Electromagnetic Theory
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Lecture - 15
Divergence & Poission's & Laplace's Equation

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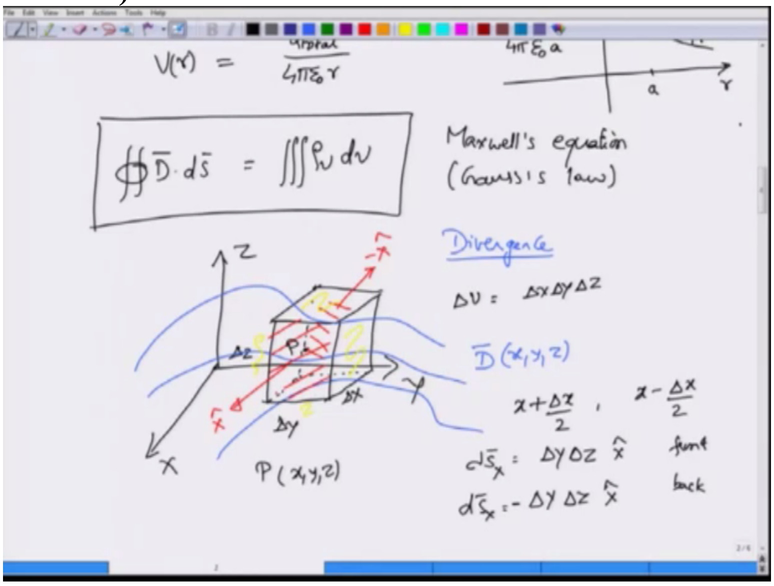

We will now start building up to what is called as point form of Gauss's law. We have already seen that Gauss's law in its integral form that is integral of D over a closed surface, which is equal to the total charge enclosed and in general the charge will be the volume charge density. This integral form is one of Maxwell's equation. This is actually Gauss's law, but in the four Maxwell's equations, this is one of the Maxwell's equation.

So, this is one of the Maxwell's equation. This of course is the Gauss's law, so that we do not forget that this was originally given by Gauss, ok. So, this is integral form. Now, is there any other way of putting the same information in terms of point or a differential volume element in the space? Why is differential volume element is important in the space is because, the differential form of the laws actually tell you that the effects are localized.

So, if there is a charge distribution that is sitting here, the differential form of the law tells you that only the regions nearby that charge distribution get affected. The fields that are generated get affected. Of course, that affection or the disturbance keeps propagating and eventually reaches to the far end. So in some sense integration is a large area effect.

You are going to take an integral over the entire space whatever that is happening you consider the entire space and then find out the totality of the effect, whereas, if you obtain differential form of laws, which involves differential equations then the effects are local. They will of course propagate, ok. So, it is important in another sense that differential forms of laws are frequently employed in numerical solution of electromagnetic problems.

Therefore, getting to know how to go from the integral form to differential form is very important. So, we are going to do that one by considering evaluating this left hand side of Gauss's law over a closed box or a closed differential volume element in Cartesian coordinates because that is kind of simplest to evaluate. Expressions for this differential form in other coordinate systems can be obtained very easily if you look at the text book formulas.

So, what is that I am going to do now? Let me assume that I am working in the Cartesian coordinate system. I want to introduce an important concept called divergence. So this entire point form is building up towards this divergence. If you are curious as to what divergence is just have a little bit of patience you are going get the divergence in a few minutes. So, I have this Cartesian coordinate system. So, I am going to consider a differential volume element.

This volume element has the height delta z, has width delta x and another dimension delta y. So, what is the total differential volume delta V that is delta x, delta y and delta z. Let us also assume that there is a D field, which is varying as a function of x, y and z, so the D field is varying as a function of x, y and z and we are going to consider the variation of D inside this volume in order to evaluate the left hand side, but we will assume that this volume element delta V is very small.

Now, if you see this rectangular volume element that we have considered there are 6 surfaces to this volume. So, there is a front surface over here. There is a corresponding back surface. The front surface is going along the x axis whereas the back surface is pointing the surface normal is pointing along minus x axis. Then, there are additional surfaces, so there is a surface over here.

There is a surface to the right and the left and there is a surface to the top and the bottom. We will solve or we will apply this left hand side of Gauss's law to the front and back surface, the other surfaces will be very easy to evaluate, so we will apply this to the front and back surfaces. This surface first of all is closed, which means that I can apply the left hand side of the Gauss's law. Now, I am going to assume the positions of these front surface and the back surfaces are at x plus delta x by 2 and x minus delta x by 2 where x is any point in space.

So, I am actually going to consider the centre of this point as X, so if I call the centre of the point as P then point P is defined by x, y and z. So, with this point you move delta x by 2 to the front and then you move delta x by 2 to the back erect the 2 planes and these 2 planes will have a width of delta y and the way I have written this delta x, delta y is slightly wrong this actually has to be delta y here and this has to be delta x.

So, this is delta x and this is delta y. So, in the front surface you have delta y and delta z. So, delta y is the width, delta z is the height so the differential surface area in the front surface will be equal to delta y and this will be pointing along the x direction. This will be delta y, delta z along X. The surface to the back side, so this is the front surface, the back side surface will have the differential surface area pointed along minus x, so this will be delta y, delta z along minus x direction.

The corresponding value of the D field in the front surface, we will assume it to be constant and the value of D field can vary with respective y and z. It can also vary with respective x.

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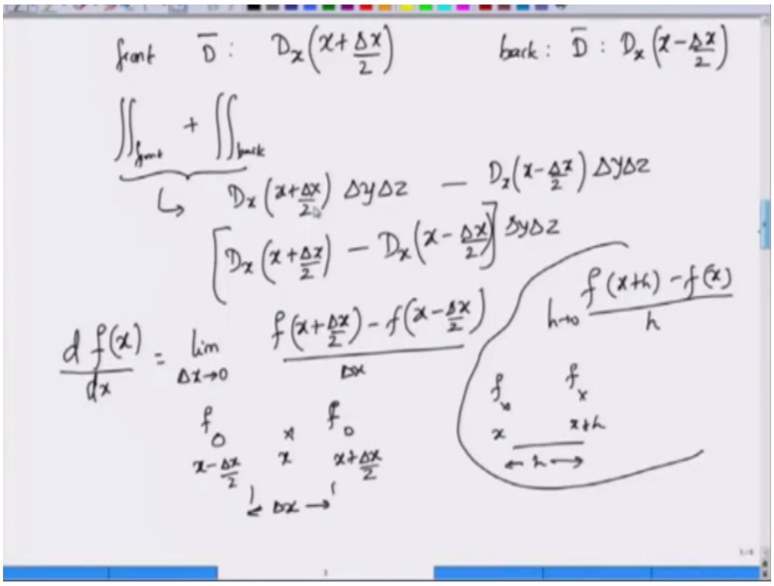

But, we are going to assume that on this surface D is constant and is given by on the front surface the value of D is constant and you are only looking at the x component. Why I am looking at only the x component? Because the front surface of the area, which is directed along x axis. So, if I have D dot DS done over there because surface area is pointing the x axis, D dot DS will also point in the- I mean the only component that will be necessary for that front surface will be the x component.

On that x component, on the front surface which is given by x equal to x plus delta x by 2 will have a value D x of x plus delta x by 2 and this is going to be constant. For the back surface what will be the value of D? D will be whatever the value of the x component of D that is there that is the only component that is of interest to us, but, this time x will be x minus delta x by 2.

What about y and z? Since y and z are constant in the constant x plus delta x by 2 and x minus delta x by 2 planes in the front and back surfaces. There is no requirement for me to right down for y and z. So, as far as the front surface and back surface is concerned y and z are constant. Of course, D is a function of all 3 coordinates x, y and z, but in this calculation on the front surface D x it has to be evaluated x plus delta x by 2.

But y and z will be constant whatever the value of y and z that is there you can put them over here. Now, look at the integration over front and back surfaces. So, if you integrate front and back surface and add the 3 integrals. What are you going to get? So, this integral on the front surface will be equal to Dx of x plus delta x by 2, delta y, delta z, correct. What about the back surface, back surface will be Dx of x minus delta x by 2, which we are assuming to be constant and delta y, delta z.

There is a minus sign here please note that this minus sign because the surface area on the back surface is actually pointing along minus x direction. Now, you can actually simplify, so this is the result of these 2 integrals. You have assumed the Dx of x plus delta x by 2, Dx of x minus delta x by 2 are constant and delta y, delta z themselves are small. So, if I write down this, I am going to get Dx of x plus delta x by 2.

That is the x component of D vector evaluated at x plus delta x by 2, minus the x component of D vector evaluated at x minus delta x by 2 multiplied by delta y and delta z. Now, if you remember, given any function f of x, the way we would define the derivative of that function would be d f by d x was defined as some limit delta tend to 0 f of x plus delta x by 2, minus f of x minus delta x by 2.

You could of course define this as f of x plus h, minus f of x divided by h and then let h go to 0. Sorry, here there is a delta x which is going to 0, but instead of you take two points one at x and one at x plus H and then find out the corresponding values of the function at these two points and divide this one by h. The same thing can be done if you take two points one at x minus delta x by 2 and the other at x plus delta x by 2.

And then find the corresponding values of the function f here and then divide this one by the separation, which is delta x and then let this separation go to 0. So, these two forms are essentially equivalent, you might have seen this form with h, but this is also equivalent to obtaining the derivative of the function f. Just take two points which are spaced some delta x apart find the corresponding values of the function that you are looking at.

And then take the difference between the two divide by delta x and then let delta x go to 0. When you are implementing this on a computer you cannot take delta x go to 0, but you are going to take delta x to be some small non-zero value. Then this becomes the numerical approximation of the derivative. A numerical approximation of derivative will become very important when we deal with how to numerically solve Laplace's equation and other equations that we are going to solve.

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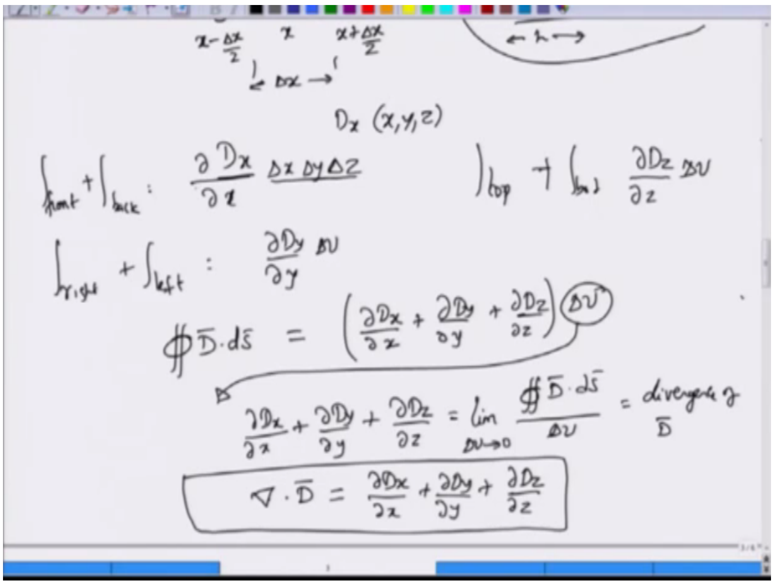

So, now remembering this definition and comparing this to the term that is there in this bracket, you can clearly see that this has to be the derivative of the x component of D. Now this cannot be the ordinary derivative because D x itself could be a function of x, y and z.

Therefore, this ordinary derivative must be replaced by partial derivative with respect to x. So, if you go back to that integration over the front.

And the back surfaces of the rectangular volume element that we have considered, this has to be equal to del Dx by del x multiplied by delta x, delta y and delta z. Why is at delta X, delta y and delta z? Because you can look at this one what we have is only the numerator part. So, if you compare these two parts, you can divide this by delta x and multiply this by delta x and then let delta x goes to 0.

So, we have not left delta x go to 0, not necessary at this point, but this is anyway going to give you the partial derivative. So, this is going to you the partial derivative then it is getting multiplied by the differential volume delta x, delta y and delta z. Similarly, the top and the bottom surfaces are also going to give you partial derivatives, so because the top and the bottom surfaces are oriented along the Z axis, this will give you del Dz by del z.

The volume element delta V still remains and then the right and the left surfaces are also going to give you a term which will be del Dy by del y with multiplied by the volume element delta V. So, in fact this is the result of applying Gauss's law, the left hand side of Gauss's law applying to the rectangular differential volume element that we have considered and this will be equal to del Dx by del x plus del Dy by del y plus del Dz by del z multiplied by the volume element delta V.

Now, what I am going to do is that, I am going to bring this delta V down to the left hand side and then rearrange the equation so that this equation comes to the left so I actually have del Dz by del z is equal to integral of d dot ds divided by delta V. Now, I am going to assume that delta V goes to 0. When I do this, what I have obtained on the left hand side and in fact what I have got on the right hand side is called divergence of D.

And, this is represented by del dot D, the left hand side of this one is represented by del dot D. The del is an operator that we have already seen earlier when we were discussing the gradient. Now, the dot operation will give you the divergence. So, del dot D is equal to del Dx by del x plus del Dy by del y plus del Dz by del z. This is divergence of the field D. If you are not seeing why this has to be you need to recall what the D field represented.

So, you need to recall what that gradient operator del represented.

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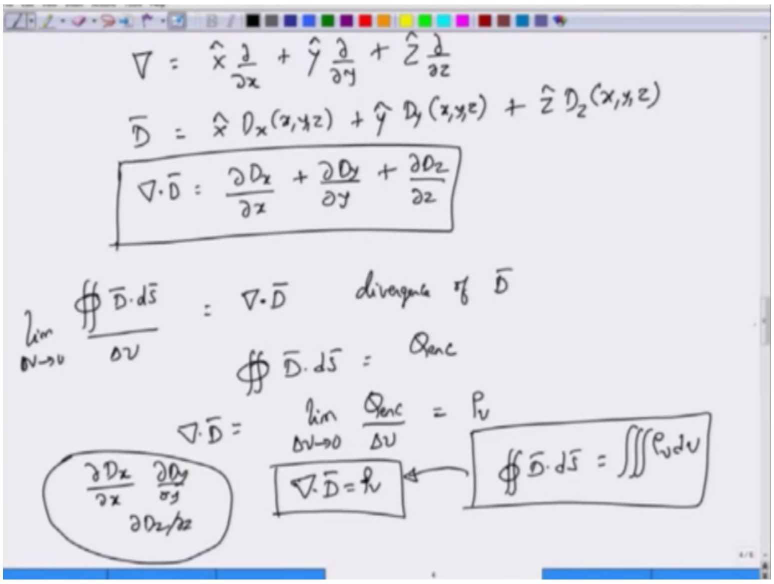

Del was x hat del by del x plus y hat del by del y plus z hat del by del z, this was an operator which we saw in gradient time as well. So, when we were looking at electric field as a gradient of potential. Now D is another vector given by the components Dx which can be a function of x, y and z. D y component can also be a function of x, y and z as well as the D z component that could be a function of x, y and z.

So, if you now take the dot product of these two, you can see that since x hat dot x hat is equal to 1 and x hat dot y hat and x hat dot z hat is equal to 0, this expression simply reduces to the expression that we have written earlier. So, what we have now, we have reduced this integral of D dot ds or we have actually written that integral of D dot ds divided by delta V in the limit of delta V going to 0 as del dot D.

And we say that this is the expression or this is the definition of divergence of vector field D. This is a definition of divergence of the vector field. Now you can transpose this limit of delta V going to 0 on to delta V on to the right hand side, you get this integral of D dot ds will be equal to since we already know that this integral of D dot ds is going to be the total charge enclosed by this differential volume element.

You can replace this D dot ds by the total charge enclosed in that volume element and then you can see that del dot D will be equal to limit of delta V tending to 0 the charge enclosed in that volume element divided by delta V as delta V itself goes to 0. So, it’s like charge enclosing the differential volume is by definition the charge or the volume charge density rho V. So, you can rewrite these two and say del dot D is equal to or rearrange these two.

And write del dot D is equal to rho V and this expression is the equivalent expression for the integral form of Gauss's law. So, this is the integral form of Gauss's law and then by defining a quantity called divergence, we have been able to rewrite this equation in terms of the differential form. Now, will this differential form always exist. What are the conditions that this differential form must exist?

Now, without going too much into the mathematical details, the requirement for del dot D to be defined is that all this partial derivatives, del Dx by del x, del Dy by del y and del Dz by del z must be finite. So, these quantities cannot be infinite. So, that's the requirement for the divergence to exist and once the divergence is there then you can actually find out del dot D which is divergence, which will you give you the volume charge density in the given region.

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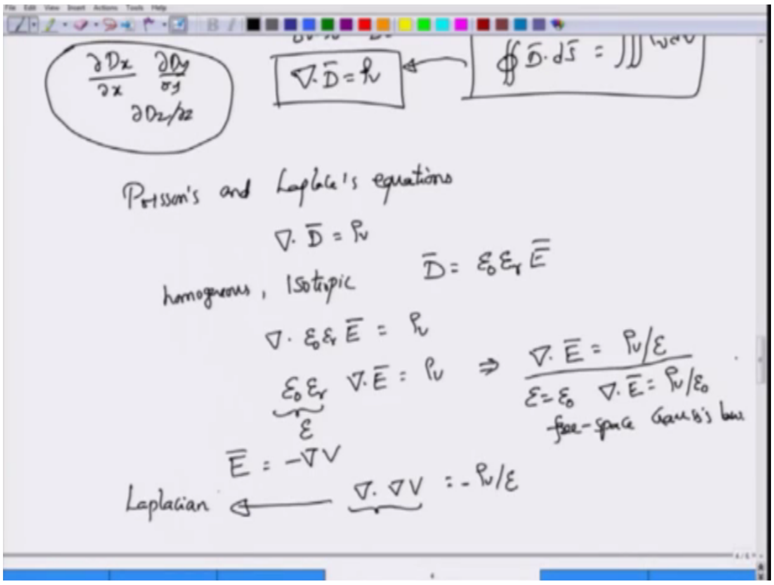

From here we will skip and hop to two of the most important equations in electrostatics and elsewhere which comes up, these equations are called as Poisson's and Laplace's equation. The development of Poisson's and Laplace's equation is fairly simple, if we know the point form, so Poisson's and Laplace's equation. So, we start by writing del dot D is equal to rho V. In a medium, which is homogenous that is to say the material properties does not depend on where you are inside the medium.

So, for example in the vacuum it does not matter where I am standing, where I am positioned because D and E will be related by a simple number. The properties are actually independent of where I am standing in the free space. So, such a medium where the material properties do not change as we go along the medium is called a homogenous medium. Moreover, if the medium properties do not change as you change the direction.

So, for example if the charge is placed here and another charge is placed here, if the interaction between the two remains the same if you switch around or if you turn the positions of the charges. If there is no directional dependence of the results, then the medium is called isotropic. So, if the material properties do not depend on the direction of the applied electric fields then it is called isotropic media.

So, if you consider such a homogenous isotropic and static media that is the medium properties are not varying with time then I can write down the relation between D and E in its general terms as epsilon 0, epsilon r into E. We will say more about this relative permittivity epsilon r later, but for now we can write down D is equal to epsilon 0, epsilon r, E and epsilon r will be just a number.

So, you can substitute that into the point form of Gauss's law and write down this as del dot epsilon 0 epsilon r, E, which will be equal to rho V. Because the medium is homogenous, epsilon r is not a function of x, y, or z coordinates. So, it can come out of the differential. Epsilon 0 is just a number; it can also come out of the integral. So, what is becomes is epsilon 0, epsilon r del dot E is equal to rho v.

Let me call this product epsilon 0 into epsilon r as sum epsilon. So, if I call this as sum epsilon then what I get is del dot E is equal to rho v by epsilon. In fact, this when epsilon is equal to epsilon 0 what you have is the point form for Gauss's law in free space. So, this is free space Gauss's law. So you have del dot E is equal to rho v by epsilon. Now I also know that electric field can be written as minus gradient of V, where V is electrostatic scalar potential.

It is an electrostatic scalar potential and in terms of that I know how to write E. E will be equal to minus gradient of V, substituting this in this equation for del dot E, you have del dot, a minus sign can be taken outside the integrals. So, I have minus del dot gradient of V is equal to rho V by epsilon. Let me remove the minus sign from the left hand side and put the minus sign on to the right hand side.

The quantity that we have written here del dot gradient of V comes up very often in electromagnetics and in other areas that is actually called Laplacian. This is also operator and this operator in Cartesian coordinate systems is very simple.

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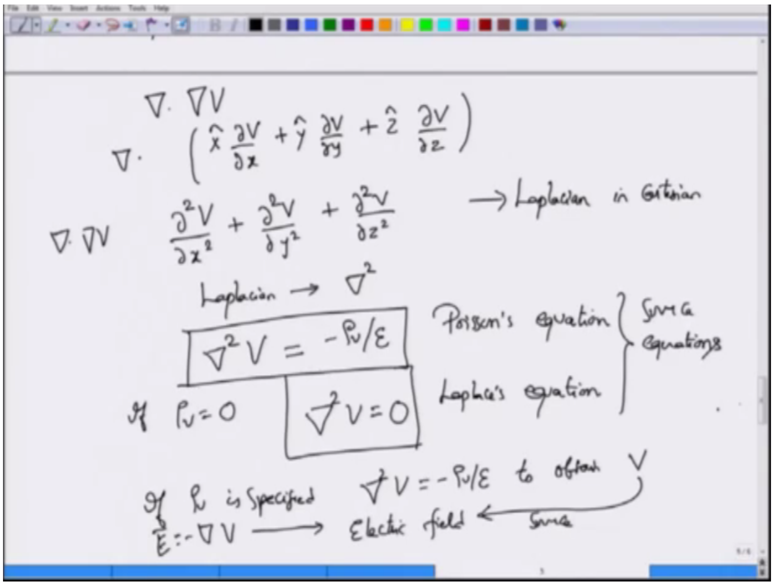

You can actually find out what is the expression for this one. What is gradient of V? Gradient of V is a vector. Remember gradient of V is a vector that will be x hat del V by del x plus y hat del V by del y plus z hat del V by del z. So, this is the gradient of V. If you now operate del dot on to this one what are you going to get? Del operator if you see that is x hat del by del x plus y hat del by del y plus z hat del by del z.

So, when you operate this del on the gradient of V, you are going to get that will be a scalar because dot product will give you scalar. What you are going get is del square V by del x square plus del square V by del y square plus del square V by del z square. This is the expression for Laplacian or the Laplacian operator in Cartesian coordinate systems. In other coordinate systems, you cannot write down like this.

You have to go back to the gradient of V expression then write down the appropriate form of divergence and then derive all that. You can refer to the textbook to find the expressions for this Laplacian's in other coordinate systems, other in cylindrical and spherical coordinate systems. So, this is the left hand side, this is Laplacian and this Laplacian is denoted by a special operator symbol called del square.

So, this Laplacian is denoted by the symbol del square, so in terms of this del square operator or the Laplacian operator the equation that we were working so far becomes del square V is equal to, so you can see this, this was the equation that we were working with and then we have to come up to the stage of del dot gradient of V is equal to minus rho V by epsilon. Now, with del square operator defined in this way, what happens to this is del square V is equal to minus rho V by epsilon.

This is known as Poisson's equation, very important equation, we are going to solve these equations later. Suppose, you are considering the region where there is no free charge that is rho V is equal to 0. If you consider performing this divergence and note taking the gradient operations and applying this point form in a region where the charge density rho V is equal to 0 you end with a simplified equation called Laplace’s equation.

This is called Laplace’s equation. These two equations can be thought of as the source equations. What are the sourcing? These are sometimes called a source equation, so it is important to know what they are sourcing. What they are sourcing is this. If rho V is specified, if rho V the charge density is specified then you can use this equation del square V is equal to minus rho V by epsilon to obtain the scalar potential V.

So, if the charge density everywhere is specified, you can use the Poisson's equation to obtain V, if of course charge density is 0 you can use this equation to obtain the potential V everywhere in the space. Now, from V by applying the gradient operator I can obtain electric field. So, I can obtain electric field from V and in this process you will realize that the source for electric field is actually the charge distribution.

We will be writing similar equations or we will be finding source equations for magnetostatics later and there we will find that the source equation for magnetostatic fields will be currents. The source for electrostatic fields is charges. The source for magnetostatic fields are currents and a time varying magnetic field can be source for time varying electric field.

And time varying electric field can be source for time varying magnetic field that will be the law of electrodynamics. We have Poisson's equation and Laplace's equation. The expressions for Laplacian's also we have written them down and we will be solving these equations later. I wanted to just show you in few steps how to get to Poisson's and Laplace's equation.

We will be solving these equations mostly in two dimensional cases, three dimensional cases are not normally sort. Here is before we break off this lecture, there is one important thing that we have not specified. We know if the charge distribution is given to us, if the charge distribution everywhere is given to us then we can find out from the source equation what is the potential distribution in the space.

From the potential distribution I can find out what is the electric field everywhere. That seems to be a very natural way of specifying the electromagnetic problem. However, in practice that is not the way problems are specified. You consider a simple case of a cathode ray tube. There is a cathode and anode, which are metal plates and then you apply a certain voltage between the two.

You can connect a battery and then you apply a voltage between the two that is all that you can specify or that is all that is known about the problem. If you make some approximation, then you can say ok this is the charge density that is there on the cathode and there is some amount of charge density on the anode. But beyond this you do not know what the charge density is and the region in between the cathode and anode plates.

You do not know what is the charge distribution. If you do not know the charge distribution, how do we obtain the potential everywhere in the space, I know only one potential that potential I know is because I have applied the potential at the two plates. I only know what potential are there in that particular boundary of the problem. This is called as boundary condition.

I know what the potential must be at the boundaries, but I do not know anything about the potential in between. The beauty of this equation is that you can actually solve for the potentials by inverting this equation or by solving this equation. If you just know the value of the potential V at the boundaries and some additional constraints about the charge distribution without knowing the actual charge distribution.

You can actually estimate what is the potential everywhere in space. This is more so in the case of Laplace’s equation, but it is also equally valid for Poisson’s equation. So from just knowing the potentials everywhere, so you have this 2 cathode ray tubes. Now let’s say I bring in 1 more electrode that become some sort of a triode system. So, what I know is what is the voltage on the cathode plate? What is the voltage in the anode plate?

What is the voltage that I am bringing in through an electrode? or I could be putting in a needle. So, I only know what these potentials are from the knowledge of these potentials or potentials at the boundary, I would be able to calculate the potential everywhere and then find out the charge distribution and from the knowledge of potential I will find out the electric field. This of course are not as simple as we are saying.

They are simple only when we consider simplistic scenarios which is what we are going to consider, but for a proper way of solving electromagnetic problems. For example, charge distribution on an antenna or rather a current distribution on an antenna, you will have to use numerical techniques and iterate them. So, there are lot of numerical techniques, we will be seeing some of those numerical techniques in the due course of this lecture.

So, with this we will close Gauss’s law, we will talk couple of things which are sort of unrelated things, you might seem at first, but then we will bring them altogether. We will not introduce anymore laws here. We will simply recap what we have done and from there we will apply that knowledge that we have learned for different systems. So, we will be looking for applications of this and then some elaborations on the topics that we have covered today in the next class.