**Electromagnetic Theory**
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**Lecture - 28**
**Solution of Laplace's equation – I**

So far we have discussed techniques of electrostatic problem solving. Now in this class we will look at solution of Laplace's and Poisson's equations. These pronounce typically classified as boundary value problems, you will soon seen why they are called as boundary value problems. Now we have of course used Laplace's equation earlier when we were calculating the capacitance of certain structures.

Like for the parallel plate capacitor, we have solved Laplace's equation in one dimension and then proceeded to find out the capacitance of that structure. We called it in fact the V method of finding the capacitance of that particular structure. We have also used Laplace's equation in that sense of solving and calculating the capacitance when we considered capacitance of a coaxial cable.

However, at that point we were not really talking anything formally about Laplace's equation whether the solutions that we found where the only possible solutions that we could find or if that is the only possible solution under what conditions will that solution be unique okay. So this question is very important because you might for example have a situation where two of you might work separately and actually get solutions of Laplace's equation with a given boundary conditions.

Now what is a guarantee that these two solutions are going to be different right, if they are same or if they are different, so if they are same then it is good because no matter what method you apply to attack the problem, you will always end up with a unique solution. However, if the solution is not unique there is no guarantee that the solutions obtain with different approaches are unique then there is really no hope of solving such a particular problem right.

So we will look at today a theorem, which tells us the conditions that are necessary to guarantee that the solution of Laplace's equation that we find by different methods. By different method we not only mean by different mathematical methods, there could be method such as experimental methods, you know the ones that we talked about by drawing the field lines and from there you know you can actually experiment and determine the field lines.

And from there try to proceed to find the capacitance of a structure if that is the problem that we are interested in or it could numerical method. So you apply a numerical method and then you find the solution, which fits the boundary conditions very well. It will also satisfy the corresponding equation Laplace's equation or Poisson's equation. So if you solve these problems in two different methods and if the solutions are not going to be unique then the situation is pretty bad for us.

However, if the solutions obtained by different methods are going to be the same when they are applied to the given problem, when it is good because you find solution in one method, which might be easier to evaluate compared to another method. Then you are guaranteed that you have actually solve the problem and you will be confident that there is no other solution that is possible for given scenario, the problem plus the boundary conditions okay.

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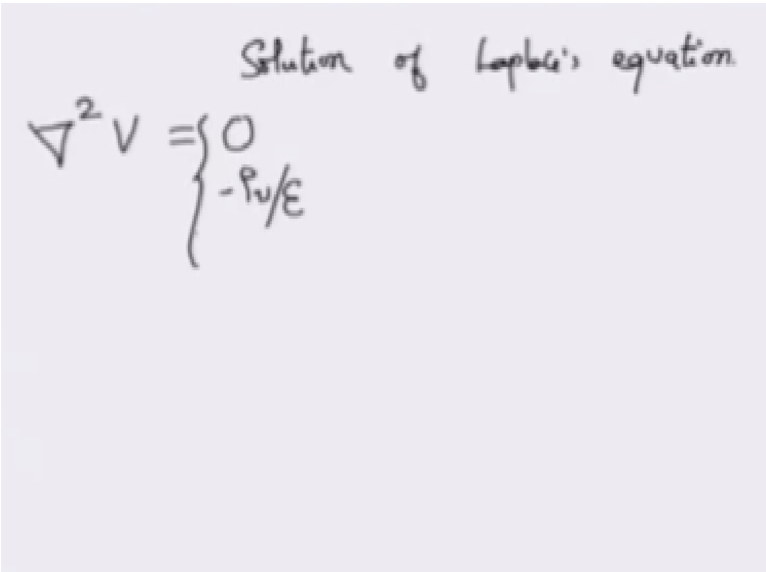


So we begin with Laplace's equation and explore what are the conditions that are necessary for this equation to have a unique solution. So if you recall what was Laplace's equation, Laplace's equation was a partial differential equation. which is del square is in fact called as a Laplacian operator and this operator operates on a scalar function or a scalar field V okay.

And for Laplace's equation, the right hand side is equal to zero and for Poisson's equation the right hand side is actually given by the volume charge density at that particular point divided by epsilon. I am going to assume that the medium that we will be working with is all uniform dielectric and this dielectric that we are going to assume will also be linear, isotropic and homogeneous.

By liner, isotropic and homogeneous, I can, what I mean is that, the dielectric will be linearly responding to the electric field; therefore, it can be characterised by a simple susceptibility parameter okay. By isotropic, I mean that the properties of the dielectric are independent of the direction of the electric fields that we apply. So epsilon is again just a scalar and by homogeneous, we mean that the dielectric properties, that is for example, the polarization would not be a function of the special coordinates.

It will not, for example, if you take a slab of dielectric, the epsilon value here will not be different from the epsilon value at any other point in the slab. So essentially what it allows us to do is that replace those polarizations by the relative permittivity epsilon and as I mentioned to you, epsilon are the relative permittivity has to be multiplied by epsilon zero, the permittivity of free space in order to give you the full permittivity epsilon.

And that is what the notation that I am going to use. Anyway Laplace's equation is much more you know solvable than Poisson's equation or occurs much more naturally than Poisson's equation. So we will be concentrating mostly on Laplace's equation. So we will be hardly be having any use for epsilon that is not given constant, that is not a constant okay.

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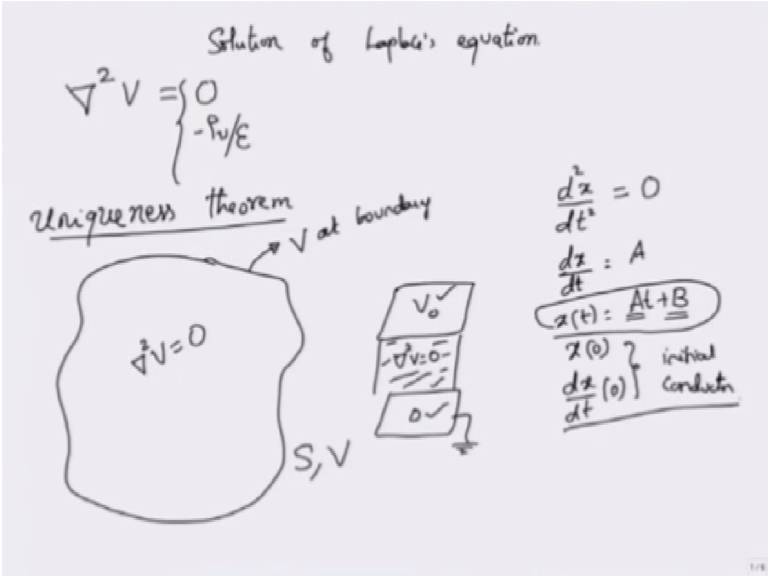


So with that in mind, let us explore what are the conditions that are necessary for this solution that we obtain of the equations to be unique okay. So this solution of the uniqueness of the solutions is actually captured by what is called as uniqueness theorem okay. We will be dealing with this uniqueness theorem in a qualitative way.

We will not be considering the mathematical problems that might really, I mean the real mathematical problems that might arise. But for our purposes the theorem will be developed sufficiently so that you will be confident that if you solve Laplace's equation or Poisson's equation with a given boundary condition then you have obtained only one solution and that solution is going to be unique okay.

So what is uniqueness theorem to develop that consider first a surface S okay, this surface S also bounds a particular volume okay, encloses a particular volume and for the moment let us not put any sources inside okay. So we have no sources, no conductors inside that second part of the theorem that we will be developing will actually have some conductors inside okay. For now, there are no conductors or no charges that are specified inside.

So clearly this is the situation where we can apply Laplace's equation because del square V will be equal to zero throughout this surface or throughout this region, which is characterised by the surface S and the corresponding enclosed volume V okay. Of course, the Laplace's equation is just that, you need to supplement this equation by certain boundary condition. By boundary conditions we mean the values of the potential at the boundary.

So if I specify the value of the potential at the boundary okay. So V at boundary, then I will be able to solve this equation and obtain a solution okay. Otherwise, the solutions that we obtain will be quite general. If you are confused about what this boundary and what the solution that is not, you know that requires the boundary condition. Just consider an analogous equation say d square x by dt square, which could for example represent the acceleration of a particle under the force or under no force.

So if I have this equation, d square x by dt square where x is a function of time. If this is equal to zero then if you integrate this equation once you get dx by dt to be a constant right and you integrate once more you get x of t, which is the displacement for example to be At plus B. Now this is all the solution can tell you.

Now unless you specify the initial conditions like you specify what would be the initial position at time t is equal to zero or at any other convenient time, you also specify what would be the value of dx by dt at time t equal to zero. You will not be able to evaluate these two constants A and B. So to evaluate the constants A and B of this solution, which is a general solution you need the initial conditions okay.

Now in potential problems, what we typically find is okay in a case of a parallel plate capacitor, which we will discuss later, you have a potential of the top surface kept at some V zero potential of the bottom surface is at zero because I am going to ground this one and I am looking at applying Laplace's equation in the region in between right.

So it will eventually reduce itself to a nice differential equation and when you solve the differential equation in the region here right. In the region between the top and the bottom plates, you will end up with two constants and to find the values of those two constants, you need to know the value of the potentials at the boundary okay. Here you were talking about initial condition because the variable was time and it is natural to think of them at time t is equal to zero or time t is equal to infinity as initial and final conditions.

However, in case of problems that are not time dependent per se, but they are dependent on space okay, the variable, the independent variable of this one is space, then it is natural to talk about boundary values rather than initial values. So this is called as an initial value problem. This is called as a boundary value problem. They are not mathematically same because there exist some important differences between the two.

In the initial value problem, you actually are given the values of x of t and the corresponding derivatives okay at a given point in the solution and then your objective would be to find the solution at all later points okay. So you start with one point over here and then you start moving towards the values of that x of t for example at different values of time. So you just start at one point and then you move to the other points trying to find the solution everywhere for time t greater than zero.

On the other hand, in a boundary value problem such as a parallel plate capacitor over here, I know what is the boundary here or the potential value here at this boundary and I know the potential value at this boundary okay and my objective is to find a function that fits into this boundary satisfying the given equation. So there is no marching, you know, you are starting at one point and marching over to the next point.

We actually have a function, which is in between and then you are trying to fit a function that satisfies the boundary condition. So they are not the same but in some of the numerical methods of solutions that we will be taking up you will actually convert a boundary value problem into that of an initial value problem. That is something for later time, so let us not worry about that for now.

The important point of the last few minutes of discussion was that it is necessary in addition to specifying the equation that governs the situation of the problem, it is necessary to specify the corresponding boundary values okay. These problems are called as boundary value problems and since we are looking at electrostatic case these problems are called as boundary value problems in electrostatics.

So with this (()) (11:47) that we have made, let us get back to the original problem. Like we have a region of space that is described by the surface S, which encloses a volume V within that surface we are applying Laplace's equation and we apply the Laplace's equation, we find a solution that solution must now fit into the boundary values that we have specified.

The question we were asking was if I give you two such functions okay or two such solutions, will they be unique or not, that is will they be equal or not, if they are equal at all points in the space including the boundaries than the solution is unique. You cannot have two solutions, which are not equal everywhere including the boundaries, but they satisfy the same set of equations okay. Alright, so to develop that we need some results from vector calculus.

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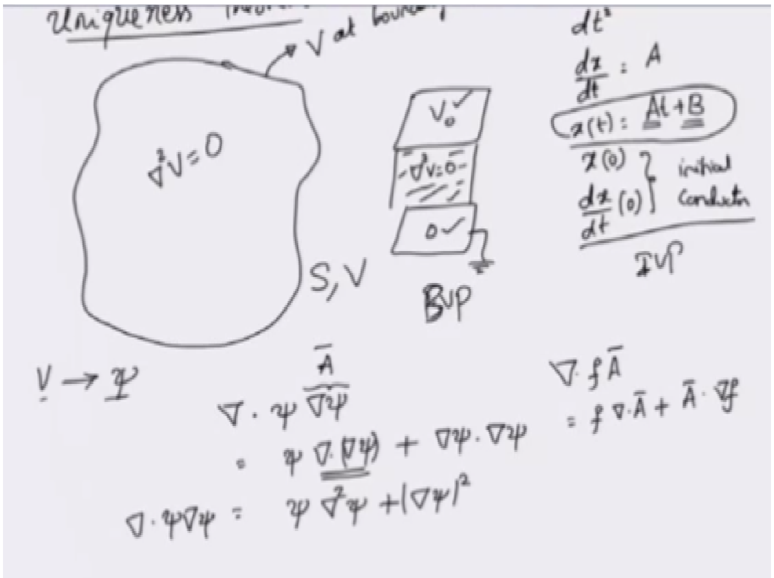


So let us look at that and in the following what I will do is instead of talking about V, I will use a function psi okay. Psi is you know I want to use this one because I want to not that because V is potential and psi is a different thing. Psi is also the same kind of function, but I want to be little more general and therefore I am considering a potential psi.

This generality may not be necessary at this point okay. However, when we talk of uniqueness theorem in the time varying case, we will see that this electrostatic potential needs to be replaced by a different kind of potential okay. So in that case, it is to be little more general. So I am going to switch the notation from V to psi okay and I want to use or I want to get a particular result from vector analysis.

And the result can be obtained by looking at this quantity. So consider divergence of psi gradient of psi okay. We have seen this earlier when we were developing, I believe curl equation or something when we were developing this. So we have seen this vector identity earlier. So if you recall we had a scalar function S and a vector field A okay.

And if you multiply the scalar function with a vector function or a vector field, you still end up being with a vector field and del dot of fA, which is this quantity, divergence of this quantity fA is given by f del dot A plus A dot gradient of f right because gradient of f gives you the vector field when you dot that with A you will get a scalar and everything else is a scalar now. So this is the vector identity that we are going to use in this context.

So you have divergence of psi gradient of psi okay. So this would be obviously equal to psi del dot gradient of psi because the vector field A is gradient of psi for us okay. So psi del dot gradient of psi plus gradient of psi dot gradient of psi. Now from the definition of Laplace's operator, I already know what is this term correct, I know what this term is and that is nothing but del square.

So I have psi del square psi plus del psi magnitude square because A dot A is magnitude of A square right. So this is the quantity that I have and if you look at this quantity you will immediately recognise that this first term has to be equal to zero.

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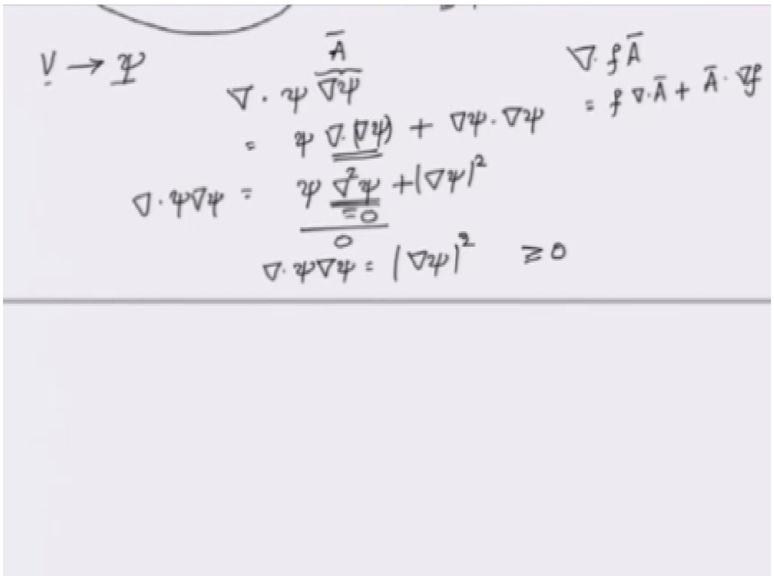

Why because in the surface that I am considering the Laplace's equation tells me that del square of the potential is equal to zero, and since psi is essentially potential that we are considering, this fellow must be equal to zero. So zero multiplied by psi is zero. So this entire term is equal to zero. So what we are left out with is a quantity, which is completely positive and greater than or equal to zero right.

It can utmost be equal to zero but it can never been negative, why cannot been negative, this is magnitude square. Whenever you take magnitude of a particular number that number will always be positive or utmost equal to zero when that number itself is equal to zero okay. Starting from this relation, del dot psi gradient psi is equal to del phi square.

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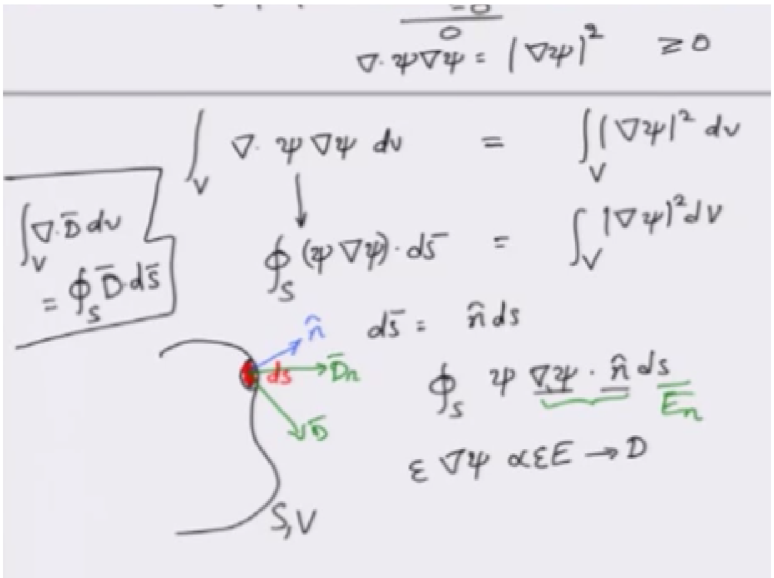

The next step for us would be to integrate over the volume that this region of space is enclosing. So if integrate this over the volume, I get on both sides I have to do this integration of course. So I integrate this one over the volume okay and if I do that on the right hand side, I am integrating this quantity gradient psi square dv over the volume V okay.

Now I know what is divergence theorem, Gauss's divergence theorem, which allows me to convert an integral of this nature into a surface integral right. So I can convert this into a surface integral except that the surface has to be closed. If you are unsure about this recall how we used this relation for developing the point form of Gauss's law, so you had del dot D dv integrated over the volume, this was equal to the charge that is enclosed in that volume.

But this quantity del dot D dv was actually can be replaced by this surface integral of the flux density right. So this was the divergence theorem that we used and using this divergence theorem, but on this quantity I can replace the volume integral of divergence of a quantity by a surface integral of that quantity okay. So this right hand side does not change, it remains the same. So I have del psi square dv.

Now we come to an important part in this theorem okay. If you look at the left hand side, you have some surface area, I am not showing the entire surface. So this is the surface that we were considering and the corresponding volume that this surfaces encloses okay. Ds bar is a vector surface element right. So if you consider on this surface, this small patch, you know, this small patch that I am considering as the differential surface element ds okay.

This surface element has an area of ds, but it would be pointing in the direction that is normal to the contour here. The surface has a certain contouring normal to that contour is the normal to the surface and the vector surface element ds bar is actually given by n hat ds, that is it has a magnitude of ds, the differential surface area of the surface that we are considering okay at that particular point.

But it would be pointing, the vector element would be pointing normally to the surface contour okay and that is given by this n hat direction okay. Now go back to this expression over here, the left hand side has a closed surface integral psi gradient of psi dot n hat ds okay. The dot operation is actually happening between this del psi and n right because that is the vector and this is the vector that you have.

So the dot product is happening between these two elements okay. If you recall what was this del psi is right. So del psi is essentially the electric field, its proportional to electric field, there is a minus sign somewhere, but that is not really important for us. What is important is that this del psi is proportional to E and in a situation where I am considering the material medium properties to be described by a constant epsilon okay.

Then if I multiply this one by epsilon, what I get is epsilon del psi to be epsilon E, which is nothing, but the flux density D okay. Now the flux density D could be coming out in any direction okay. So if this is my flux density D, it could be coming out in any direction on these surface element ds. So then D dot ds will simply tell me what is a normal component of the flux density D okay.

So this is, this del psi dot n is actually the normal component of the flux density that you are looking at of course after multiplying by epsilon. So it is clear that even if it is not dn if the material medium is there then it would be proportional to En. So this del psi dot n could very well be a quantity which is En, the normal component of the electric field at that surface area okay.

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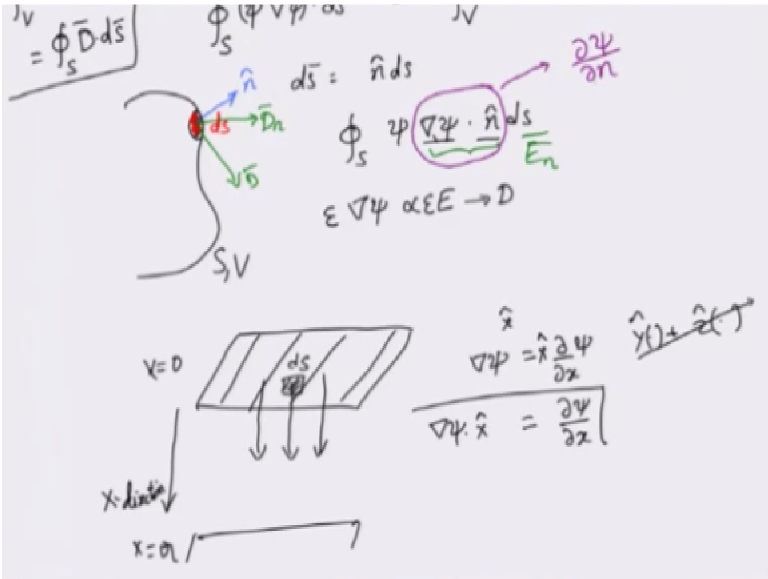

So this is one thing, you just keep this one in mind, we will be requiring it after a certain time. There is another interpretation over here okay. Consider the parallel plate capacitor that we talked about right. So I have a parallel plate capacitor of here. I know that the electric fields would all be uniform assuming that the plates are quite wide in width and length compared to the separation between them.

So the electric fields would all be vertical and would be pointing downwards right. So what is if I considered this as my surface area or a small area around this one as a surface area and lets I place the capacitor in x direction okay. That is the plates are separated at some x equal to A and x equal to zero okay. So this has the two plates. Sorry, I should place the top surface at x equal to zero and the bottom surface at x equal to A.

So since the direction of x I assumed was downwards. So what is the normal to the top surface of the elemental area that I have considered, the normal is actually along x direction correct and what is gradient of psi okay. Well one interpretation was in terms of the electric field. So clearly that should give you the normal component of the electric field, but that is one thing.

If you expand gradient of psi itself in the Cartesian coordinate system, you will see that there are terms with y and terms with z. Let us not worry about those terms because when you take the dot product these terms will be left out right. So you have del by del x of psi in the x direction and when you take the dot product of del psi with the normal to the surface, which happens to be along x direction now, you get del psi dot x hat to be del psi by del x right.

So in general if the normal is along n what this quantity del psi dot n is telling you is, so let me highlight this one, so this del psi dot n is actually giving you the derivative of the potential function psi with respect with the normal coordinate right. If n is any general direction for the normal then del psi dot n is this del psi by del n okay, very interesting.

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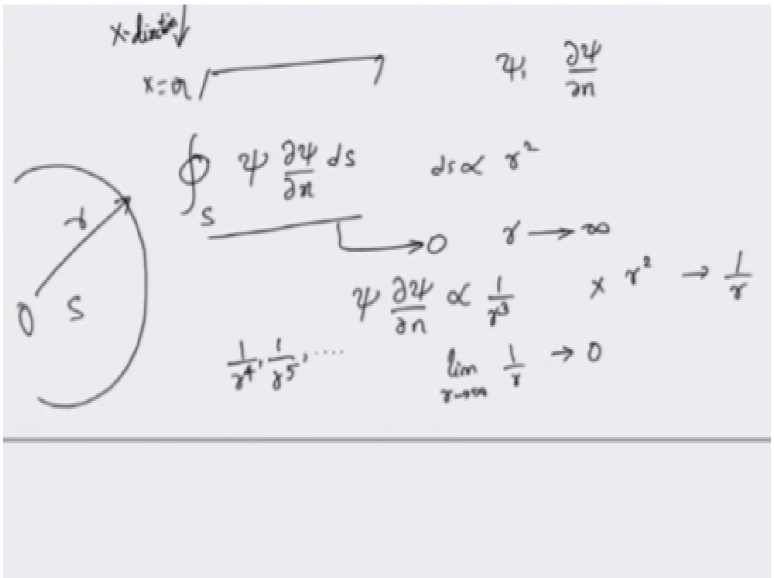

Now I can rewrite the integration over the surface as the closed surface okay as the closed surface, I have psi here and del psi dot n can be replaced by del psi by del n ds okay. So whatever the surface that we were considering over this entire surface s. This is the integration that we are performing okay. The surface has to be of course closed. Now look at what is happening inside.

You have a potential psi, you have the field del psi by del n, at least the normal component of the field del psi by del n and there is a surface ds that you are considering. Now if I imagine that this surface is not just any surface, but some sort of a spherical surface okay so this must be a spherical surface of certain radius small r okay. Having a radius measured from some origin okay.

Then the surface area would actually be proportional to r square correct. The surface area of a sphere is essentially r square. Now I ask you this question, when will this integral go to zero or when will this contribution of this entire thing go to zero, well I cannot just say this to be zero at a particular value of r. My qualifying condition is that if I start increasing the radius of the sphere to infinity, that is imagine that I take a sphere and keep on increasing this distance go to infinity.

Then when will this quantity be equal to zero. It will be equal to zero provided this psi del psi by del n go at least as 1 by r cube correct. It must go at least as 1 by r cube. So that when you multiply this 1 by r cube by r square, which is the proportionality factor for the surface element.

Surface area goes as r square, if this product goes as 1 by r cube, then the product goes as 1 by r and in the limit of r tending to infinity 1 by r goes to zero and your integration would essentially go to zero. So in other words, on the boundary right, the potential times the normal derivative of the potential. This is called as a normal derivative of the potential at least in my book.

So you can see that if this product goes at least as 1 by r cube. Of course it can go as 1 by r power 4, it can go as 1 by r power 5, it can go as any other, you know higher order term, but it must at least go as 1 by r cube. Of course, it can go as 2.1 and so on. But let us just focus on this terms going as integer values okay. So in this situation, what I have is the contribution of this integral over the sphere of radius tending towards the infinity will turn out to be equal to zero.

Now this is something that you have seen and you have experienced it when we solved electric fields earlier. Consider a point charge, how does the potential of a point charge go. If the point charge is q, the potential of the point charge is q by 4 pi epsilon r. So the potential is going as 1 by r. What about the electric field of a point charge, well you can take the gradient of the potential or you know already from Coulomb's law that the electric field goes as 1 by r square.

So this product of this point charge potential and the point charge electric field go as 1 by r cube. So if you now imagine a sphere of radius r, then on that boundary of the sphere, the field is actually 1 by r and as the boundary starts increasing in other words the sphere radius starts increasing, this 1 by r term keeps going to zero. So you will be integrating over an infinite radius of the sphere but the values of the fields at each point will essentially be equal to zero.

So that the total contribution to the integral itself is zero. This is very very crucial. So the point that we have made is that the potential must go as 1 by r at least then the gradients of the potentials along that normal direction would go as 1 by r square okay at least on the spherical surface and for all practical purposes this works out very well.

So the conditions at infinity that is as r tends to infinity is that potential must drop to zero and this del psi by del n must drop to zero even faster than the potentiality. For a point charge, it works out very well. For a line charge, when we talked about finding the potential of a line charge we found that I cannot take infinity as the point of reference for a line charge, and the reason is precisely because of this.

The potential of a line charge, an infinite line charge I am talking about does not go to zero as you go towards infinity why because the potential is some log of r and log of r actually increases towards infinity does not diminish towards infinity. So it is pretty bad in terms of that one and you can see why this is not going to infinity because this is log of r not going to zero, because this is log of r, the potential is log of r. The field is 1 by r.

So when you multiply them utmost they would cancel out each other right. Log of r for very large r is something, in fact they will not really cancel out each other, but you can think of them as been cancelling, you know large quantity, inverse quantity essentially multiplied together at large values of r would be as a constant okay. When you multiply that with the surface area, that is r square, you will see that, the total has actually jumped up to r square.

That is the product has jumped up to the power of r square al least and when you integrate r square over you know an infinite radius that fellow will diverge. So essentially give you non-zero value for the contribution of the sphere. The same thing happens in the field of a plane sheet of charge which is infinitely everywhere okay because there the potential goes linearly with respect to the distance, the electric field does not even vary with the position.

It will remain constant. It will not diminish at all right. Well these problems true, they are mathematically not very nice because they are going off to infinity and their behaviour at infinity is not very nice, but those situations are also unlikely to occur, I mean how much work you require to actually assemble a line charge of infinite length.

Even if the density of the line charge would be some non zero finite value, what would be the total charge on that infinite length line. It would be infinite. Similarly, you cannot actually fabricate an infinitely large plane charge, you know the plane sheet of charge. So it is just not possible in practice to get to those situations.

Mathematically you subvert those problems by postulating that the fields are not required to go to infinity at that stage, but they have to go to infinity at a point of reference okay. So there are certain certainties of this theorem, which we are not talking about, so as I said earlier, but for most applications, this theorem works as long as the potential is going as 1 by r and the field is going as 1 by r square okay.

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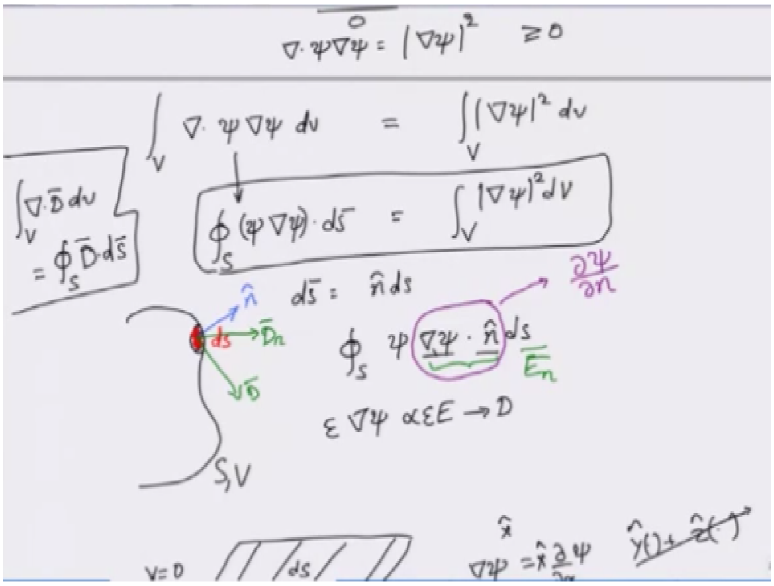

So let us come back to this. The whole point of motivating this one was to show that the left hand side of this equation, you know, this equation which I have can be made equal to zero. So this equation, the left hand side goes to zero as r tends to infinity. What about the right hand side?

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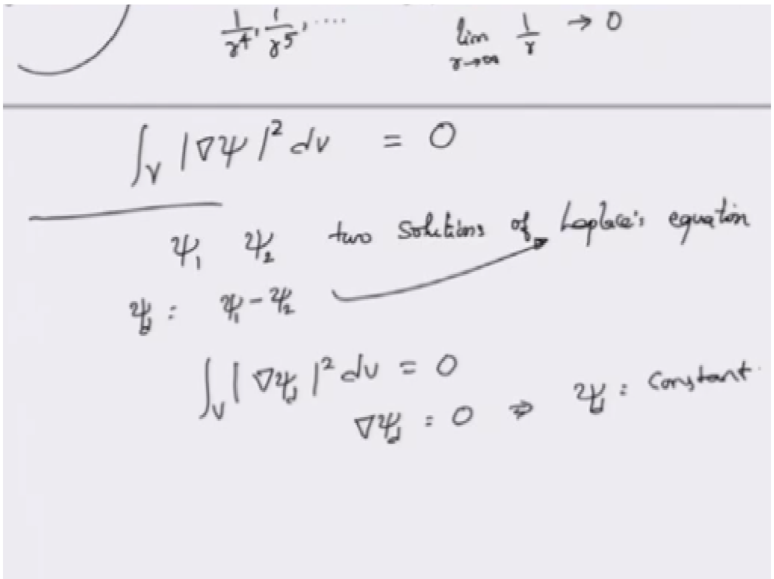


Well, if the left hand side is going to infinity that simply means that the volume integral of this quantity del psi square dv must also be equal to zero right. On the sphere of infinite radius, this must certainly be equal to zero okay. Now let us get back to the potential function, now let us say that psi 1 and psi 2 represent two solutions of Laplace's equation, which has been found by two different methods.

For example, psi 1 was found by you, psi 2 was found by your friend okay. Now because of the linearity of Laplace's equation this difference, psi 1 minus psi 2, which we will call as psi d is also a solution of Laplace's equation okay because of the linearity of Laplace's equation, the difference of the individual solutions are also the solution okay and this solution must be valid everywhere okay.

So this expression is completely independent of what type of potential function I choose okay. So I can put this psi d into this expression and see that this del psi d gradient square dv must be equal to zero. Now the only way this can happen is when this gradient of the difference potential is equal to zero, which immediately tells you that this psi d must be equal to constant right. So when there, it is a constant then the gradient of that constant which is essentially the slope of that surface would be equal to zero.