**Electromagnetic Theory**
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**Lecture - 27**
**Capacitor – II (Contd.) and Equipotential Surfaces**

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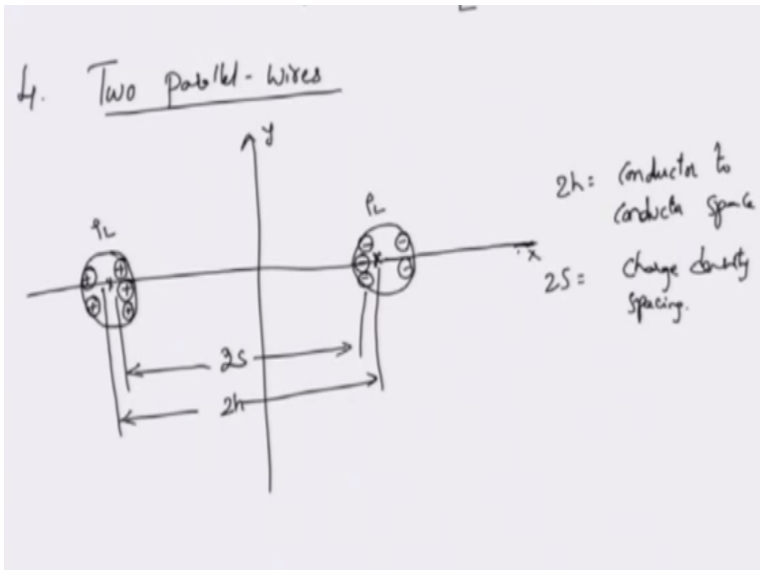


Two parallel wires, as their name suggests, are two parallel wires, which are of certain radius separated by a certain distance. So, let us say that the center to center conductor, these are 2 conductors, the center to center conductor is spaced at 2 h. And you want to find, so let's also call this as the x axis and this is the y axis and you want to find out the potential. Now, this case is slightly interesting because if you apply a potential difference between these 2 cylinders.

What happens is that the charges get concentrated heavily towards the sides that are facing to each other. There would be a heavier concentration or a denser concentration of charges near the surfaces which are facing with each other and there will be a weaker charge distribution at the faraway ends. The faraway ends of the cylinders have a weaker charge distribution. In net what happens is that the charge distribution is not symmetric or uniform.

However, I can restore some semblance of uniformity by considering a line charge not exactly at the center to center spacing but slightly off set from this. Slightly off set from this we will consider them to be the charge distributions. So, if you do that one, sorry this x must be the charge must be kept closer to the, this thing. This center to center and this is the charge. The distributions are at or the line charge distribution is at 2s separation compared to 2s separation of the conductor to conductor spacing.

So, 2h is conductor to conductor spacing, whereas 2s is the charge equivalent charge density to density spacing. The line charge density is still the line charge density. So, we can assume a uniform line charge density Rho L. However, they are not located at the center of the conductors. How do we now proceed?

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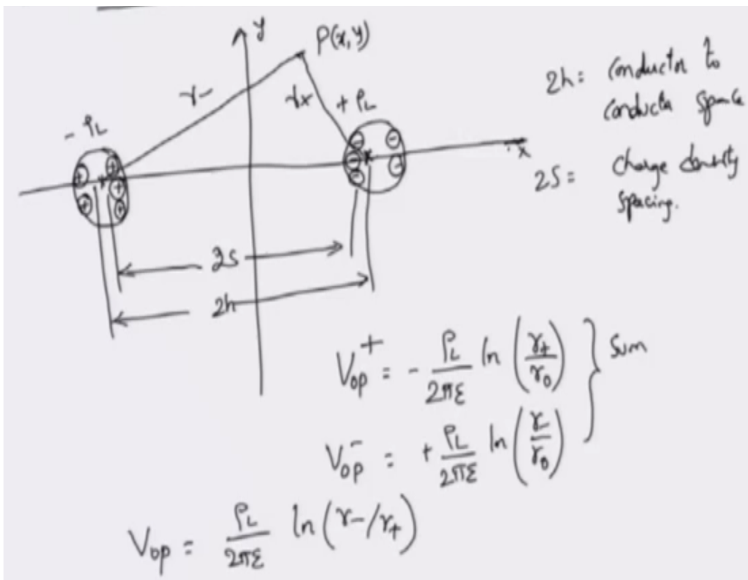


Well to proceed, I need to use the potential. I need to find the potential at any point on the xy plane. Now, there are 2 cylinders. The potential would obviously be because of the superposition of the potentials. Considering only one cylinder at a time, so let's say + Rho L cylinder we will consider at a time, the potential of this with respect to certain reference point or the origin point will actually depend on this distance r + from the point s where the line charge is located to the point P (x, y).

And this potential we already know because of the cylindrical thing, Rho L by 2 phi Epsilon r. We can already calculate this one. This would be equal to rho L by 2 phi epsilon log of r + by r 0. Now, this P stands for potential at point P. Now, there is another potential because of the, so let us call this as V op +, okay, similarly there will be V op - which would be because of the negative line charge density rho L which is at a distance of r – from the point P.

So, this V op – is given by – rho L by 2 phi epsilon log of r - by r 0. The total potential is the sum of these 2. The potential at point P is the sum of these 2 and it is given by rho L by 2 phi epsilon, so when you take log of r + by r 0 – log of r- by r 0 from the log rules, I know that this can be written as r + by r 0 divided by r – by r 0 and this becomes log of r - by r +. So, the potential becomes, the potential here is +, the potential here is -, this is because of the one by r L.

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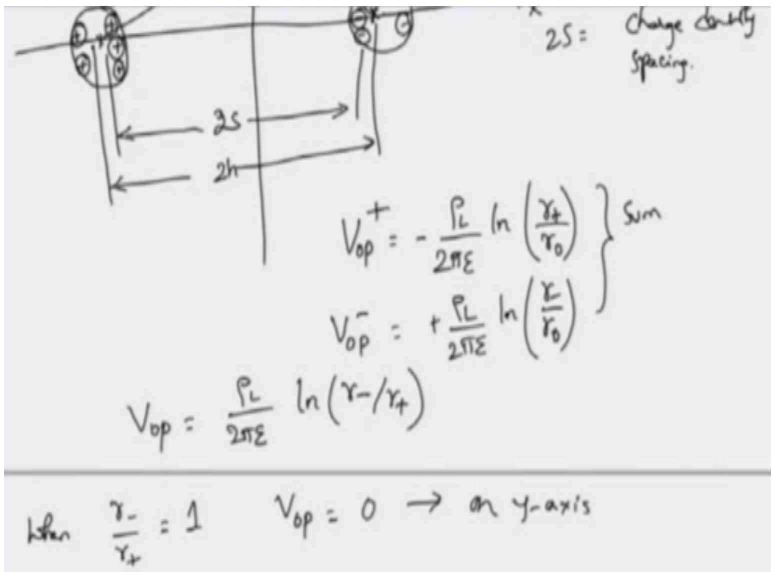


So, the potential due to the P + is actually rho L by 2 phi epsilon log of r 0 by r +, whereas the potential because of the minus charge density is rho L by 2 phi epsilon log of r - by r 0. So, only then when you do this V op the sum of these 2 potential that would be log of r – by r +. Now, what is important to note here apart from my small blunder of – and + confusion is that the potential would be exactly equal to 0, when the ratio r – to r + is equal to 1.

When can r – by r + be equal to 1? It means that all the points, which are equally distant from the positive charge density and the negative charge density, when they are equal, then this ratio will be equal to 1 and the potential V op, the potential at the points, which satisfy this condition will be equal to 0.

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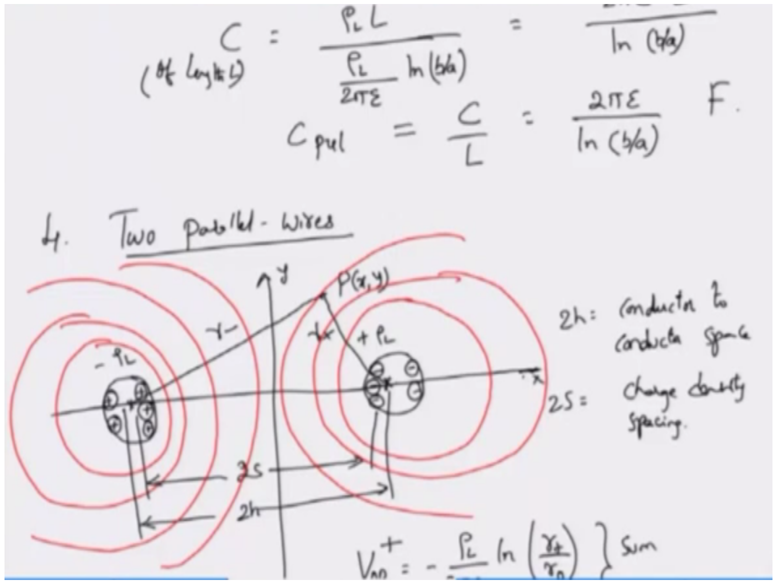


In other words, this happens on the y axis, because all points on the y axis for this case that you can see they would all be there equally distant from minus as well as the, from the negative as well as the positive charge densities. So, because of this when the ratio is equal to 1 the potential will be equal to 0 and y axis plays the 0 potential that we are considering. The right r - will be greater than the r + therefore the potentials will be positive.

To the left – will be small compared to r + and therefore the potential will be negative. So you can see that r - is larger here in the right hand side compared to r + and therefore these potentials will all be positive. So, this potential surface would all be positive. These are the equipotential surfaces that I am drawing. To the left these would all be negative. And you can see that these equipotential surfaces are centers but their centers are not exactly concentric.

They are slightly different, right? So, this is the equipotential surfaces and you can see that if you want to calculate the potential you need to actually calculate the potential to take any point P(x, y) and corresponding mirror image point. So if you calculate the potential at the mirror image point that will required for calculating the capacitance. We will come to that in a minute. So y axis is the zero potential point and the potential is log of r – by r + dependant.

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$V_{op} = \frac{\quad}{2\pi\varepsilon}$

When $\frac{r_-}{r_+} = 1$ $\quad V_{op} = 0 \rightarrow$ on y-axis

Equipotential Surfaces

$$\frac{r_-}{r_+} = k \qquad r_-^2 = (x+s)^2 + y^2$$

$$r_+^2 = (x-s)^2 + y^2$$

$$\frac{r_-^2}{r_+^2} = k^2 = \frac{(x+s)^2+y^2}{(x-s)^2+y^2}$$

$$x^2 + y^2 + s^2 - 2sx\left(\frac{k^2+1}{k^2-1}\right) = 0$$

Add to both $\quad s^2\left(\frac{k^2+1}{k^2-1}\right)^2$

First we will find the equipotential surfaces. To find the equipotential surfaces, I need to look at cases, where this potential is constant, at all points on that particular surface the potential must be constant. Let us denote this r – by r + as some k, where k is a constant and if I start giving different values to k, then I will get different constant potentials or equipotentials, right?

So, this if it is equal to k what it means is, in terms of x and y is that, r –, I already know is given by x + s whole square + y square, r + square that is the distance from the positive line charges is x – s whole square + y square. Therefore, the ratio r – square by r + square which is equal to k square is given by x + s square + y square divided by x - s square + y square. So, you can rearrange this equation, after rearranging you will see that this would be x square + y square + s square – 2 s x (k square + 1 by k square – 1).

Remember k can be both greater than 1 as well as less than 1. So, if it is greater than 1 the potentials are positive; if it is less than 1 the potentials are all negative. So this equation that would be equal to 0 is the equation that relates r – and r + to the constant value k. I can rearrange this equation or sorry, I can complete the square for this equation and to complete the square I need to add and subtract or add to both sides this s square * (k square + 1 by k square – 1) square.

Why because this is looking like x square + s square – 2sx multiplied by some quantity. So, if I take this quantity square times s square, I will actually be able to simplify this equation.

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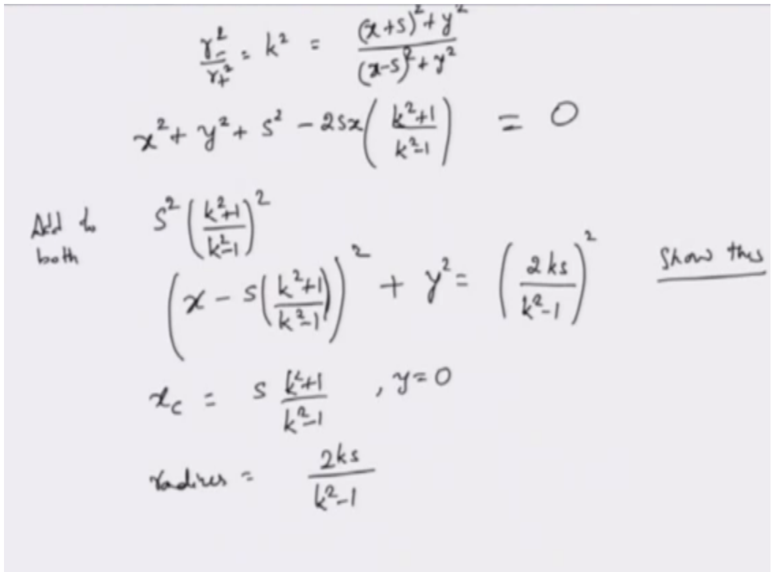


So, if I add to both sides this quantity and simplify the resulting equation, you can see that this would be equal to (x – s (k square + 1 divided by k square – 1)) whole square + y square is equal to (2ks by k square – 1) whole square. You can show that this is valid, so show this. I will leave this as an exercise to you, you can show that this is the case and what you see is that we have obtained the equipotential surfaces.

And those equipotential surfaces have turned out to be circles of center located at s * (k square + 1 by k square – 1) on the x axis and y is equal to 0. This is the center of the circles and their radius is given by 2 k s divided by k square – 1.

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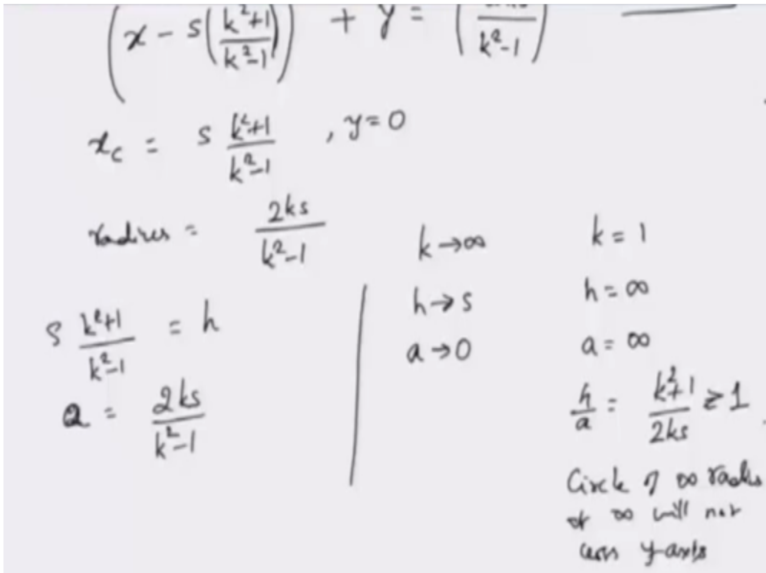


We denote this s * (k square + 1 by k square – 1) as some h. That would be the center of the circles that we are going to consider and the radius a, we will denote this by the radius a as 2ks by k square – 1. Let us look at what happens as k changes. So first consider what happens when k goes to infinity. As k goes to infinity, h goes to s. So, h goes to s why because on the numerator k square + 1 becomes k square, denominator it becomes k square – 1.

Therefore, k square and k square cancel each other; h becomes equal to s, so h turns to s. So, the line charge can be thought of s centering at the origin for k going off to infinity. What happens to the radius? Denominator becomes k square; the numerator is k, so one k gets cancelled. However, radius is going as one by k now so as k goes to infinity, radius goes to 0. So the circle radius keeps on getting smaller and smaller, while the equipotential point or the center of the circle moves towards s.

So this is very important for you to remember that. Consider what happens as k tends to 1 or when k is equal to 1. When k is equal to 1, we immediately find that h will be equal to infinity that is (()) (11:48). So, a is the radius of the circle and we will look at the 2 cases k tends to infinity, h tends to s, a tends to 0. The circle radius goes to 0; the circle radius becomes smaller and smaller and goes to 0, while the center of the circle moves from h to s.

When k is equal to 1, this is an interesting case, when k is equal to 1; h goes to infinity because k square – 1 is going towards 0. So, at k is equal to 1 the denominator is 0, so h becomes infinity and a also becomes infinity. Now, what would be the relationship between h and a? Whether h approaches a faster or approaches infinity faster. In other words, what happens to the ratio of h by a? Whether it would be greater than 1 or it would be less than 1?

You can show that when you actually do this h by a, you can calculate this; this is equal to k square + 1 by 2ks and at k is equal to 1 you can see that the ratio h by a, would actually be greater than 1. So, the ratio of h by a will be greater than 1. What it means to us is that the center of the circle keeps going towards infinity. The center of the circle goes towards infinity, whereas the radius of the circle.

So that is the center and from there the radius of the circle you are looking at and the radius of the circle would be less than the distance from the origin to the center. So, since this distance from the origin to the center is less than the radius of the circle, the circle would never cross the y axis. That is important. So circle of infinite radius at infinity will not cross y axis.

So, the potential to the entire region of the space towards the right side of the y axis would always be at a positive potential and similar argument for the other case, when k goes to, when you consider the negative potential that is 1 by k, you will see that the potential would always be in such a way that the circle of infinite radius would not cross the y axis. So there is no overlapping of the positive and negative equipotential circles.

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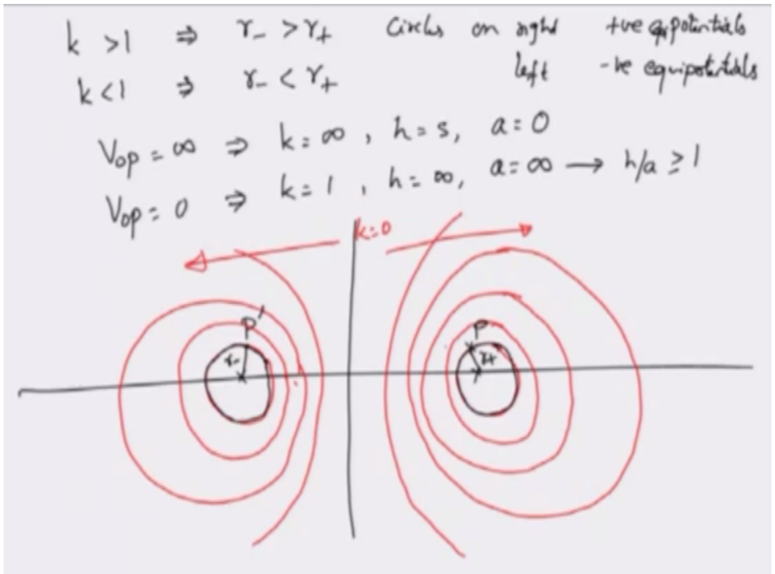


So, with this we are now ready to consider the capacitance structure. So let's summarize that. k greater than 1 implies r – greater than r + and circles will be on the right giving you positive potentials, positive equipotentials and when k is less than 1, this implies that r - is less than r +, the radial distance r – is less than r + and all circles are on to the left and they give you negative equipotentials.

And V op goes to infinity, implies k has gone up to infinity, in that case h is equal to s, a is equal to zero. The center of the circle h has moved to s and the radius of that has become equal to zero. V op equal to zero, which is really the y axis in our picture implies k is equal to one, h is equal to infinity, the center of the circle is towards infinity and the radius of this is also infinity. Now, a circle with an infinite radius is essentially a straight line.

We will see this infinite radius circles later when we discuss bit chart, when we discuss transmission lines. So, however in this last example, h by a will always be greater than one, indicating that the circles would not actually crossover between the two. Now our problem is very simple. Let us begin just by writing the equipotential surfaces. So, I have equipotential surfaces, which are on this axis, x and y axis, located in this way.

I have two equipotential surfaces, one for positive and one for negative. Not two, there are multiple. But one to the right side of the y axis could be all positive equipotentials, to the left could all be negative equipotentials. So, I am going to draw some and beware that my drawing is not very accurate and I am slightly exaggerating all these results to show you the zero potentials and everything.

So, these are the constant equipotential surfaces that we are considering. They are all circles. We can see that the center is moving towards infinity, whereas the radius is moving towards zero. And then I have and you can clearly see why symmetry was not helping us, Gauss law could not be used here, because the equipotential surfaces are not really symmetric. They are not concentric circles. They are all changing their centers depending on what value of k you have.

So, this is the k is equal to zero circle and potential progressively increases over here and potential progressively decreases to the left. Now, what do I do? All I have to do is consider to any of this constant potentials, I can replace those potentials by conducting surfaces. I can do that because conductors are equipotential surfaces. All I have to do is find two conductors, one conductor I put it at whatever potential I want and I will take the conductor and put it at the mirror potential that I want.

So, I take the conductor here, let us say, this is the conductor I want, of a certain radius a and I will consider the equip, you know, equal mirror potential on the right hand side and consider another potential and I get these two conductors now. Now, this becomes my transmission line. This is my two wire, parallel wire transmission line. All I have done is, I have taken these conductors and inserted them in the equipotential surfaces.

And by doing so, I have not changed the electric field. This is very important. A metal can be introduced at an equipotential surface without changing the electric field. That is very critical thing that you have to remember, even later when we discuss transmission lines. So I have taken two circles, which are, I mean I have taken two metals and I have placed them at appropriate radiuses. For example, if the radius is given as a, you take the two equipotential surfaces of radius a, and place the two metal surfaces.

Having done that, I need to still find out what is the potential difference between the two and from there calculate what is the capacitance? Let us say I pick point P over here, which is unfortunately at a very short distance r +, I will also pick a corresponding mirror point, call this as P prime, which would be at r -. This would be at r -.

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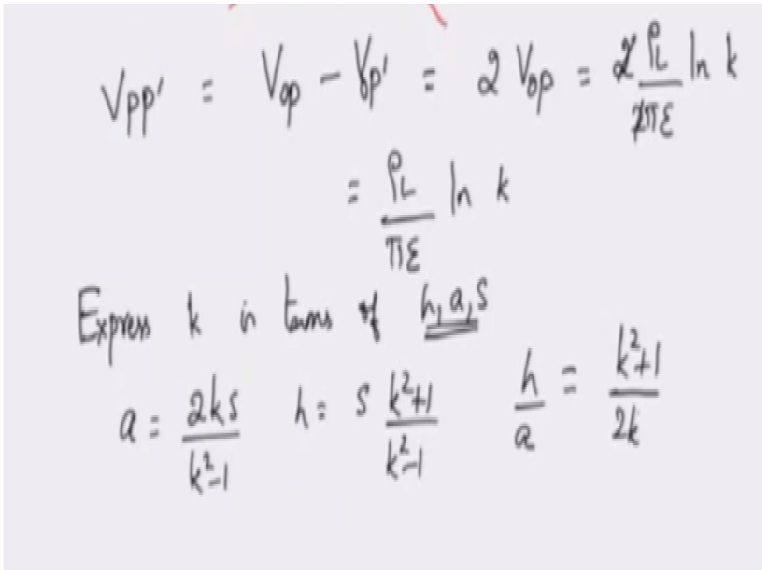


And if I now ask, what is the potential difference between the point P and P minus, as P and P prime, that potential difference will be V pp prime, given by V op, which is the potential of the point P with respect to a certain origin minus potential of point P prime with respect to the origin. Because of symmetry, both P and P prime are at equal, but opposite potentials. Therefore, this must be equal to two times the potential of the point P.

This is at hundred volts and this is at minus hundred volts, the potential difference is two hundred volts, which is two times the potential of one of the conductors. So this is, V pp prime is equal to two times potential V op. But what is this V op? V op, we have already seen, is equal to rho L/ two pi epsilon, log of k. This is something that we have already seen. And two in the numerator cancels with the two in the denominator to give you rho L/pi epsilon, log of k.

But what is k, in terms of s and h. So, express k in terms of the parameters h, a and s. These are important for us because these are the ones, which are determining where we have to place. So, it is good to know what is the value of k in terms of h, a and s. And you can do that very easily, if you recall a, the radius of the equipotential surface as 2ks / k square - 1 and h as s into k square + 1 / k square - 1. You can solve for these two, by finding h/a. h/a is k square + 1 divided by 2k.

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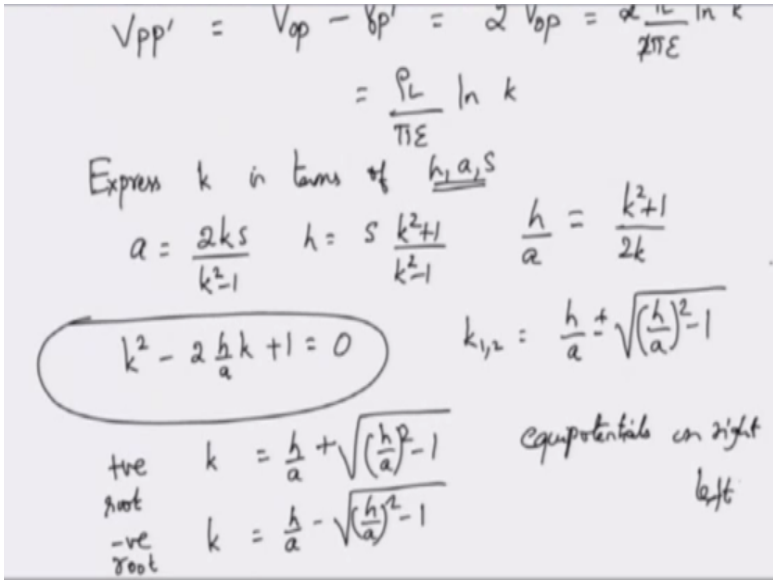


And then, invert this relationship, we will get a quadratic equation in k. So, you get k square - 2h/a * k + 1 is equal to zero. So, this quadratic equation that you obtain will relate k to h and a. The solution of this quadratic equation is, it has two solutions, and it is given by h / a plus or minus square root of h / a square minus 1. This is the solution of this one. And if you choose positive cases, if you choose positive root of the quadratic equation, you get k given by h/a + square root of h/a square - 1.

And this corresponds to the equipotentials on the right side. If you choose negative root, your k will be equal to h/a minus square root of h/a square minus 1. This will give the equipotentials on to the left. So, that will give you the equipotentials on to the left and let us call this as the k and k prime, just to indicate that this a positive root and this a negative root.

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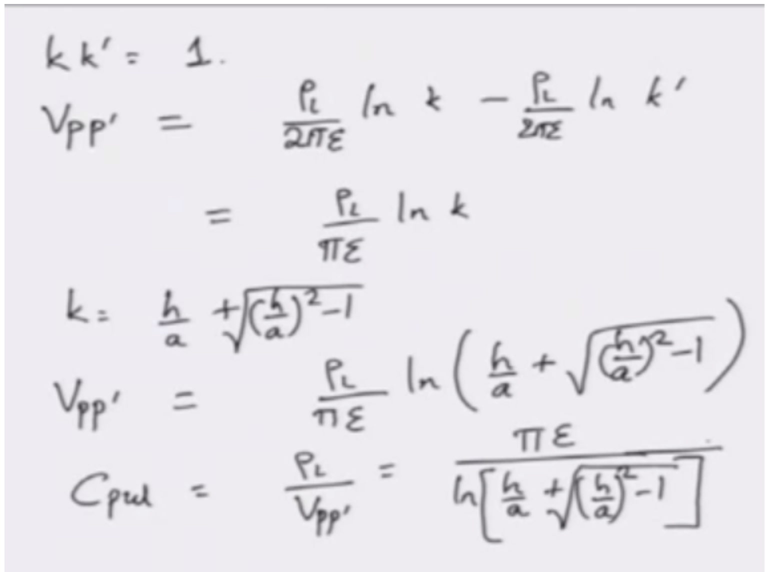


And you can show that k into k prime will actually be equal to one. So, going back to the potential difference V pp prime, the potential difference between the point p on the conductor with a positive charge density and the conductor on the negative charge density point p prime, you see that V pp prime is equal to rho L/2pi epsilon log of k - rho L/2pi epsilon log of k prime. Because for the point p, we have a potential equipotential on to the right that you are calculating and you have to choose the positive potential.

For the left, you have to choose k prime. But, because k and k prime are related in this expression, k is equal to 1/k prime, you can substitute for each other and you will see that this is equal to rho L/pi epsilon log of k, something that we have seen already earlier. Now what is k. k is equal to h/a, because it is a positive root, plus h/a square minus 1 under root. So this is the value of k.

And you get p to the potential difference V pp prime is equal to rho L/pi epsilon ln of or log of h/a plus square root of this quantity. And capacitance per unit length of these two parallel wires is given by the charge density rho L divided by the potential difference V pp prime. And this potential difference is given by so this is rho L, pi epsilon goes to the numerator and I get pi epsilon divided by log of h/a + h/a square - 1 under root. So, this is the expression for the capacitance per unit length.

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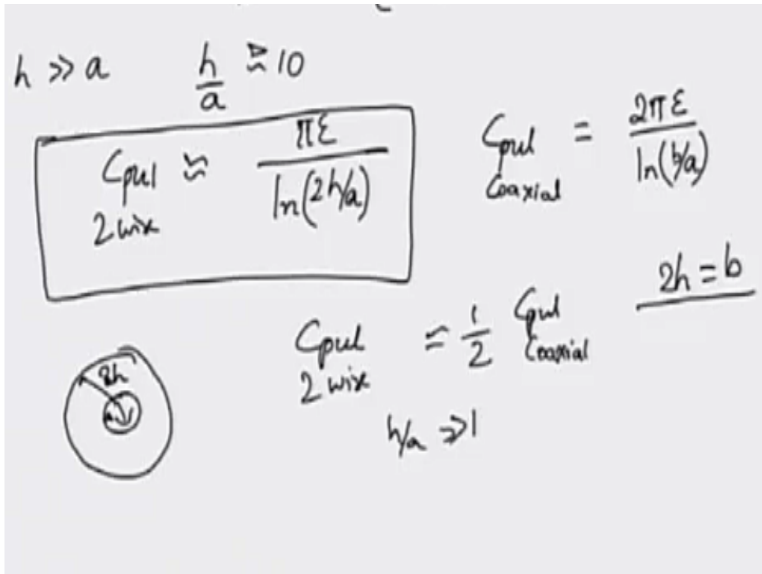


This expression for capacitance per unit length can actually be simplified considerably provided h is much larger than a. It happens that if h/a ratio is around 10 or more, greater than or 10, then the quantity h/a square can be considered, I mean will become very much larger than this one and the root can be cancelled out on that one, root can be cancelled out on the square, so capacitance per unit length can be approximately written as pi epsilon divided by log of 2h/a.

And if you compare this expression for capacitance per unit length with that of the capacitance per unit length of the coaxial cable, for the coaxial cable we had 2pi epsilon divided by log of b/a. If I recall correctly, let us actually go to the top and find out that it is actually the correct expression. So the expression for this one was 2pi epsilon log of b/a, where b was the radius of the outer cylinder and a, was the radius of the inner cylinder.

So, this is the coaxial cable and this is the two wire transmission line and you will see that the capacitance per unit length of the two wire transmission line is approximately half the capacitance per unit length of the coaxial cable, provided you take this 2h as b. So, if you assume that the two wire can be equivalently represented as a coaxial cable of radius a, inner radius a, and an outer radius of 2h.

Then, you can see that the capacitance of the two wire line is approximately half the, it is actually equal to half the capacitance of coaxial line, provided that h/a is much larger than 1. So, this completes our introduction to capacitance and calculation of capacitances. More sophisticated capacitances, which involve multiple dielectrics, we will calculate them, but after we have formally introduced to how to calculate or how to solve Laplace's and Poisson's equation.

So, the next topic that we will be considering in the class will be how to solve Laplace's equation and Poisson's equation. There we will revisit some of the capacitance calculations. However, the primary goal there would be to show you that in the electrostatic case, you can actually solve the Laplace's equation and Poisson's equation and the solutions that you will obtain will all be unique.

And the other thing that I would like to convince you is that the analytical form of the solutions that you will be able to obtain is actually limited to a very small class of problem. So far whatever we have done, we have used lot of closed form expressions, but there is strictly speaking not valid for a large class of electromagnetic problems and solving Laplace's or Poisson's equation also falls in the same category.

However, it would be, it is comforting to note that if you solve Laplace's and Poisson's equation by whatever means, it could be numerical means, it could be graphical means or it could be analytical means, the moment you obtain the solution, which satisfies the boundary conditions and the other considerations of the problem, then that solution will be unique. So you do not have to worry, if you have used a numerical method that whether the solution I obtain will be unique or not, whether there is another solution that I have missed.

It tells out that we will prove theorem called as uniqueness theorem. It will tell us that the solution of Laplace's and Poisson's equation once you have obtained by any means will actually be unique. Of course that unique solution must be satisfying the appropriate boundary conditions.