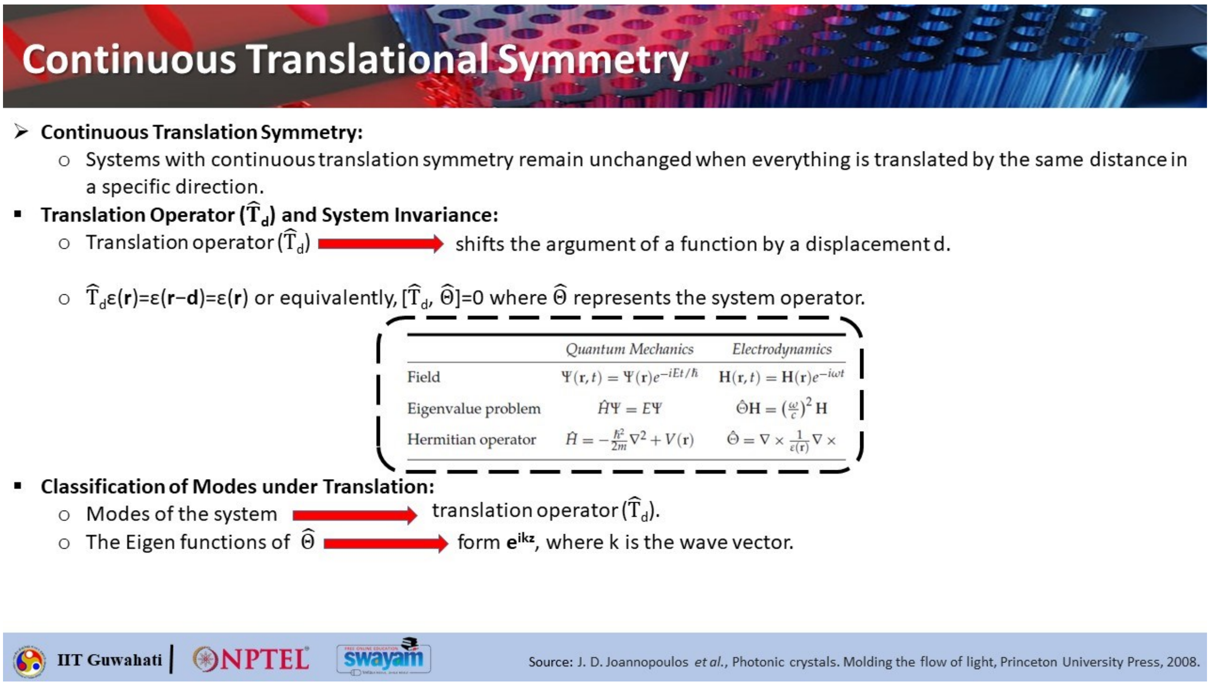
But one has a symmetric distribution, while the other has an asymmetric distribution. So, this inversion symmetry implies that if a particular mode pattern denoted by $\mathrm{H(r)}$ is identified with a frequency ω, its symmetric counterpart... Which is $H(-r)$; this part will also have the same frequency of omega.

However, this overall odd and even mode may have different frequencies, okay? So, now let us look into, you know, the modes with the same frequencies, okay? Those are basically called degenerate modes, right. So, if a mode HR is not part of the degenerate family, then its symmetric counterpart, h minus r, with the same frequency must be exactly identical. So, this would indicate that h of minus r is basically a scalar multiplied by the actual field h of r. So, you can say that, depending on the type of symmetry—whether it's even or odd—this alpha can be plus 1 or minus 1. Now, we have seen in the previous slide that an alpha value of 1 signifies even modes, which are invariant under inversion.

You can consider alpha equal to minus 1, which characterizes an odd mode that becomes opposite under inversion, right? So, what happens if you invert the system two times? So, if you invert the system twice, the system will return to its original function, leading to the condition that alpha squared times h times r is equal to h times r, right? This way, you can classify non-degenerate modes. Non-degenerate modes can be classified based on their response to inversion symmetry. You can think of $H(-r)$ as being equal to $\mathrm{H(r)}$ and odd modes, which are essentially -$H(-r)$ being equal to $\mathrm{H(r)}$. So, this classification provides insights into how modes behave under the system's symmetry operations. So, we have already discussed this in the previous slide.

Continuous Translational Symmetry

IIT Guwahati | NPTEL swayam



So, moving on to the next type of symmetry, we will discuss continuous translational symmetry. So, continuous translational symmetry exists. So these are the symmetry conditions under which the system would remain unchanged when everything is translated by the same distance in a particular direction. Okay. So, you can think of the translational operator $(\hat{T}_d)$ and system invariance like this.

You apply this translational operator to a system that is equivalent, and you know all the functions being displaced by a displacement of d. So, you can write that $\hat{T}_d\varepsilon(\mathbf{r})$ will be equal to $\varepsilon(\mathbf{r}-\mathbf{d})$, which is the same as $\varepsilon(\mathbf{r})$. In that way, it exhibits translational symmetry. It means that if you move your system a distance d, the same properties will repeat. So, you can consider, you know, $\left[\hat{T}_d,\hat{\Theta}\right]$ to be 0, okay? So, this represents that the system remains unchanged, where theta cap represents the system operator, right? So, you can see this particular comparison between quantum mechanics and electrodynamics, where you can express the field or.

.. You know the potential in this particular form in quantum mechanics, and you can write the magnetic field in a similar form in electrodynamics. If you represent this as an eigenvalue problem, you can see that you can write $\hat{H}\Psi = E\Psi$, where you can see that $\hat{\Theta}\mathrm{H}=\left(\frac{\omega}{c}\right)^2\mathrm{H}$. "Square it into h. So, this is also in the form of an eigenvalue equation." The sentence is already grammatically correct.

However, if you're looking for a slight variation, you could say: "And what is this?" This is basically Maxwell's operator, which we will discuss in more detail in the next lecture, right? So, you can actually find $\hat{H}$ as in the Hermitian operator, and similarly, your $\hat{\Theta}$ is that of the Maxwell operator. So, you can classify the modes using translation. So, the modes of the system can now be classified based on their behavior under the translational operator $\left(\hat{T}_d\right)$. The eigenfunctions of $\hat{\Theta}$ can be chosen to be the eigenfunctions of all these $\left(\hat{T}_d\right)$. So, this leads to a z dependence in the form of e to the power of ikz, where k is the wave vector.

Continuous Translational Symmetry

- **Eigen function of Translation Operator ($\hat{T}_d$) in z Direction:**
  - $\hat{T}_{d\hat{z}}\, e^{ikz} = e^{ik(z-d)} = (e^{-ikd})e^{ikz}$.
  - The corresponding eigenvalue is e^{-ikd}.
- **Classification by Wave Vector (k):**
  - Modes of the system can be classified by the values of *k*, the wave vector, indicating the *z* dependence of the functional form e^{ikz}.
  - In an infinite system, k must be a real number, ensuring that modes have bounded amplitudes at infinity.

IIT Guwahati | NPTEL swayam — Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

So, we discussed that the eigenfunction of the translational operator, denoted as $\left(\hat{T}_d\right)$ in the z direction, can be written like this. So, you apply this translational operator to the parameter e^{ikz} okay? So, that becomes $e^{ik(z-d)}$. Therefore, you have this term, okay? So, this multiplied by this. So, basically, if you have one operator operating on e^{ikz}, you get this particular eigenvalue and the parameter itself. So, e^{-ikd} is basically an eigenvalue, right? The modes of the system can be classified by the value of k, which is the wave vector, and it indicates the z-dependence of the functional form.

So, it is e^{ikz}. If you consider an infinite system, k must be real, which ensures that the modes have bounded amplitude at infinity. So, if you consider a system with continuous translational symmetry in all three directions, it actually becomes a homogeneous medium. This can be characterized by a constant permittivity epsilon, which is typically considered to be one for free space

Continuous Translational Symmetry

- **Homogeneous Medium and Continuous Translational Symmetry:**
  - A system with continuous translational symmetry in all three directions is a homogeneous medium, characterized by a constant permittivity (ε), often equal to 1 for free space.

- **Mode Form in Homogeneous Medium:**
  - Modes in a homogeneous medium : $\mathbf{H_k(r)} = \mathbf{H_0}\mathbf{e^{ik\cdot r}}$, where $\mathbf{H_0}$ is any constant vector. These modes are plane waves, and their polarization aligns with the direction of $\mathbf{H_0}$.

- **Transversality Requirement and Restriction on Wave Vector:**
  - Imposing transversality requirement, $\mathbf{k}\cdot\mathbf{H_0} = 0$, further restricts the possible wave vectors ($\mathbf{k}$). This condition ensures that the plane waves satisfy the essential properties for electromagnetic waves.

IIT Guwahati | NPTEL swayam Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

So, how does the mode form look in a homogeneous medium? So, if you consider modes in a homogeneous medium, you can write $\mathbf{H_k(r)} = \mathbf{H}_0 \mathrm{e}^{\mathbf{ik\cdot r}}$,

Here, $\mathbf{H}_0$ is a constant vector. So, these modes are basically plane waves, and their polarization aligns along the direction of h naught. Now, if you impose the transversality requirement, you can say that k dot H naught will be equal to 0, which will further restrict the possible wave vectors. So this particular condition will ensure that the plane waves satisfy the essential properties of electromagnetic waves, right? So, for plane waves, you can start discussing the dispersion relation. So, this is the master equation. So this basically comes from Maxwell's equations, and this particular term, $\left(\frac{\omega}{c}\right)^2$, can also provide a result if you apply this specific master equation

Continuous Translational Symmetry

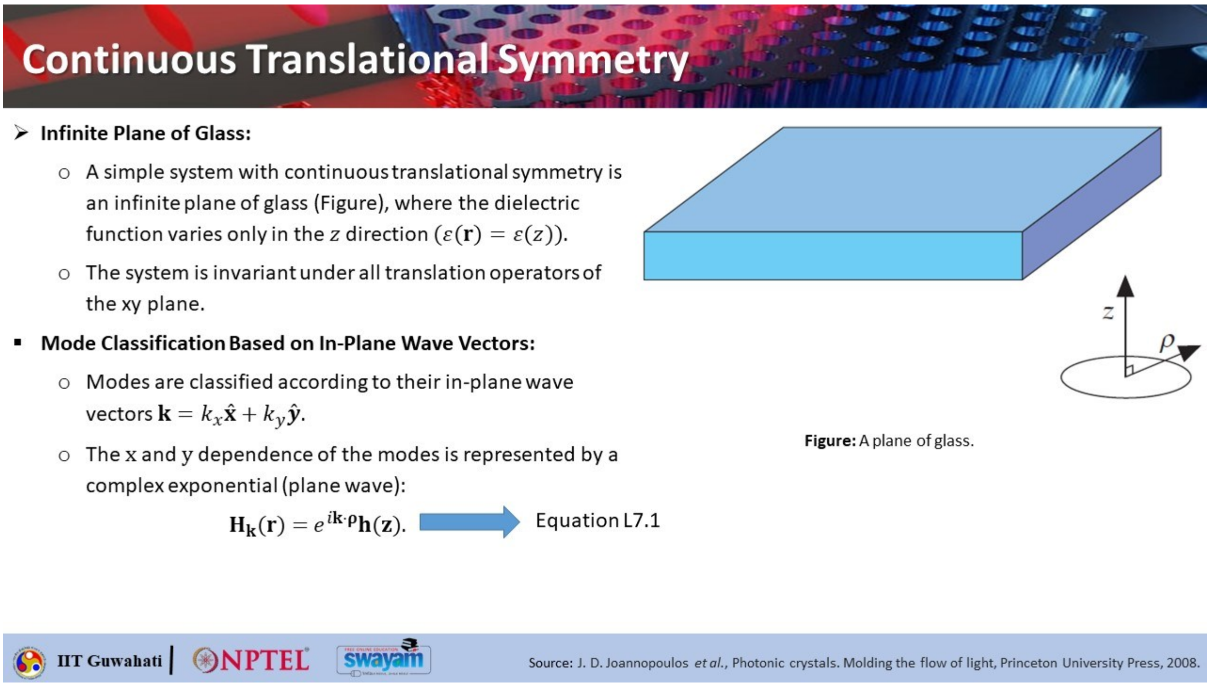
- **Dispersion Relation for Plane Waves:**
  - Master equation → $\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r})\right) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r})$ → (solution) The plane waves, $\mathbf{H}_{\mathbf{k}}(\mathbf{r})$
  - Eigenvalues given by $\left(\frac{\omega}{c}\right)^2 = \frac{|\mathbf{k}|^2}{\epsilon}$.
  - The resulting dispersion relation: $\omega = c|\mathbf{k}|/\sqrt{\epsilon}$, where ω is the angular frequency, c is the speed of light, $|\mathbf{k}|$ is the magnitude of the wave vector, and ϵ is the permittivity.

- **Classification by Wave Vector (k):**
  - Wave vector $(\mathbf{k})$ → (Classifies) Plane waves, $H_k(\mathbf{r})$

So, you can find the plane waves, which are basically h, k, and r; they are the solutions of this master equation. Okay, and these will be the eigenvalues, right? So, omega squared divided by c squared looks like the eigenvalue, which is basically given by the modulus of k squared divided by epsilon. So, the relationship between omega and k in this particular medium with permittivity epsilon is okay. So, that is the dispersion relation.

(The sentence is already grammatically correct.) So, what is that relationship? Here you can see that omega equals c times the modulus of k divided by the square root of epsilon. So, what is Omega? It is the angular frequency, c is the speed of light, and k is the modulus, which is essentially the wave number. Alternatively, you can say that it is the magnitude of the wave factor, and epsilon is the permittivity. The sentence "right." is already grammatically correct as a single-word response.

However, if you intended to provide a longer context or a complete sentence, please provide more information. So, now we can classify according to the wave vectors. So, we understood that the plane waves are classified by their wave vectors, which specify how the mode transforms under a continuous translation operation. Also, the wave vector plays a crucial role in determining the direction and characteristics of the plane waves in a homogeneous medium.



So you can consider an infinite plane of glass. So, this is a simple system with continuous translational symmetry, right? Where the dielectric function varies only in the z direction, right? You can say that E of r is basically, or sorry, you can say that epsilon r is basically epsilon z. So, along with all the others you know, or you could say in the azimuthal direction or azimuthal plane, it is the same, right? So, you can say the system is invariant under all translation operators of the x-y plane, you know. So, it only changes along the z direction, right? You can see from the figure itself that the glass extends much further along the x and y directions. In the z direction, you can consider it to have finite thickness. You can say that epsilon r is essentially varying only along the z direction, and there is no dependence on the in-plane coordinates, which are like rho.

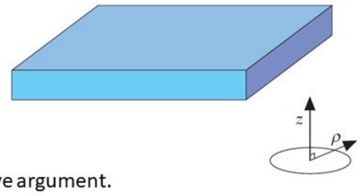
It can be x or y, but there is no dependence, right? So, if you now try to classify the modes according to their in-plane wave vector, okay? The in-plane wave vector $\mathbf{k}$ can then be written as $k_x\hat{\mathbf{x}} + k_y\hat{\boldsymbol{y}}$. .The xy dependence of the modes can be represented by a complex exponential, which is a plane wave. So you can write $\mathbf{H}_k(\mathrm{r})$ to be equal to $e^{i\mathbf{k}\cdot\rho}\mathbf{h}(\mathbf{z})$, okay? The function $\mathbf{h}(\mathbf{z})$ that you see here, which is basically dependent on K, cannot be determined solely by this reasoning. Because the system lacks translational symmetry along the z direction.

So, you have to impose the condition of transversality. That condition imposes a restriction on the function H. So you can take $\mathbf{k}\cdot\mathbf{h}$, which will be equal to $i\dfrac{\partial h_z}{\partial z}$. Now, if you apply the symmetry arguments, You can say that you know the mode can be described by $\mathbf{H}_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\rho}\mathbf{h}(\mathbf{z})$, okay? So if you then put non-collinear neighboring points at the same z value, they must be treated equally due to symmetry. So, that actually sets the phase relation between the points and effectively specifies kx and ky universally for this particular plane. And along the z-axis, this particular restriction does not hold.

Continuous Translational Symmetry

- **Role of Function $\mathbf{h}(\mathbf{z})$:**
  - The transversality condition imposes a restriction on $\mathbf{h}$

$$\mathbf{k} \cdot \mathbf{h} = i\frac{\partial h_z}{\partial z}.$$



- **Symmetry Argument :**
  - The modes described by $\mathbf{H}_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\boldsymbol{\rho}}\mathbf{h}(\mathbf{z})$ can be understood through an intuitive argument.
  - Non-collinear neighbouring points at the same z value must be treated equally due to symmetry, setting the phase relationships between points and effectively specifying k_x and k_y universally to the plane.
  - Along the z direction, this restriction does not hold, allowing for different amplitudes and phases.

IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

So, it allows for different values of amplitude and phase. So, when you classify by wave vector k and band number n, you can represent each mode using k and n. And in case there is degeneracy, you can use additional indices to distinguish those degenerate modes, which have the same n and k values. So, here you can see a bench structure that is basically a dispersion relation. So, the bench structure is basically a plot of the wave vector versus the frequency for that particular plane of glass. So, we are basically talking about the same system that we have seen here, which is the band structure or dispersion relation.

So here, different bands correspond to lines that rise uniformly in frequency as the band number increases. This band structure provides insights into the allowed modes and their frequencies in this particular system. The sentence "Right." is already grammatically correct. If you would like to provide more context or another sentence, I'd be happy to help! So here a couple more pieces of information are also available.

Continuous Translational Symmetry

➢ **Mode Classification by Wave Vector (k) and Band Number (n):**

- Each mode —Classified by→ (**k**, n)
- If there is degeneracy, additional indices may be included to distinguish degenerate modes with the same n and k.

▪ **Band Structure and Frequency:**

- The band structure (or dispersion relation) : wave vector versus mode frequency.
- Different bands correspond to lines rising uniformly in frequency as the band number (n) increases.
- The band structure provides insights into the allowed modes and their frequencies in the system.

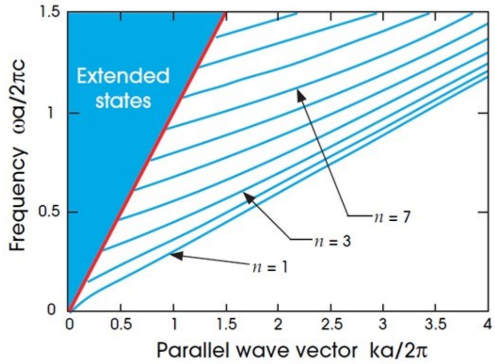


Figure: Harmonic mode frequencies for a plane of glass of thickness a and ε=11.4.

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

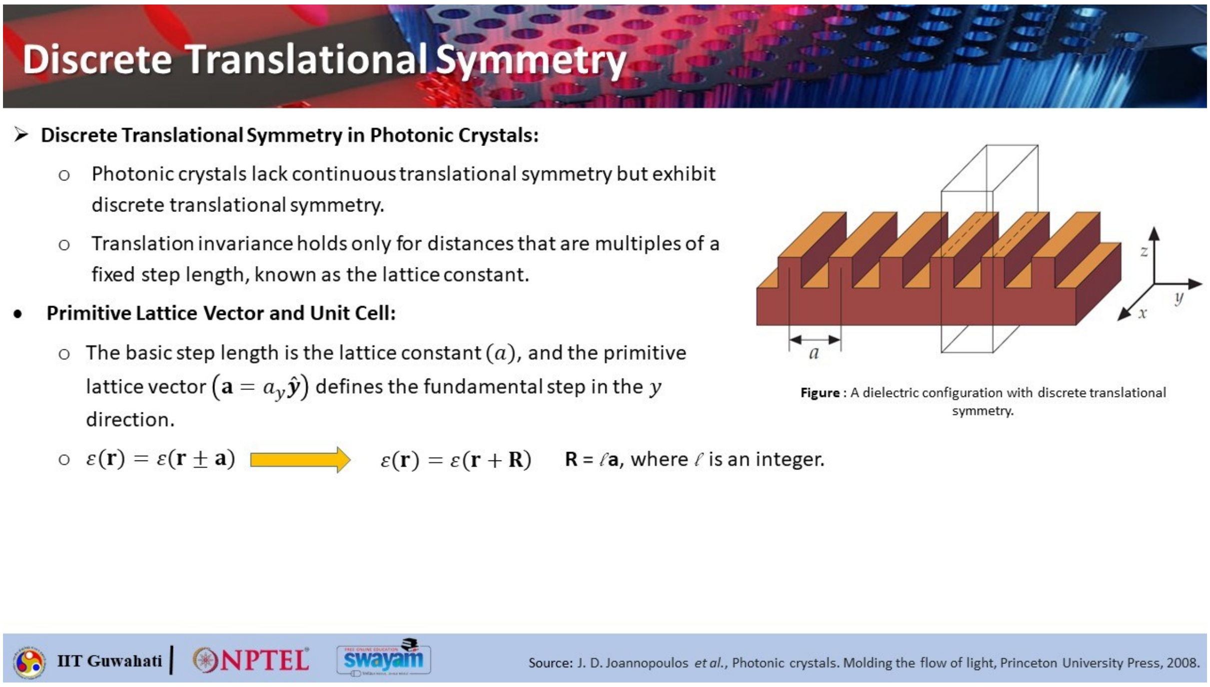
As you can see, the frequency is essentially a normalized frequency. Similarly, the parallel wave vector is normalized as well. So this is done for a plane of glass with a thickness A and a permittivity of epsilon, which is taken to be 11.4. Okay, so these blue lines correspond to modes that are localized inside the glass.

You can see different mode numbers over here. And you can see a red line that basically marks the light line, so this is the line that shows you the dispersion relation of omega equals c k. The sentence "Right." is already grammatically correct. If you have a longer sentence or additional context that needs correction, please provide it! So the shaded blue region is basically a continuum of states.

The sentence "OK." is grammatically correct as it stands. If you have a longer sentence or another one to correct, please provide it! They extend into both the glass and the air around it. But here you have those discrete states. The sentence "OK." is already grammatically correct.

So, this particular plot shows you the modes with only one polarization. Here, H is essentially perpendicular to both the Z and K directions.

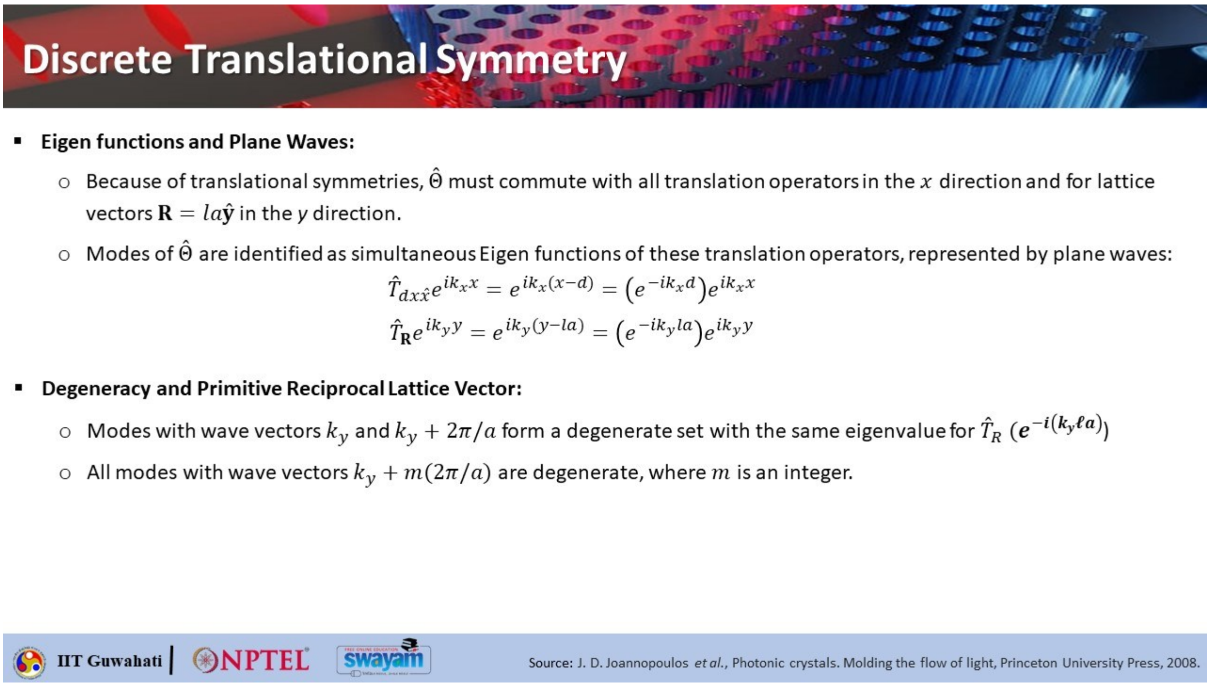
Discrete Translational Symmetry



So, with that, we will now continue our discussion on discrete translational symmetry. So you can think of a figure of grating like this. So, this dielectric conjugation has discrete translational symmetry. So, this is one type of photonic crystal, and it is very relevant to photonic crystals because they.

.. Also, they lack continuous translational symmetry, but they exhibit this discrete translational symmetry correctly. Because of this discrete symmetry, you can write $\varepsilon(\mathbf{r}) = \varepsilon(\mathbf{r} \pm \mathbf{a})$, where a represents that periodicity. By repeating this translation, you will see that epsilon r is basically epsilon r plus capital R, where capital R is the integral. Multiple periods of A are shown here.

The repeated dielectric unit is highlighted in this box. You can call this a unit cell, which is repeated periodically in one dimension to form the entire structure. The sentence "right." is grammatically correct as it stands. However, if you intended to provide a complete thought or context, please share more details for further assistance. So, here it is basically an xz slab of dielectric material that has a width of a in the y direction.



So, that way you can define this correctly, okay? So, this is what we discussed. (The sentence is already grammatically correct.) So, R, as I mentioned, is basically L, and L is any integer. So, because of these translational symmetries, theta cap must commute with all translational operators in the x direction. For lattice vectors, that is $\mathbf{R} = la\hat{\mathbf{y}}$, that lies along the y direction.

Now, the modes of theta cap can be identified as simultaneous eigenfunctions of these translational operators, which are typically represented by plane waves. So you can see that the operator on this $e^{ik_x x}$ can be adjusted, allowing x to be written as x minus d. So, this term results in the eigenvalue, and you obtain this equation again. The same thing also happens when you use the lattice vector. So, you can because this is continuous, while this is discrete, as r is only an integral multiple of the lattice period.

So, when you replace this equation dx with R, you can see that d will be replaced by la. So it is a discrete step, you know, through which you are translating along the structure. And you can see that this is how the discrete translational symmetry operation looks. We can begin to classify the modes by specifying, you know, k_x and k_y, okay, as you see here.

However, not all the values of k_y will yield eigenvalues. So, let us consider two modes: one with a wave vector of k_y and another with a wave vector of $k_y + 2\pi / a$. And if they form a degenerate set with the same eigenvalue for this particular operation. So, we can say that k y and know $k_y + m(2\pi / a)$ would be degenerate, where m is basically an integer, okay?

Discrete Translational Symmetry

- **Linear Combinations and Expansion Coefficients:**
  - Linear combinations of degenerate Eigen functions yield modes in the form

$$\mathbf{H}_{k_x,k_y}(\mathbf{r}) = e^{ik_x x} \cdot e^{ik_y y} \cdot \sum_m \mathbf{c}_{k_y,m}(z) \cdot e^{imby}.$$
$$= e^{ik_x x} \cdot e^{ik_y y} \cdot \mathbf{u}_{k_y}(y,z)$$

  - The expansion coefficients ($\mathbf{c}$) are determined through explicit solution, and $\mathbf{u}(y,z)$ is a periodic function in y, satisfying $\mathbf{u}(y+la,z) = \mathbf{u}(y,z)$.

- **Periodic Modulation and Bloch State:**
  - The y dependence of $\mathbf{H}$ is a periodic modulation, expressed as $\mathbf{H}(\dots,y,\dots) \propto e^{ik_y y}.\ \mathbf{u}_{k_y}(y,\dots)$.
  - This result, known as Bloch's theorem, is a fundamental concept in solid-state physics and mechanics, providing insight into the behavior of waves in periodic structures.

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

? So, since any linear operation, or you could say any linear combination of these degenerate eigenfunctions, yields modes in the form of this one, okay? So, we can take the linear combinations of our original modes and express them in the form below. So, you have

$$\mathbf{H}_{k_x,k_y}(\mathbf{r}) = e^{ik_x x} \cdot e^{ik_y y} \cdot \sum_m \mathbf{c}_{k_y,m}(z) \cdot e^{imby} = e^{ik_x x} \cdot e^{ik_y y} \cdot \mathbf{u}_{k_y}(y,z)$$

, and then you can actually take a linear combination of them.

Degenerate eigenfunctions allow you to represent any mode in this particular form. So, here you can see that the expansion coefficient c can be determined through an explicit solution. However, u(y,z) is basically a periodic function in y that satisfies the particular condition that $\mathbf{u}(y+la,z)$ is the same as $\mathbf{u}(y,z)$. We are not talking about the dependency here; whatever is there will simply be translated along y, fine. The discrete periodicity in y leads to a y dependence for H, which is simply the product of a plane wave and a y-periodic function, something like this. We can think of it as a plane wave, as it would be in free space, but it is essentially modulated by a periodic function due to the periodic lattice.

So, this particular result is also known as the Bloch theorem. And it is one of the fundamental concepts in solid state physics and mechanics because it provides insight into the behavior of waves in periodic structures. So, what does it mean if there is a plane wave? When the waves meet a periodic structure, their amplitude will pick up the periodicity of the structure. This is how you can explain it in simple words. So, now let us take a look at rotational symmetry.

Rotational Symmetry

Rotational Symmetry

- **Symmetries in Photonic Crystals:**
  - Photonic crystals may exhibit various symmetries beyond discrete, translations, including rotational symmetry, mirror reflection, or inversion symmetry.
- **Rotational Symmetry Operator:**
  - Suppose the operator (3×3 matrix) $\mathcal{R}(\hat{\mathbf{n}}, \alpha)$ rotates vectors by an angle α about the $\hat{\mathbf{n}}$ axis.
  - Abbreviate $\mathcal{R}(\hat{\mathbf{n}}, \alpha)$ by $\mathcal{R}$. To rotate a vector field $\mathbf{f}(\mathbf{r})$, we take the vector $\mathbf{f}$ and rotate it with $\mathcal{R}$ to give $\mathbf{f}' = \mathcal{R}\mathbf{f}$.
  - We also rotate the argument $\mathbf{r}$ of the vector field: $\mathbf{r}' = \mathcal{R}^{-1}\mathbf{r}$.
  - Therefore $\mathbf{f}'(\mathbf{r}') = \mathcal{R}\mathbf{f}(\mathbf{r}') = \mathcal{R}\mathbf{f}(\mathcal{R}^{-1}\mathbf{r})$. Accordingly, we define the vector field rotator $\hat{O}_{\mathcal{R}}$ as

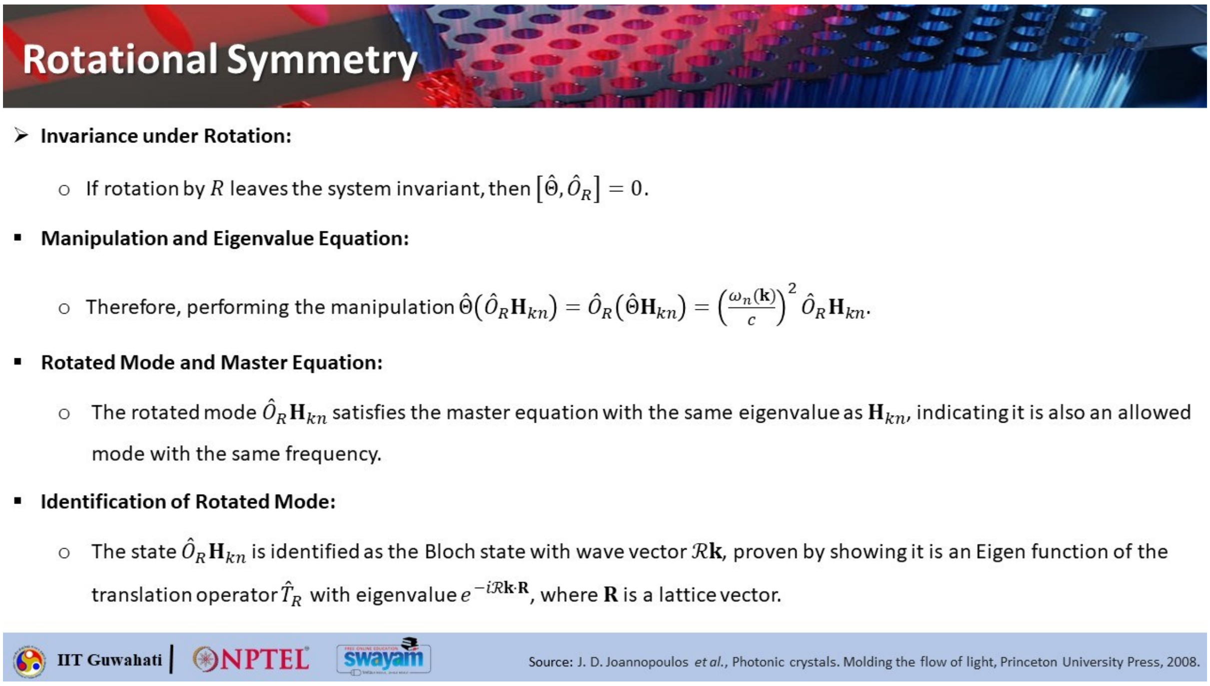
$$\hat{O}_{\mathcal{R}} \cdot \mathbf{f}(\mathbf{r}) = \mathcal{R}\mathbf{f}(\mathcal{R}^{-1}\mathbf{r}).$$

IIT Guwahati | NPTEL swayam Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

So, photonic crystals might have symmetries other than discrete translations. So, a given crystal might have, you know, other symmetries, such as rotational symmetry, mirror reflection, or inversion symmetry, right? To begin, we will examine the conclusions that can be drawn about the modes of a system with rotational symmetry. So suppose there is an operator, a three by three matrix of $\mathcal{R}$, which represents the rotational symmetry operator. Okay, it is basically a rotation. So, there has to be an axis and there has to be an angle by which you are rotating.

So, you can represent it as $\mathcal{R}$. It means that this operator rotates vectors by n. Angle alpha about the n-hat axis, right? So, you can abbreviate this $\mathcal{R}$ simply as $\mathcal{R}$, okay? So, what you know makes life easy for us, and if you want to rotate a vector field, that's okay. You take that vector $\mathbf{f}$ and rotate it with this curly $\mathcal{R}$, and you get $\mathbf{f}'$, which is basically $\mathcal{R}$, right? So you can also rotate the argument, which is the space. Okay, you can write it as $\mathbf{r}'$, which will be the $\mathcal{R}$, okay? So if you put $\mathbf{f}'$.

.. You can write it as $\mathcal{R}$, which can be expressed as $\mathcal{R}$ $\mathcal{R}$, okay? It means you can define this as a vector field operator $\hat{O}_{\mathcal{R}}$. So, this operator, when it is operating on a vector field $\hat{O}_{\mathcal{R}}$ $\mathbf{f}(\cdot) = \mathcal{R}$ $\mathcal{R}$, is it okay? So, this is the operation. (The sentence is already correct.



) So, $\mathcal{R}$ $\mathcal{R}$. That is basically the operation on the argument. So if the rotation $\mathcal{R}$ leaves the system invariant, then you can say that the $\left[\hat{\Theta}, \hat{O}_R\right]$ and this operator will give you 0, right? By performing the manipulation, it goes like this: you take the field $\mathbf{H}_{kn}$ and apply this rotational operator. And then you already have the theta cap, which is the Maxwell operator. You can do this because these two can be interchanged, and you can write it like this. So you already know this from the master equation.

So what you are doing here is the rotated mode, which is basically this one. You can see that it satisfies the master's equation, and what you have as your eigenvalue is the same as $\mathbf{H}_{kn}$, right? So, you can see that it is allowed. The mode with the same frequency indicates that it has rotational symmetry, so the state $\left(\hat{O}_R \mathbf{H}_{kn}\right)$ can be identified as the block wave. With wave vector $\mathbf{k}$, it is proven by showing that it is an eigenfunction of the translational operator $\hat{T}_R$ with the eigenvalue of $e^{-i\mathcal{R}}$.

Dot R, where R is the lattice vector

Rotational Symmetry

- **Eigenvalue Calculation:**

$$\hat{T}_{\mathbf{R}}(\hat{O}_{\mathcal{R}}\mathbf{H}_{\mathbf{k}n}) = \hat{O}_{\mathcal{R}}(\hat{T}_{\mathcal{R}^{-1}\mathbf{R}}\mathbf{H}_{\mathbf{k}n})$$
$$= \hat{O}_{\mathcal{R}}(e^{-i(\mathbf{k}\cdot\mathcal{R}^{-1}\mathbf{R})}\mathbf{H}_{\mathbf{k}n})$$
$$= e^{-i(\mathbf{k}\cdot\mathcal{R}^{-1}\mathbf{R})}(\hat{O}_{\mathcal{R}}\mathbf{H}_{\mathbf{k}n})$$
$$= e^{-i(\mathcal{R}\mathbf{k}\cdot\mathbf{R})}\hat{O}_{\mathcal{R}}\mathbf{H}_{\mathbf{k}n}.$$

Since $\hat{O}_{\mathcal{R}}\mathbf{H}_{\mathbf{k}n}$ is the Bloch state with wave vector $\mathcal{R}\mathbf{k}$ and has the same eigenvalue as $\mathbf{H}_{\mathbf{k}n}$, it follows that

$$\omega_n(\mathcal{R}\mathbf{k}) = \omega_n(\mathbf{k}).$$

- **Frequency Band Redundancies:**
  - The conclusion is that when there is rotational symmetry in the lattice, the frequency bands $\omega_n(\mathbf{k})$ exhibit additional redundancies within the Brillouin zone.

- **General Symmetry Operations:**
  - For photonic crystals with rotation, mirror reflection, or inversion symmetry, the functions $\omega_n(k)$ also exhibit that symmetry. This set of symmetry operations (rotations, reflections, and inversions) is termed 'point group' of the crystal.

IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

So, you can calculate the eigenvalue. So, you can take $\hat{T}_R$, which is a translational operator, and apply it to this operator, okay? So, you can write the rotational operator, and then you can write $\left(\hat{T}_{\mathcal{R}} \quad \mathbf{H}_{\mathbf{k}n}\right)$, okay? Finally, by doing these steps, you can say $e^{-i(\mathbf{k}\cdot\mathcal{R}}$, and then you specifically get this operator. So, what you see from here is that this is the rotational operator acting on this particular field, okay? It gives you the block state with the wave vector $\mathbf{k}$, and it has the same eigenvalue $\mathbf{H}_{\mathbf{k}n}$.

So, it basically follows this form: $\omega_n(\mathcal{R}$ is essentially $\omega_n(\mathbf{k})$. The corrected sentence is: So the conclusion here is that when there is rotational symmetry in the lattice, the frequency band exhibits additional redundancies within the Brillouin zone. So, this will be very important when analyzing photonic crystals, where we will be using rotation and mirror reflection. Inversion symmetry, and you will see that the function $\omega_n(\mathbf{k})$ exhibits symmetry. This set of symmetry operations, such as rotation, reflection, and inversion, is termed the point group of the crystal.

Rotational Symmetry

- **Symmetry of Frequency Functions:**
  - The functions $\omega_n(\mathbf{k})$ exhibit the full symmetry of the point group of the crystal.
  - Due to the symmetry, it is unnecessary to consider $\omega_n(\mathbf{k})$ at every k point in the entire Brillouin zone.
  - The smallest region within the Brillouin zone where $\omega_n(\mathbf{k})$ functions are not related by symmetry is termed the irreducible Brillouin zone.
  - For a photonic crystal with the symmetry of a simple square lattice, the Brillouin zone is a square centred at $\mathbf{k} = 0$.
  - The irreducible zone in this example is a triangular wedge, covering 1/8 of the area of the full Brillouin zone.
  - The remaining portions of the Brillouin zone consist of redundant copies of the irreducible zone.

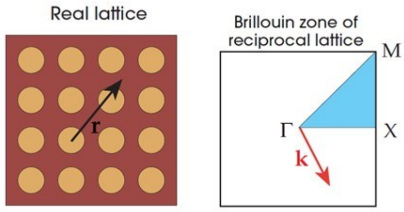


Figure: Left: A photonic crystal made using a square lattice. An arbitrary vector r is shown. Right: The Brillouin zone of the square lattice, centred at the origin (Γ). An arbitrary wave vector **k** is shown.

IIT Guwahati | NPTEL swayam Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

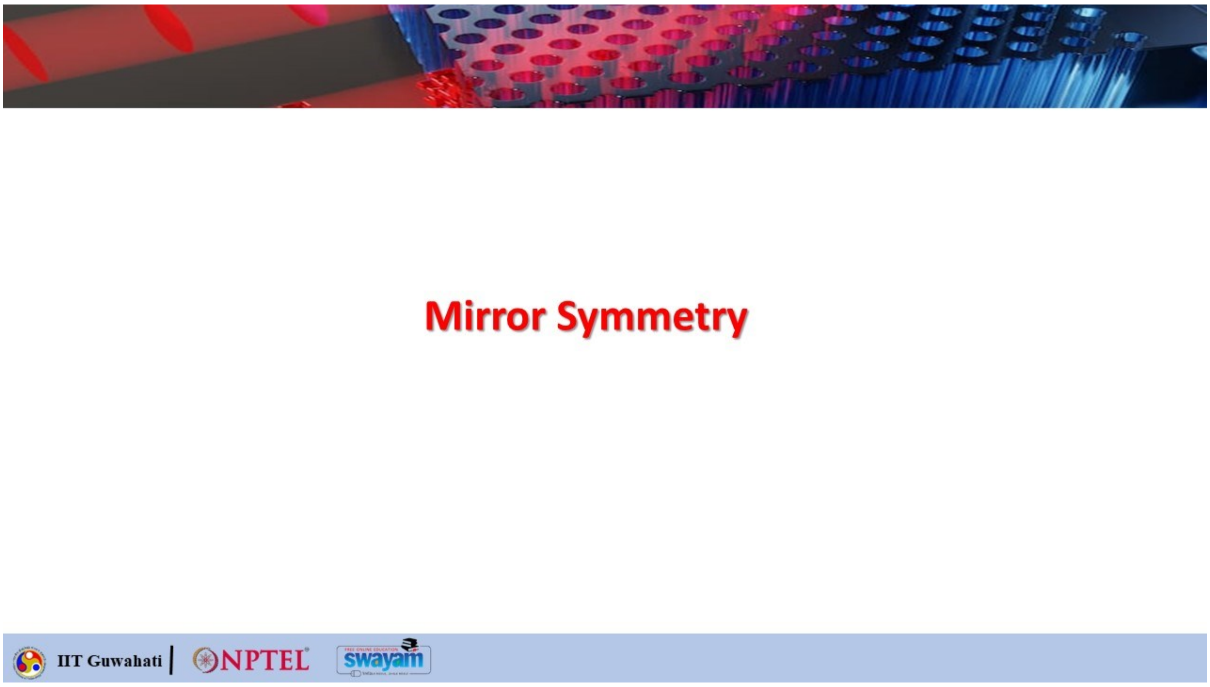
Now, here you see a real lattice, okay? (The sentence is already grammatically correct.) And this is a photonic crystal where you have a square lattice, correct? And this is the Brillouin zone, okay? Which has an origin marked as gamma, okay? The two other important points marked as X and M will come to this one. This is the brilliant zone of the reciprocal lattice of this particular real lattice. So, we will discuss the real and reciprocal lattice in the next lecture, okay? So, here we will look into a couple of important factors regarding what is called the irreducible Brillouin zone. But this triangular wedge can actually recreate the entire Brillouin zone by using all those symmetry operations that we understood.

So, here there are three important points: gamma, m, and x. When you connect the origin of the Brillouin zone, or the center of the Brillouin zone, with the... The midpoint of the sides is x, and when you take it to the corners or edges, that is m.

So here, when you discuss the symmetry of... The frequency function is such that the functions omega and k will exhibit full symmetry of the point group of the crystal, and because of that, you do not need to consider, you know. For every k point in the entire Brillouin zone, you can only look for the smallest region within the Brillouin zone where omega and k values are not related by symmetry. Here, all these omega and k values will be unique, and you can only consider this particular range because once you know the information here, you can.

.. Use the symmetry to recreate your entire reciprocal lattice Brillouin zone. So, for a photonic crystal with the symmetry of a simple square lattice, you can see that the Brillouin zone is square. So, this is the brilliant zone, and it has a center marked as gamma. Here you can see that the irreducible brilliant zone is essentially only one-eighth of the area of the full brilliant zone. So, this remaining portion is basically redundant copies of this irreducible Brillouin zone.

So, how does it help? (The sentence is already grammatically correct.) So, when you want to compute the band structure of your photonic crystal, if you are able to identify the irreducible Brillouin zone, And only compute the band structure for this one, as you know the remaining Brillouin zones are basically redundant copies. So, you can use those symmetries to recreate the band gap for the entire crystal or the Brillouin zone



Mirror Symmetry and the Separation of Modes

- **Mirror Reflection Symmetry:**
  - Mirror reflection symmetry in a photonic crystal is notable for its ability to separate the eigenvalue equation for $\hat{\Theta}_k$ into two distinct equations, each corresponding to a specific field polarization.
- **Polarization Separation:**
  - Under mirror reflection symmetry, conditions arise allowing the separation of modes, where in one case $\mathbf{H}_k$ is perpendicular to the mirror plane and $\mathbf{E}_k$ is parallel, and in the other case, $\mathbf{H}_k$ is in the plane, and $\mathbf{E}_k$ is perpendicular.
- **Convenience and Information:**
  - This simplification is advantageous as it provides immediate information about the mode symmetries and facilitates numerical calculations of their frequencies.

IIT Guwahati | NPTEL swayam Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

. So, now let us look at another important symmetry, which is mirror symmetry. So, mirror reflection symmetry in a photonic crystal is notable for its ability to separate the eigenvalue for theta cap into two distinct equations, each for its particular polarization.

So, we can do polarization separation, something like, you know, under mirror reflection symmetry. The conditions that arise will allow it. The separation of modes occurs in two cases: in one case, $\mathbf{H}_k$ is basically perpendicular to the mirror plane, while $\mathbf{E}_k$ is parallel; in the other case, it is reversed, with $\mathbf{H}_k$ in the plane and $\mathbf{E}_k$ perpendicular. This simple approach, or you could say this simplification, is advantageous as it provides immediate information about the mode symmetries and facilitates the numerical calculation of the frequency. So, if you consider this system again, which is basically a dielectric configuration with discrete translational symmetry, okay? So, what is this structure? This is basically a notched dielectric, isn't it? So this is invariant under mirror reflection along the YZ plane. So, if you cut it like this, and if you place a mirror along the YZ plane, you can actually see mirror symmetry in the XZ plane.

Mirror Symmetry and the Separation of Modes

- **Illustrative Dielectric System:**
  - Considering a dielectric system, such as the notched dielectric in figure , which is invariant under mirror reflections in the yz and xz planes.
- **Mirror Reflection Operator $\hat{\mathbf{O}}_{M_x}$:**
  - We define a mirror reflection operator $\hat{O}_{M_x}$ corresponding to reflections in the yz plane.
  - It reflects a vector field using M_x on both its input and output: $\hat{O}_{M_x}\mathbf{f}(\mathbf{r}) = M_x\mathbf{f}(M_x\mathbf{r})$.
- **Eigenvalues of $\hat{\mathbf{O}}_{\mathbf{M_x}}$:**
  - The possible eigenvalues of $\hat{O}_{M_x}$ are +1 and -1 , representing restoration to the original state after two applications of the mirror reflection operator.

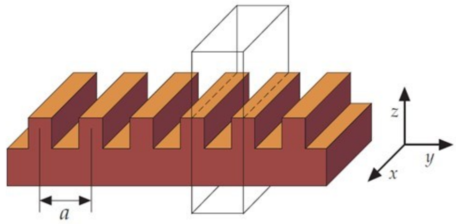


Figure : A dielectric configuration with discrete translational symmetry.

IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

So, you can name an operator like $\hat{\mathbf{O}}_{M_x}$, which is the mirror reflection operator. So, once you define the operator corresponding to reflections in the y-z plane. So, what you can write is that this particular operator will reflect the vector field using mx on both its input and output.

Therefore, you can $\hat{O}_{M_x}\mathbf{f}(\mathbf{r})$, so $\mathbf{f}(\mathbf{r})$ is the... The vector field reflects the vector field, so you have $M_x\mathbf{f}\left(M_x\mathbf{r}\right)$okay? So, how do you find the eigenvalue? So, you can see that this can be either plus one or minus one. So, the two possible eigenvalues of this mirror reflection operator can be +1 or -1. So, that represents the restoration of the original state after two reflections of the mirror reflection operator, right? So, due to the system's symmetry, $\hat{\mathbf{O}}_{M_x}$ will commute with theta cap. So, you can write this as equals 0, okay? So, now if you operate on HK with the commutator, which is demonstrated as you know, $\hat{O}_{M_x}\mathbf{H}_\mathbf{k}$.

Mirror Symmetry

- **Commutation with $\hat{\Theta}_{\mathbf{k}}$:**
  - Due to the system's symmetry, $\hat{O}_{M_x}$ commutes with $\hat{\Theta}_k$: $[\hat{\Theta}_k, \hat{O}_{M_x}] = 0$.
- **Operator Application to $\mathbf{H}_{\mathbf{k}}$:**
  - Operating on $\mathbf{H}_{\mathbf{k}}$ with the commutator, it is demonstrated that $\hat{O}_{M_x}\mathbf{H}_{\mathbf{k}}$ corresponds to the Bloch state with the reflected wave vector $M_x\mathbf{k}$, expressed as

$$\hat{O}_{M_x}\mathbf{H}_{\mathbf{k}} = e^{i\phi}\mathbf{H}_{M_x\mathbf{k}}. \longrightarrow \quad \text{(Equation L7.2)}$$

where ϕ is arbitrary phase

- **Phase ϕ Consideration:**
  - The arbitrary phase ϕ in this relation does not impose significant restrictions on the reflection properties of $\mathbf{H}_{\mathbf{k}}$ unless $\mathbf{k}$ is aligned in such a way that $M_x\mathbf{k} = \mathbf{k}$.

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

Now, this will correspond to the block mode that will have a reflected wave vector of $M_x\mathbf{k}$. So, it can be written this way. So, $\hat{O}_{M_x}\mathbf{H}_{\mathbf{k}}$ is basically $e^{i\phi}\mathbf{H}_{M_x\mathbf{k}}$. So, what is $e^{i\phi}$? So, not theta, phi. So, here, phi is basically an arbitrary phase. So, this arbitrary phase in this relation does not impose significant restrictions on the reflection properties of $\mathbf{H}_{\mathbf{k}}$ unless k is aligned in such a way that you.

.. Know that $M_x\mathbf{k}$ turns out to be k. So, what happens when $M_x\mathbf{k}$ is essentially k? The equation that you see here becomes an eigenvalue problem, and $\mathbf{H}_{\mathbf{k}}$ must then satisfy the particular condition that this mirror symmetry operator imposes. This should give you plus or minus 1, and this field itself. So, you can write $M_x\mathbf{k}$, and $M_x\mathbf{r}$ is the argument. The electric field $\mathbf{E}_k$ can also follow a similar equation, resulting in both electric and magnetic fields being either even or odd under this mirror symmetry operation.

Mirror Symmetry

- **Eigenvalue Problem under Mirror Reflection Symmetry:**
  - When $M_x\mathbf{k} = \mathbf{k}$, the equation (L7.2) becomes an eigenvalue problem, and $\mathbf{H_k}$ must satisfy $\hat{O}_{M_x}\mathbf{H_k}(\mathbf{r}) = \pm\mathbf{H_k}(\mathbf{r}) = M_x\mathbf{H_k}(M_x\mathbf{r})$.
  - The electric field $\mathbf{E_k}$ also follows a similar equation, resulting in both electric and magnetic fields being either even or odd under the $\hat{O}_{M_x}$ operation.
- **Field Components for Even and Odd Modes:**
  - Since $M_x\mathbf{r} = \mathbf{r}$ within the dielectric, and considering the transformation properties of electric and magnetic fields, the only nonzero components for an $\hat{O}_{M_x}$-even mode are H_x, E_y, and E_z, while $\hat{O}_{M_x}$-odd modes are described by the components E_x, H_y, and H_z.
- **General Conditions for Mode Separation:**
  - In general, the separation of polarizations is only possible under specific conditions, i.e., when M $\mathbf{k}$= $\mathbf{k}$ for a reflection M such that $\left[\hat{\Theta}_k, \hat{O}_M\right] = 0$.

IIT Guwahati | NPTEL swayam Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

So, since m x r equals r within the dielectric, that’s okay. So, considering the transformation properties of the electric and magnetic fields, and that the only non-zero component for the OMX even mode is basically Hx, Ey, and Ez are considered while you examine the odd modes of rotational symmetry; you can see that they can only be described by Ex, Hy, and Ez. So, in general, the separation of polarization is possible under specific conditions. So, this is the condition that you know: $\left[\hat{\Theta}_k, \hat{O}_M\right]$ such that these two will commute. So, you have $\hat{\Theta}_k$ and $\hat{O}_M$, which give you this 0. So, what is the applicability of this to two-dimensional photonic crystals? So, in the case of a two-dimensional photonic crystal, these conditions can always be met.

These crystals are periodic in a plane but uniform along the axis perpendicular to that plane. So, you can consider a symmetry operation to be something like how Z cap can be replaced by minus Z cap. So, if you perform this operation, it is basically a symmetry for any choice of origin in the 2D crystals. We also understood that in two dimensions, or you could say that every two-dimensional photonic crystal can classify its two distinct polarizations. So, it can either be even modes, which are represented by Ex, Ey, and Hz, or it can be odd modes, represented by Hx, Hy, and Ez, okay? So, how do you define the transverse electric (TE) and transverse magnetic (TM) modes in this case? In the case where the electric field is primarily confined to the XY plane, we can refer to it as transverse electric (TE) modes.

Mirror Symmetry

- **Applicability to Two-Dimensional Photonic Crystals:**
  - For two-dimensional photonic crystals, the conditions can always be met.
  - These crystals are periodic in a plane but uniform along an axis perpendicular to that plane.
- **Symmetry Operation $\hat{\mathbf{z}} \rightarrow -\hat{\mathbf{z}}$:**
  - The operation $\hat{\mathbf{z}} \rightarrow -\hat{\mathbf{z}}$ is a symmetry for any origin choice in two-dimensional crystals.
- **Polarization Classification in Two Dimensions:**
  - Every two-dimensional photonic crystal can classify its modes into two distinct polarizations: either (E_x, E_y, H_z) or (H_x, H_y, E_z).
- **Transverse-Electric (TE) and Transverse-Magnetic (TM) Modes:**
  - The former, where the electric field is confined to the xy plane, is called transverse-electric (TE) modes.
  - The latter, where the magnetic field is confined to the xy plane, is called transverse-magnetic (TM) modes.

IIT Guwahati | NPTEL swayam

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

And later, where the magnetic field is confined to the XY plane, you can refer to it as the transverse magnetic mode. So, what is the XY plane? You can go back and see it here, okay? Let us quickly go back to the structure. In the XY plane, you can see this one.

Time Reversal Symmetry

Time Reversal Symmetry

- **Time-Reversal Symmetry and Complex Conjugate:**
  - Time-reversal symmetry, is a globally significant symmetry in the context of electromagnetic systems.
- **Complex Conjugate of Master Equation:**
  - Taking the complex conjugate of the master equation for $\hat{\Theta}$

Master Equation → $\nabla \times \left(\frac{1}{\varepsilon(\mathbf{r})} \nabla \times \mathbf{H}(\mathbf{r}) \right) \Big) = \left(\frac{\omega}{c}\right)^2 \mathbf{H}(\mathbf{r}) \Big)$

  - Note that the eigenvalues are real for lossless materials.
- **Resultant Equation:**
  - The complex conjugate of the master equation, when eigenvalues are real, yields a specific mathematical expression.

$$\left(\hat{\Theta}\mathbf{H}_{\mathbf{k}n}\right)^* = \frac{\omega_n^2(\mathbf{k})}{c^2}\mathbf{H}_{\mathbf{k}n}^*$$
$$\hat{\Theta}\mathbf{H}_{\mathbf{k}n}^* = \frac{\omega_n^2(\mathbf{k})}{c^2}\mathbf{H}_{\mathbf{k}n}^*.$$

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

Okay. So, we will discuss the last important symmetry, which is time reversal symmetry.

So, time reversal symmetry is a globally significant symmetry in the context of electromagnetic systems. So, if you take the complex conjugate of the master equation for theta cap, you will note that the eigenvalues will be real for lossless materials. Now, if you take the complex conjugate of the master equation when the eigenvalues are real, you will get something like this. You are taking the complex conjugate, and the eigenvalues are real. So, they only apply to the function, and through this manipulation, the $\mathbf{H}_{\mathbf{k}n}$ conjugate satisfies the same equation as $\mathbf{H}_{\mathbf{k}n}$ with the same eigenvalue, right? So, you can say that if you write $\mathbf{H}_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{u}_{\mathbf{k}}(\mathbf{r})$, it is evident that the conjugate of $\mathbf{H}_{\mathbf{k}n}^*$ will also form a block state at -k

Time Reversal Symmetry

- **Equation Resulting from Manipulation:**
  - Through the manipulation, $\mathbf{H}^*_{\mathbf{k}n}$ satisfies the same equation as $\mathbf{H}_{\mathbf{k}n}$ with the same eigenvalue.
  - Using equation:

$$\mathbf{H}_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{u}_{\mathbf{k}}(\mathbf{r})$$: it is evident that $\mathbf{H}^*_{\mathbf{k}n}$ is a Bloch state at $-\mathbf{k}$.

  - The consequence of the manipulation is the relationship $\omega_n(\mathbf{k}) = \omega_n(-\mathbf{k})$, which holds for almost all photonic crystals.
  - The frequency bands exhibit inversion symmetry even when the crystal itself does not possess inversion symmetry.
- **Complex Conjugate and Time-Reversal Symmetry:**
  - Taking the complex conjugate of $\mathbf{H}_{\mathbf{k}n}$ is equivalent to reversing the sign of time t in the Maxwell equations:

$$\nabla \cdot \mathbf{H}(\mathbf{r}) = 0, \nabla \cdot [\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})] = 0$$

- **Time-Reversal Symmetry Consequence:**
  - The relation $\omega_n(\mathbf{k}) = \omega_n(-\mathbf{k})$ is considered a consequence of the time-reversal symmetry inherent in the Maxwell equations.

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

So, you can say that the consequence of this manipulation is that this one, $\omega_n(\mathbf{k})$, can be written as $\omega_n(-\mathbf{k})$. And this holds true for all the photonic crystals. So, what do you observe about the frequency bands that exhibit inversion symmetry, even when the crystal itself does not possess inversion symmetry? So, this is something very important that will help us calculate the band structure very quickly. So, taking the complex conjugate of $\mathbf{H}_{\mathbf{k}n}$ is equivalent to reversing the sign of time in Maxwell's equations, okay? So, the equations remain as they are, and this particular relation between omega and k can be written as omega n minus k. This can be considered as the time-reversal symmetry inherent in Maxwell's equations, right? So, that is a consequence of time reversal symmetry.

	Quantum Mechanics	Electrodynamics
Discrete translational symmetry	$V(\mathbf{r}) = V(\mathbf{r}+\mathbf{R})$	$\varepsilon(\mathbf{r}) = \varepsilon(\mathbf{r}+\mathbf{R})$
Commutation relationships	$[\hat{H}, \hat{T}_{\mathbf{R}}] = 0$	$[\hat{\Theta}, \hat{T}_{\mathbf{R}}] = 0$
Bloch's theorem	$\Psi_{\mathbf{k}n}(\mathbf{r}) = u_{\mathbf{k}n}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}$	$\mathbf{H}_{\mathbf{k}n}(\mathbf{r}) = \mathbf{u}_{\mathbf{k}n}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}$

Figure : Quantum mechanics vs. electrodynamics in periodic systems.

Source: J. D. Joannopoulos *et al.*, Photonic crystals. Molding the flow of light, Princeton University Press, 2008.

Now, here you can see a comparison between quantum mechanics and electrodynamics. So when we talk about discrete translational symmetry, you can see how it occurs in periodic potential in quantum mechanics and in photonics. You can say that you can write $\varepsilon(\mathbf{r}) = \varepsilon(\mathbf{r}+\mathbf{R})$ for crystals. This is the commutation relation for Hermitian operators that commute with the translational operator.

Here, the Maxwell operator commutes with the translational operator. And this is the block theorem. So what you see here is a comparison of the system containing an electron that propagates in a periodic potential. The system considered in quantum mechanics is then compared with the system of electromagnetic modes in a photonic crystal. So, in both cases, as you can see here, the systems have translational symmetry. In quantum mechanics, the potential $V(\mathbf{r})$ is periodic, and in the case of electromagnetism, the dielectric function $\varepsilon(\mathbf{r})$ is periodic.

This periodicity implies that the discrete translational operator commutes with the main differential operator of the problem. Whether it is the Hamiltonian in one case or Maxwell's operator in the other. We can index the eigenstates as $\Psi_{\mathrm{k}n}(\mathbf{r})$, or you can write $\mathbf{H}_{\mathbf{k}n}$ using the translation operator eigenvalues. This can be expressed in terms of the wave factors and bands in the Brillouin zone. So, all of the eigenstates can be expressed in block form, which is essentially a periodic function modulated by a plane wave, right? So, the field can propagate through the crystal in a coherent manner, such as a block wave, and this enhances our understanding of block waves.

Electrons explained one of the greatest mysteries of 19th-century physics. It is like asking why electrons behave like free particles in many examples of conducting crystals. So, in a similar way, a photonic crystal could provide a synthetic medium in which light can propagate. But in ways that are quite different from light propagation in a homogeneous medium. So, that is why you can think of the similarity shown between quantum mechanics and electrodynamics in a periodic medium here.

So, with that, we will come to an end of this lecture. We have discussed all about the symmetries for the classification of electromagnetic (EM) phenomena. Modes, and if you have any queries about this particular lecture, you can drop an email to this email address. Thank you. (The sentence is already correct.)