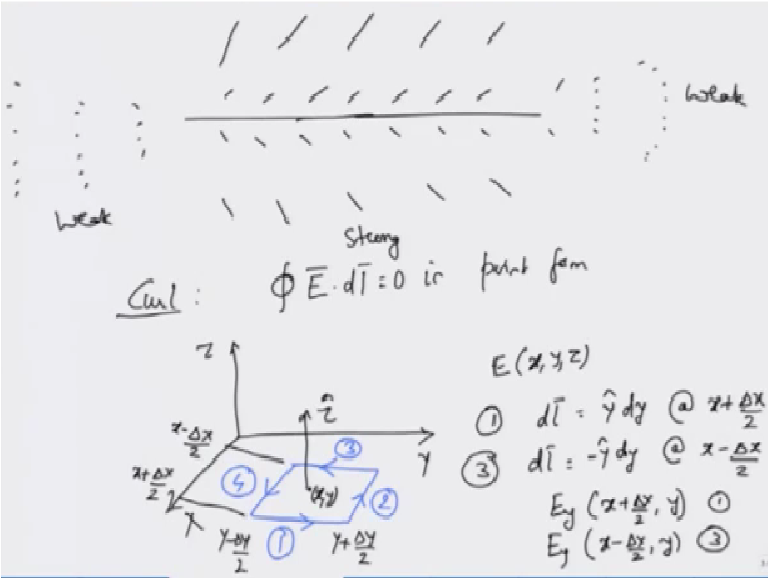
Electromagnetic Theory

Prof. Pradeep Kumar K
Department of Electrical Engineering
Indian Institute of Technology – Kanpur

Lecture - 17
Vector Analysis – III (Curl and its significance)

The next vector operation that we are going discuss is called curl and it allows me to represent or the line integral of the electric field around the closed path in point form oaky. In electrostatics, this is fairly the integral of the E dot dL over a closed path is equal to zero its very trivial because integral of E dot dL over any path will tell me the potential difference and all the disequation is telling is that as long as the electric field remains static okay.

(Refer Slide Time: 00:16)



If you go to a closed path and evaluate the line integral, this line integral will be equal to zero because they are looking at potential difference of one point with respect to the same point okay. Now do we put this in the point form, well to put that in the point form is slightly different from the divergence because curl operation is a vector operation that is it takes vector as input and gives you a vector as output okay.

So it will have three components at least in the Cartesian coordinate system it will have three components x, y and z and have to evaluate the curl all the three components by going to a certain definition oaky. So let us look at a path which lies in the x, y plane, you can image a path that is given by here at this point along this way, the path is oriented in the sense that you need to specify the direction of the path.

Now for any closed path, you have two directions, one direction is as you along clockwise direction or counter clockwise direction and the other direction will be the clockwise direction okay. What we do is we take the right hand rule into account, we say that the path should have its orientation such that if you traverse the path you will be forming a right-handed coordinate system.

So you go from x to y along this counter clockwise direction, the corresponding vector would be along the z axis okay. So this is the path that we consider and we will label this path as path 1, 2, 3 and 4 okay and we will evaluate this line integral over this path okay. So let's get started how to evaluate the line integral. Well before that, we need to know what the coordinates of the path are.

The line integral and the evaluating eventually hoping to get the point form will be at this point x, y okay and I am going to assume that the width of the path and the height of the path is going to be very small okay and the area eventually goes to zero okay. Where is the area vector directed for this path, the area vector is directed along z axis, so which means that I am actually evaluating the z component of this operation curl okay.

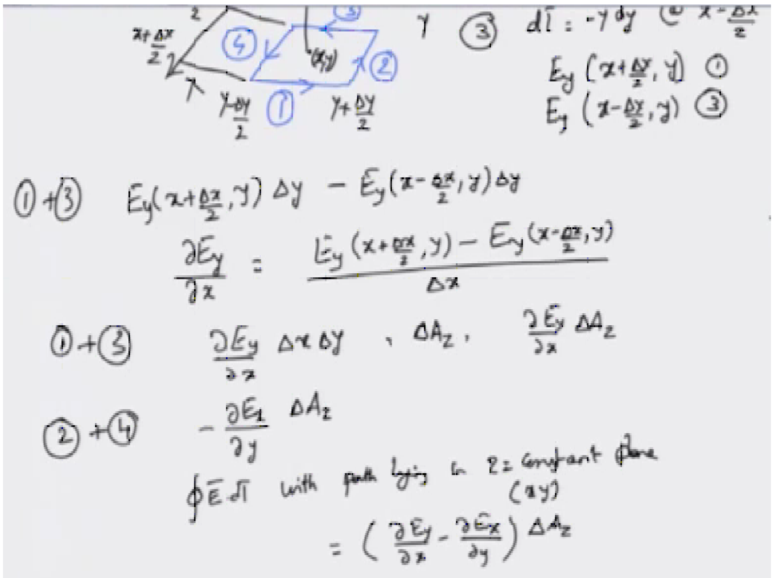
Look at path one, along path one if you start from the left edge here okay and move to the right edge over here along path one what you are seeing is that along this path the y value will be changing; however, the x value will be constant right, if the electric field is a function of all three components at x, y and z along path one okay, which is located at x plus delta x by 2 and this is located at x minus delta x by 2 okay.

So along this path that we have on the path one the value of x is constant; however, the value of y will change from y minus delta y by 2 to y plus delta y by 2 okay. So it will change from y minus delta y by 2 to y plus delta y by 2, we will assume that electric field is going to be constant okay. The electric field is going to be constant over this length; however, the electric field has to be evaluated at point x plus delta x by 2.

So along path one, the direction of the line element is along y axis okay, y at dy, but this is at a fixed value of x, which is x plus delta x by 2. Where x y is the point at the centre of this path and this edge path one will be along x plus delta x by 2. Along path three, if you look at the line element, the line element will be directed along minus y direction. See this is in the direction opposite to path one direction.

So it will be minus y at dy what is the value of x here, it is x minus delta x by 2 okay. On these parts no other component of electric field is required except the component of electric field along y okay. So you want Ey on path one and two okay, at these parts you want to evaluate Ey at one it will be Ey at constant value of x plus delta x by 2 okay, y change and the same Ey at path three will be given by x minus delta x by 2 and it will change along y.
So this is for path one and this is for path three, so if I now evaluate these line integral okay.

(Refer Slide Time: 05:36)



If I evaluate this line integral, the contribution from path one or the path one-line integral will be equal Ey x plus delta x by 2, y okay multiplied by delta y because as you integrate from y minus delta y by 2 to y plus delta y by 2, this will get multiplied by delta y and contribution from path three will be Ey evaluated at x minus delta x by 2 okay along y delta y, once I integrate there will not any y dependence so this is essentially the contribution from path 1 and path 3.

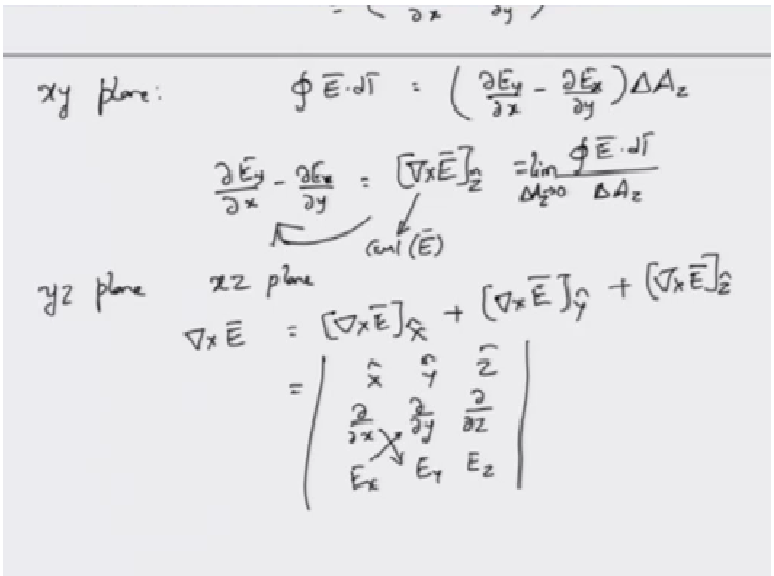
Now you rather suspect that there is some part sort of slope coming up and you will be right, this is the electric field component y at two different points x plus delta x by 2 and x minus delta x by 2 right. There are getting multiplied by delta y, the same multiple around both sides, so if you just recall the expression for partial derivative of E okay, this is the y component of electric field with respect to x, this is precisely equal to Ey x plus delta x by 2 whatever the value of y that is there okay and Ey x minus delta by 2 at the point y divided by delta x.

Delta x because delta x by 2 minus delta x by 2 will be equal to delta x okay. So this is the width of this one. So it is delta x, so you can use this definition and replace this Ey of x plus delta x by 2, y and Ey of x minus delta x by 2, y from this expression and we will obtain the contribution from path one and path 3 can be rewritten as del Ey by del x multiplied delta x delta y okay and if I called this delta x delta y as some delta area along z the contribution from path one and path three is basically del Ey by del x delta z.

Similarly, you can show that the contribution from path two and four will be equal to del Ex by del y with a minus sign if you do that when you will see why there is a minus sign multiplied by the same area delta z. Therefore, the integral of this E dot dl with path lying in the z equal to constant plane, which is x y plane okay, which is the z equal to constant plane is given by del Ey by del x minus del Ex by del y multiplied by the area delta z.

Now what we will do is we do whatever we did very similar to divergence okay. So in divergence, what we did was we obtained the closed surface integral and divided by the volume and let the volume go to zero. In this case, what we will do is we will let the area go to zero okay. We will let the area go to zero, so you can go back and write down for the path. This is x y plane, this integral of E dot dl okay is given by del Ey by del x minus del Ex by del y multiplied by delta area along z okay.

(Refer Slide Time: 08:47)



Now divide both sides of the delta area z and then take that limit of the area to zero, so what we get is del Ey by del x minus del Ex by del y, which is defined as the z component of the quantity curl okay of the electric field E, which is returned as del cross E of z okay. So this del cross E of z which is the z component of the curl of E, this fellow in the bracket is called the curl of E okay.

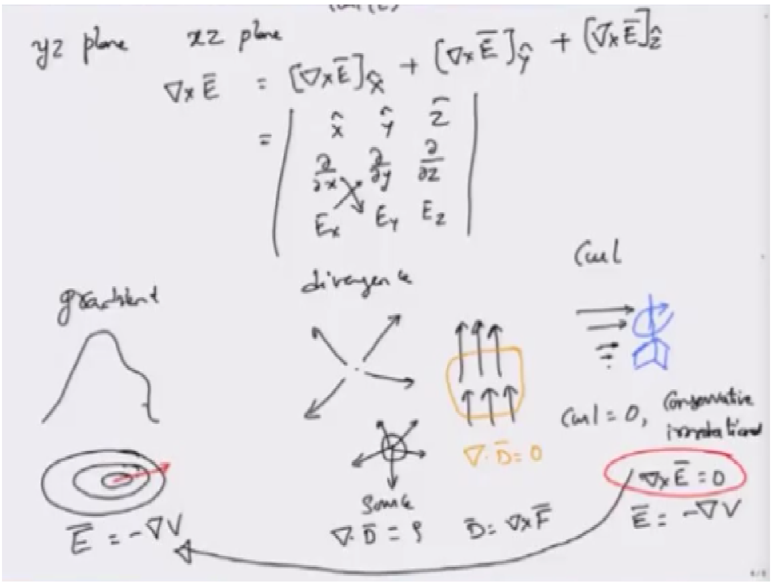
This z component is given by del Ey by del x minus del Ex by del y and operationally it is given by integrating the line integral along the path, which is lying in the x y plane okay. So you need to make this area size go to zero okay. So this is the point form for the curl in the point x y at a given value of z. Of course of you can try pass in y z plane, you can try pass in x z planes as well and will arrive at the result for del cross E, which will have three components.

So there will be a component along x, there will be a component along y and there will be a component along z okay. All three components will be there and we have just evaluated the z component, you can evaluate the x and y component by choosing appropriate paths and there is actually an easier way to remember this curl okay. If you know how to calculate determinants, then this quantity can be obtained by this particular determinant of a 3 by 3 metrics.

Here the partial derivatives okay, note that this row does not make sense unless it operates on some electric field components and you put those electric field components here okay. So if you evaluate the determinant of this, you will get the curl, which will have all three components x, y and z. For example, you can check this right. So I want to find the z component, I have to take for the z component.

I need to look at the determinant here so that would be del Ey by del x minus del Ex by del y just as we have obtained. However, I would like to you to go back and this derivation I wanted to understand. So that you can actually reproduce the required results for curl without having to remember the determinant okay or you can use the determinant as a sanity check that you have obtained the corrected value for curl okay.

(Refer Slide Time: 11:48)



Doing this in other coordinate systems is also possible they are slightly tricky and we will not do that over here, you can refer to the textbook for the corresponding formula okay. We have now introduced you to three different vector operations. One vector operation is the gradient operation, the other vector operation is the divergence operation and the other vector operation is the curl operation.

What are the physical meanings of this operation? The gradient as we remarked would tell you that if you consider contours on a hill and the consider the contours of constant height, we will see that the contours would tell you and if you go along the contours or go along the directions in which the slope changes maximum, you will reach the top of the hill or bottom of the value. So this is essentially the gradient physical example.

So this lines which we have drawn are the contours of constant height and at any point, the gradient actually points to the direction of maximum change of slope. So it is a vector which tells you how the function is changing along a particular direction. Sometimes this is also called as directional derivatives okay and to reach the top of the hill or bottom of the valley you would be wise to point or you would be wise to proceed along the direction in which the gradient is maximum.

So gradient tells you how a function is changing not in one dimension, but in three dimensions okay. So this is the gradient operation and why was gradient operation important for us, if you take this contours as constant potential contours that is the value of the potential is constant along these contours, then the electric field was given by the gradient of this potential contours right.

The magnitude of the electric field would be at any point will be the maximum value of the gradient at that particular point. So you can evaluate gradient at different directions where the direction is maximum that would be the direction for electric field okay. The second operation that we looked at was the divergence operation. This operation tells you that if you take a closed surface okay.

And then there are flux lines that are coming out of than one okay and as you let the surface go to zero, is the value remains constant okay, then this forms a source. In the context of electrostatics, the divergence of the quantity flux density gave us the total charge enclosed or the total charge density at that particular point okay. The final operation that we introduced you to was a curl operation.

In order to understand or appreciate curl, you can imagine a field, which looks like this okay. It could be a water that is moving in a river okay, if the water level is or the velocity is pretty high at the top and as you go down to the bottom, the velocity would be very weak. Now if you imagine that, I have some sort of a paddlewheel here okay. I can imagine a paddle wheel okay; I have been pretty poorly drawing these paddlewheels.

What would happen if on the top of the paddlewheel? There is a larger push because of the water current. However, at the bottom of the paddle, there is not much of a push because the velocity here is small. So what would happen is the paddlewheel would rotate okay. Thus fields which actually rotate are described as having non zero values of curl; however, fields that would not rotate are called as irrotational fields okay.

So when curl is equal to zero such fields are called as conservative fields or irrotational fields okay and it is actually an interesting thing that this curl equal to zero is the condition that is required for us to write down electric field as a gradient of a potential function okay. The fact that curl equal to zero right at any point for electrostatic case, allows me to express electric field as gradient of V okay.

Only when this condition that curl of E is equal to zero, you can actually, or the regions where this curl of E is equal to zero, you can define the scalar potential. Unfortunately, this equation writes this del cross E equal to zero is not valid in the time varying case. In the time varying case, this curl of E will have a non zero value indicating that there is a sort of a source okay for the electric field, which will give the rotation to the electric field okay.

So if you have a field which has no divergence, for example you had a field which we considered which was constant everywhere right and it was completely independent of any of the coordinates. So in this case if you try and consider a closed surface, there will not be any non-zero value of the divergence okay, as a start reducing this one you can see that there is as much flux entering and as much flux as it is leaving.

In fact, this condition that when del dot some vector field is equal to zero, we say that this field is solenoidal or divergence free field okay. So we say that d has no divergence and in fact that will allow me to express this d as a vector as the curl of another vector field f, will in electrostatics have no case to use this particular equation. I am just giving you this as a consequence of vector calculus okay.

The consequence of this del dot d is equal to zero is that I can actually write down d as curl of f okay. Although in this case, we are not going to do that one for the magnetic field case when we discuss we will see that del dot b will be equal to zero and in that case I can use b to define a new vector quantity called as vector potential okay. We will see that once we look at magnetostatic case.

So I hope that the physical meaning of gradient divergence and curl are clear. Gradient tells you the directional dependence of the slope of a function in three dimensions of course we can expand it to any n dimensional cases okay. So that is what the gradient tells you and gradient is related to the electric field because minus gradient of V is the electric field. The minus sign remember is only for our reference which tells us that as we go against the field the potential would rise.

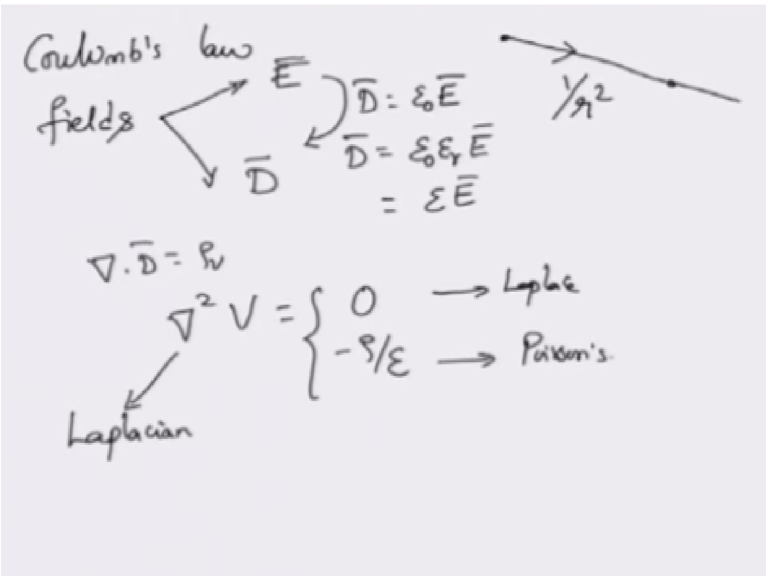
So this is the gradient operation, divergence has two things, when the divergence is non-zero it means that there are some charges at that particular region of space that you are considering and that would act as a source or the sink of the electric flux lines okay and that gives that the relation between the divergence and the charge density at a point is called Gauss's law and is given by del dot d equal to rho.

In case, del dot d is equal to zero then it simply indicates that there is as much lines that are going in as much flux lines that are coming out. However, mathematically it allows me to write down d as curl of some other vector field f. Coming to curl itself is a vector operation which kind of measures the amount of rotation of the field okay. So a simple example of a field which has non zero curl I have shown here.

And if you imagine putting up a paddlewheel you know it is like this kids which would play with these paddlewheels as kids would play with that one. If you put that on wind, if the velocity of the wind is different at different points then the paddlewheel would rotate okay indicating that the wind has or the wind that we are looking at has a non-zero value of curl okay.

And if the curl of a vector quantity is equal to zero then we call that field as conservative or irrotational, it is not rotating and in that case, and in the only case then I can write down the electric field or any vector quantity that I am considering can be written down as the gradient of a scalar quantity. Remember V scalar. So only when del cross E is equal to zero, then it allows me to write down the electric field as gradient of potential okay.

(Refer Slide Time: 20:21)



So let us look at couple of other equations in electrostatics that we have sort of remember that one was the Coulomb's law okay, Coulomb's law tells me how to find out the force on one charge distribution on a test charge okay and we know that this force would act in the direction that would join the tool charge distributions okay, it would act in the line that joins the two charge distributions.

At the same time, the field of a point charge goes as 1 by r square. For Coulomb's law, it was especially applied for point charges and we see that the field would go as 1 by r square. Instead of dealing with Coulomb's law, we have introduced fields okay. In electrostatics, we have introduced two fields, one is electric field E and the other is the flux density D. In the two cases, we related them in free space, D was epsilon zero E.

And in case of a medium, we said D is equal to epsilon zero, some Epsilon r E or in general, simply some epsilon times E. However, we did not clarify what we meant by Epsilon or nor we told you how to obtain this value of Epsilon r for different materials and that is precisely the subject of next class okay. From del dot d equal to rho v, we obtained two more equations, which are very very important.

We will be seeing these equations later when we calculate capacitances, those equations are called as Laplace equation and Poisson's equation. Laplace's equation is del 2 V equal zero and this del square was a vector operator called Laplacian okay. Laplacian of a scalar quantity, it is equal to zero for Laplace's equation, this is Laplace and this one is the Poisson's equation.