Electromagnetic Theory
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Lecture - 29
Solution of Laplace's equation – II & method of images - I

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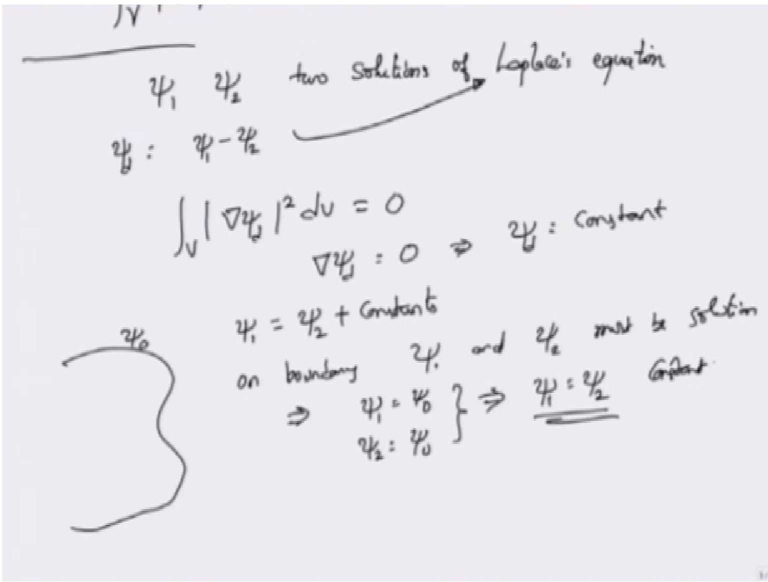

Now, psi d is psi 1 minus psi 2 so it tells you that you and your friend could be having two solutions and the difference between the two solutions would be some constants. Now here itself you can argue that constants do not really matter to me why because it is only the potential difference that matters to me right. So if I have found a solution, which is differing from your solution with a constant.

I can always you know, do not consider the constant as the reference and talk about the potential difference, the potential difference is what is counting to me and the constant does not really come into picture at all. Yes, however, this constant can be shown to be zero for our situation, how, on the boundary, right that we have considered in the original problem on the boundary, the potential is actually specified to some value psi zero right, the potential is actually specified.

And we have just said that psi 1 also is the solution psi 2 is also solution; therefore, both psi 1 and psi 2 must be the solution on the boundary itself. So on the boundary, both psi 1 and psi 2 must be the solutions right. Must be solution which implies that psi 1 is equal to psi 0 and psi

2 is equal to psi 0 right. Now it is very clear that if two quantities are equal to each other or equal to third quantity. This implies that psi 1 must be equal to psi 2 right.

And the constant must essentially go to zero because if you try doing that one anywhere else the constant would actually be equal to zero. So the point here that, for the case that we considered, no conductors inside, the Laplace's equation, the solution of Laplace's equation are actually unique okay. Even the constant of integration does not really matter.

The solutions are unique so you might for example, be very clever mathematically, you come up with a solution okay and your friend may not be mathematically that clever, but he or she is very good in numerical techniques and they come up with a solution. Provided both the solutions satisfy Laplace's equation as well as satisfy the boundary condition, these solutions are unique okay.

So if you are unable to solve Laplace's equation problem with the mathematics, you can try graphical methods, you can try experimental methods, you can try numerical methods okay. Along with whatever the methods of advantages and disadvantages, if you find solution by any of these methods which satisfies the equation as well as the boundary condition then that solution is unique.

You can go home and rest comfortably knowing that you have solved Laplace's equation once and for all, for a given boundary condition. The only conceptual problem that might arise is that you cannot really always consider cases where there are no conductor right. So you take a parallel plate capacitor, and then, how do I generate a voltage between the two, I need to apply a potential difference.

You know connect a battery between the two plates. The moment I do that there are metallic conductors, there is a battery lead, there is a battery lead on to the other side, inside whatever happens inside the battery we do not really care about but there are conductors now. Now if I consider a sphere or a close surface which also includes the conductors are the conditions for uniqueness going to change.

If they change, how are they going to change. Again we will not be too detailed in mathematics. We will give you the qualitative understanding of what happens when you have some conductors in the region okay.

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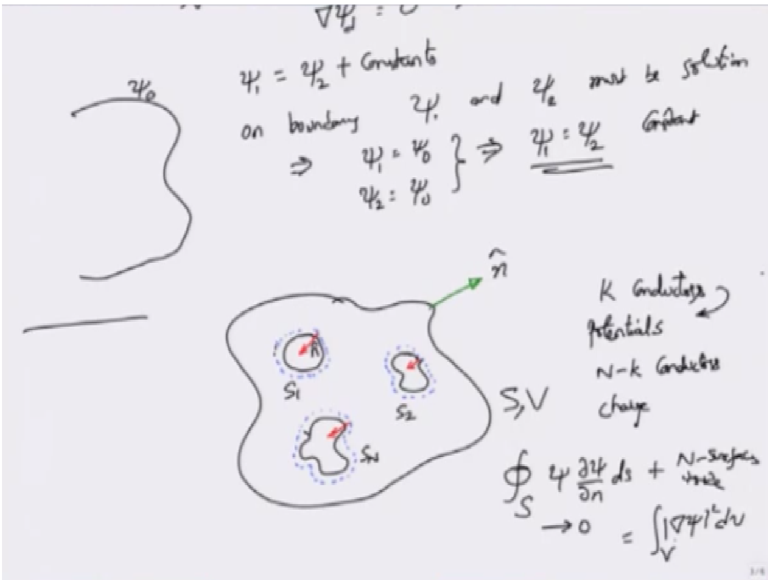

So this is the second part of the uniqueness theorem. Consider an orbitary surface S, which bounds a volume V. Now here I might actually have some conductors okay. The conductors of course inside can also be completely orbitary okay. Now surrounding each conductor, I can actually put a surface okay, a Gaussian surface, I will put a Gaussian surface around each conductor.

Let us say there are some n conductors inside here okay. There are now two kinds of surface normal's that you have to consider. One surface normal points outside on the bigger surface that is there and there are little normals, which are pointing inside okay. The idea being we want to find out what is the net flux outside of the surface okay. There would be some, but I want to find out the net outward flux okay.

Now what do I specify on this surfaces. Let us also label this surfaces okay or the conductors S1, S2 and in general SN, the normal outside is n, the normal here inside is also n okay. The normal I am considering is to be n, the directions of the normal need not be the same. Amongst these conductors, let us say I pick K conductors okay and specify potentials on these conductors.

For the remaining N minus K conductors, I will specify the charge or at least I would be specifying the surface charges and I want to find out if I have two solutions psi 1 and psi 2 which satisfy this condition that is on K conductors they actually have the potential values okay given and on the N minus K conductors where I am specifying charge, the solutions have to be just compatible with that, you know some sort of Poisson's equation.

So they have to be compatible. Our equation is still the same, so I have this integral over the surface S and this psi del psi del n ds and as before psi goes as 1 by r at least n then goes as 1 by r square on the outer sphere allowing me to show that for r tending to zero, this integral will be equal to zero, but now I also have to consider the contributions for the inner surfaces, right. So now there are N surfaces inside would also contribute and the overall result would be equal to del psi mod square dv over the volume that this encloses.

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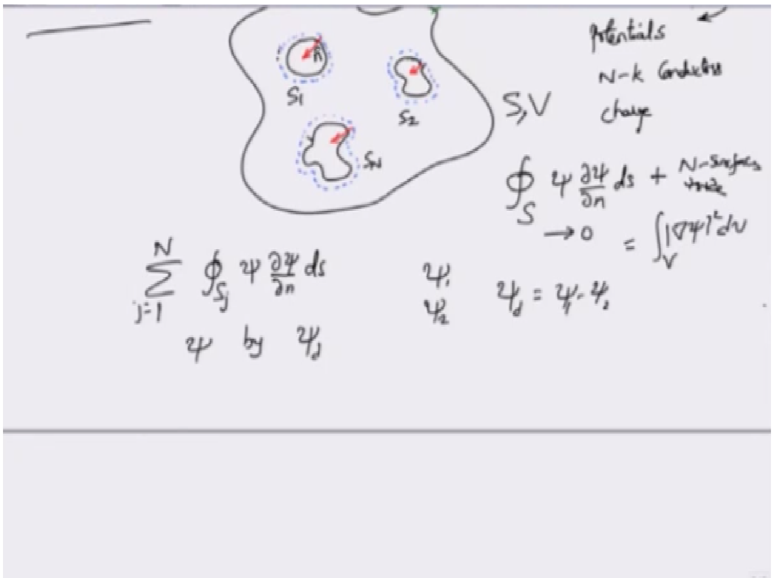

This is like consider n islands of the surfaces inside the bigger surface S and find out the contribution of the surface S itself and the inner surfaces S 1 to S N okay. So as before I have psi 1 and psi 2 as two conductors, all I have to do is I have to find out the contribution of the surfaces inside. The contribution of the surfaces inside is given by summation, so if I call them as surface 1 to surface N, there is a small integration, close surface integration that I have to perform on each of the surfaces.

I have to perform this psi del psi by del n on each of the surfaces. So if I do that one I get psi del psi by del n ds where you need to understand that this normal is actually pointing towards inside. This is like, this was pointing outward, this is pointing all inward okay. So this is the contribution; however, to this contribution okay, if I replace psi by psi d that is I have found two solutions, psi 1 and psi 2, I have formed a different solution psi d, which is psi 1 minus psi 2.

Both psi 1 and psi 2 satisfy Laplace's equation as well as they also satisfy all this boundary conditions right. So if I know that they also satisfy boundary condition then on to the left hand side of this equation, this term goes to 0 because as I keep increasing surface S, the contribution over the outer sphere goes to 0 and all I am left out with is the contribution of these n conductors right. That must of course be equal to this gradient of psi mod square dv.

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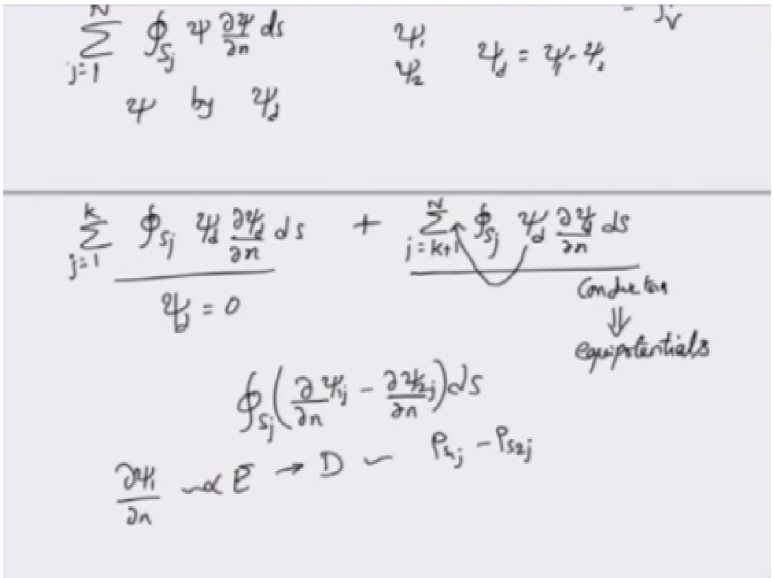

What is the contribution of this left surface, you break up this into two parts, I have j equals 1 2 K okay, integral over the surface sj psi d del psi d by del n ds plus the remaining surfaces, so j equals K plus 1 2 N okay and I have over the same n minus k conducting surfaces, the same integrals, psi d del psi d by del n ds okay?

However, on the conductors, we have actually said that psi 1 and psi 2 are the solutions. which means that on the conductors psi d is equal to 0 correct. On these conductors psi must be equal to 0 because on the surfaces, I have specified potentials. So psi 1 will be equal to that specified potential, psi 2 will also be equal to the specified potential. So the difference between the two must be equal to zero on the potentials.

So this entire term to the left of this would vanish that is the contribution of the K conductors to this integral would completely vanish because on the boundaries psi d will be equal to 0 and remember these integrations are happening on the boundary of those surfaces. What about this other term, the second term, well on this terms the potentials need not be same, but they have to be constant because these are all conductors.

Conductors mean they are equally potential surfaces okay. They might have some charge distribution no doubt, which we have induced or which we have kept over here, but because they are all equipotentials on the surfaces, I cannot have different potential values right. I have to have a same or a constant potential. It is like keeping one battery lead at one potential, another battery lead at another potential and they are conductors.

Therefore, the potentials on the conductors will be constant, which allows me to actually pull this psi d out of the integral okay. So I can pull psi d out of the integral and see what I actually get remind. So I have del by del n psi 1 minus del psi 2 by del n ds. Now here is where our observation that del psi 1 by del n being proportional to electric field, which is proportional to the d field and hence this would be the charge because of the charge on the potential that we are considering.

So on the jth charge. So this is on the jth charge that I am actually looking at. So because of the solution psi 1 del psi 1 j by del n on that particular jth conductor would actually give me the surface charge density on the conductor. Similarly, this would be the surface charge density because of the second solution. Now on this conductor you cannot have two charge distributions just because your psi 1 and psi 2 are different.

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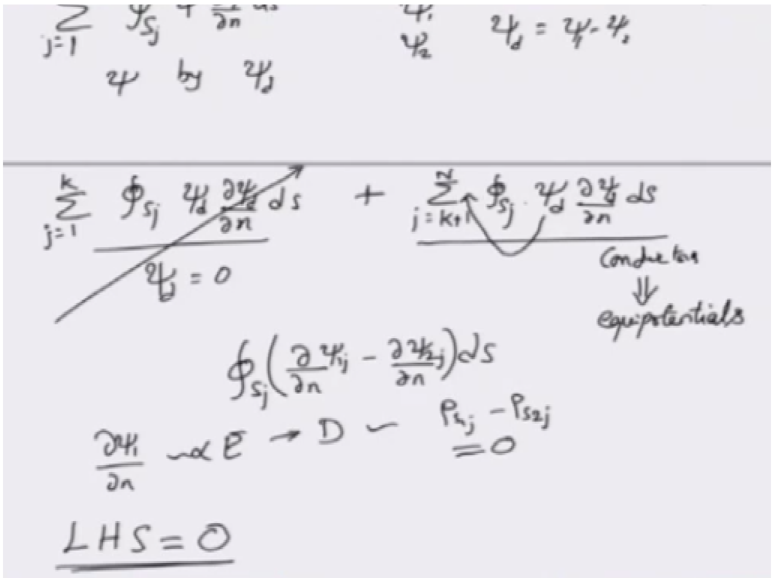

So it simply means that these quantities must be equal and hence this fellow must be equal to zero, the difference must be equal to zero, indicating that the entire left hand side is equal to zero. This is very very crucial right. So this happened because on the conductor surfaces the potential is constant right, the potentials are the same, therefore the difference in the potential will be equal to zero.

So this term integration gave you no contribution. On the conductors where the chargers were specified, the surface charges were specified. There again because the solutions are to be compatible with the given charge distribution, the solutions were in such a way that the integration of those would also give you zero. So the left hand side is zero, which simply implies that the right hand side is also zero.

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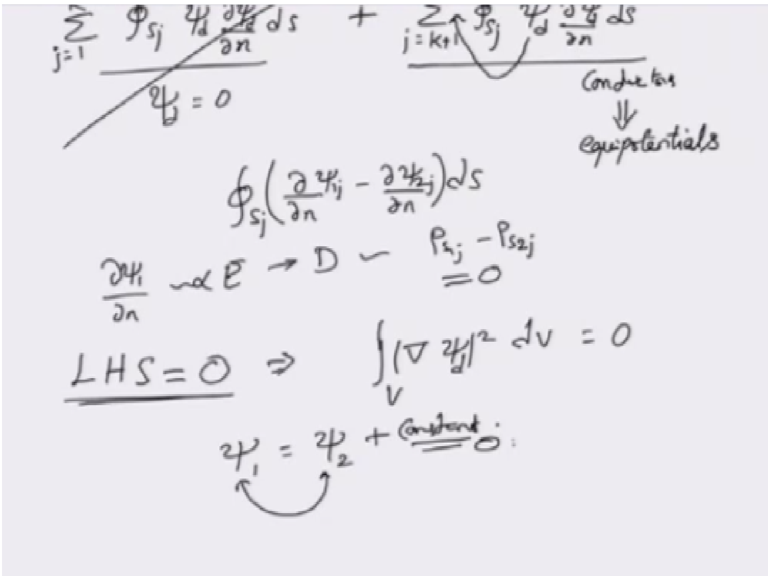

So I go back to this same condition that I obtained earlier okay. So this is also equal to zero. So del psi d square is equal to or integrated over the volume equal to zero, this gives you the same kind of arguments that psi 1 must be equal to psi 2 plus some constant okay and again this constant will be equal to zero because on the surface of the conductors where I know the potentials, these two must be equal to each other.

So the constant will be equal to zero. So no matter if your surroundings or if your region contains conductors. If you have solved Laplace's equation which satisfy the boundary condition and the charge conditions or the potential specifications, you have found the solution which is unique once and for all okay. So this is the end of uniqueness theorem that we wanted to prove okay.

And we have sort of looked at the uniqueness theorem and we have observed that if I solve the original problem that given problem, which satisfies the equation as well as satisfies the boundary condition then that solution is unique for us okay. That solution is unique to us.

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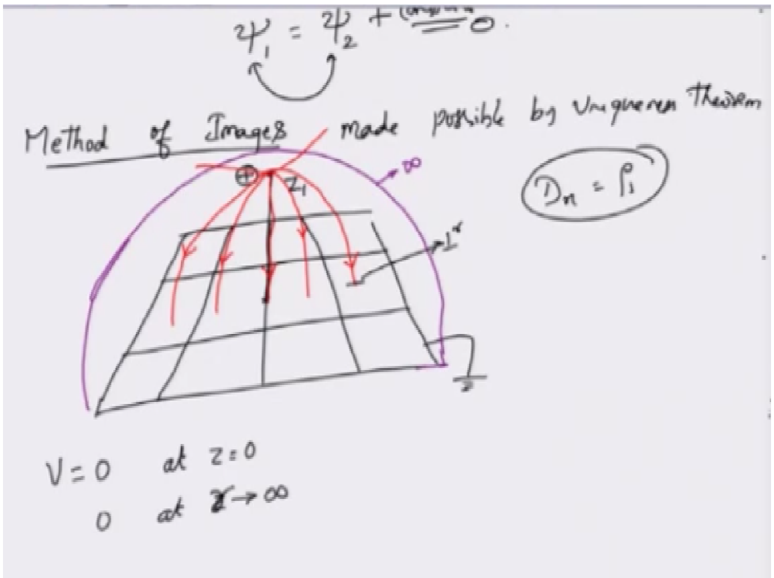

This actually allows us to consider a very very powerful method of solving these problems of electrostatic problems called as method of images. We will simply describe the method of images and then comment about the applicability of this method okay rather than considering it to be general. Now before we talk of method of images, I want to tell you that method of images is a very powerful method, but that is not very general method.

That is you cannot apply this method of images to every electrostatic problem. Only when the problem has certain characteristics then this method of images is applicable. In fact this is a common problem with all analytical techniques. Any analytical technique that you consider in solving these electrostatics or magnetostatics or in general electromagnetic problems, they are all applicable only in certain restricted class of situations okay.

They are not general enough that you can approach every problem in electromagnetics by these methods okay. So do not rely too much on this method of images thinking as this is one solution that you can apply, one method that you can apply to solve this problems, they are only applicable to certain classes of problems okay and this method of image is actually made possible okay because of the uniqueness theorem. You will soon see why so.

So consider a plane okay. Let us place this plane at z equal to zero. Let me also ground this plane so that the potential on this plane is actually equal to zero. So I have this plane okay and on this plane at some point I go up, you know at a certain distance z 1, I keep a positive charge okay. Now what I want to find out is what is the potential everywhere. This problem is actually quite difficult if you work to go by Laplace's equation earlier okay.

Because look at what is the field lines that are coming out of this problem okay. If you look at the field lines, down the field line would start from the positive charge and it would drop down perpendicularly on to the plane okay and at all field lines would actually converge or curve towards the conductor and they are all terminating on the plane okay. So you can see all these red lines which I have drawn are the fields.

And they are all terminating on the plane, very interesting right. They are not going to zero there, because they are actually going to induce some charges on the plane okay. They would induce certain charges; however, the point is that the field lines have to be perpendicular at the surface, so let me emphasise that point over here. The field lines have to be perpendicular at this point okay. Why they have to be perpendicular because this is a conductor.

And for a conductor, you have seen that there cannot be tangential component at the boundary, they have to be always perpendicular. The electric field has to be perpendicular. So and in fact, on the conductor Dn is equal to surface charge density. So if I know what is the normal d field that is hitting the conducting plane, then I also know what is the surface charge density at that particular point okay.

Now with this if you try to apply a Laplace's equation as such, you will be having difficulty in solving that okay. But let us fix up what are the conditions that I am looking at okay. I know I have to consider a certain region of space right and then I have to let that region of space towards infinity okay. The plane itself extends towards infinity on all directions therefore I have one condition, v is equal to zero at z equal to zero. This is one boundary condition that I already have okay.

Now to get another boundary condition so as to speak, I know imagine that I have sphere of some radius okay initially and then I let this radius of the sphere go towards infinity. Now if you are very far away from this situation at infinity, all the field that you are looking for are only because of these positive charges right. So there are some field lines, which are going towards infinity okay not exactly like this, but they are essentially going towards infinity.

And these fields because of the point charge vanishes by r square, I mean go as 1 by r square therefore if you now consider a hemisphere okay of certain radius r and then let the radius go towards infinity then the values that you are going to see would all be dropping down to zero because that psi del psi by del n will be going as 1 by r cube in this particular case. So the potential is basically because of the point charge. So I have another condition that v is equal to zero at z tend are radial distance r tending towards infinity okay.

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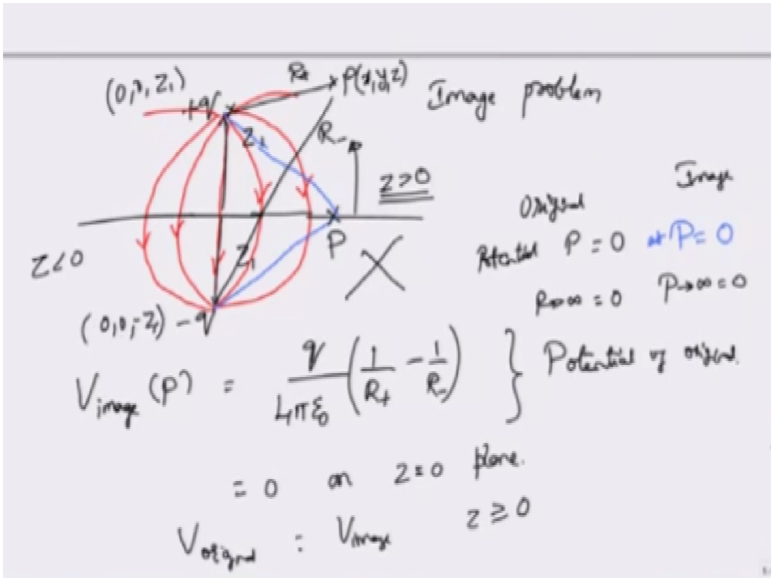

Now what method of images does is very interesting. It actually removes the conducting plate entirely. So if you remove the conducting plate and but you still of course need to retain the charge, because if you remove conducting plate and charge you have no problem left with it okay. That would be very trivial and interesting problem right. There is no charge nothing is happening.

However, this charge was there at some height z 1 right. At the same height z 1 below the conducting plate where the conducting plate originally existed, you put a negative charge minus q. Now from your experience of all the classes that you have seen and listened in this course, you know that the field lines would be directed from positive to negative charges and they would all terminate over here in this nice curved fashion right.

So they would all terminate here, this nice curved fashion, this would the fields. Now if you just look at the upper space that is if this is since this is z greater than zero region, if you just look at the upper region okay and you look at the original problem, they all look alike right. They both look alike and the upper region, there are field lines okay. They are all curving towards and they are getting terminated over here.

Some fields may turn up towards infinity later you know they would all travel all to infinity and then come back, but essential point is that the upper part of the problem is exactly the same as the original problem right. So in fact this problem that we have here is called as the image problem okay and image problem is the one in which we have removed the conductor, but put a negatively charged here at exactly the same distance z 1.

Now why should it be exactly at the same distance, consider what is the potential here, in the original problem, point p potential at p was actually equal to zero correct because it was a grounded conducting plane. In the image problem what is the situation, in the image problem if you want to find the potential at these two points and you know the potential is actually 1 by 4 pi epsilon r with infinity as the reference point.

Then the potential because of this charge plus q and the potential because of the charge minus q both will contribute to 0 correct because the distances are equal. Since the distances are equal the potential would also be equal to zero. So the image potential okay is also equal to zero. You go to infinity then the potentials would all be equal to zero because this would be 1 by 4 pi epsilon r with a minus sign, but for large r they would essentially cancel out with respect to each other.

So the potential goes to infinity at even in the original as well as the image problem. So potential at infinity at as p goes to infinity is also equal to zero and the same thing for the image problem as well okay. So it is interesting that the image problem satisfies the same boundary conditions as the original problem; therefore at least on the upper side of the region z greater than zero whatever the potential that we obtain must be the same as the potential of the original problem right.

So what is the for the image problem, what is the potential, let us call this as some v image potential at any point. So if this is the point p the potential will be equal to 1 by 4 pi epsilon 0 or q by 4 pi epsilon 0 and the radial distances right. So this is r plus and this would be r minus right. So this would be equal to 1 by r plus minus 1 by r minus and this image potential would actually be equal to zero on the z equal to zero plane okay.

In fact, this would be the potential of the original problem as well. Now you might ask what about the solution for z less than zero, it is completely rubbish, because I should not be seeing any potential up there or you know I should not be seeing anything over here. Then why is that not zero. That is rubbish, but it is alright because I am not concerned about what happens down here okay.

My only interest is in the upper portion of the problem and in the upper portion of the problem, the image potential is exactly equal to the original potential right. So v original the potential function for the original will be equal to the image potential for z greater than or equal to zero and that is sufficient for me. For concreteness sake, if I assume that this point p was lying at some x, y and z right and this charge was lying at z 1, 0 0 z 1 and this fellow was lying at 0 0 minus z 1.

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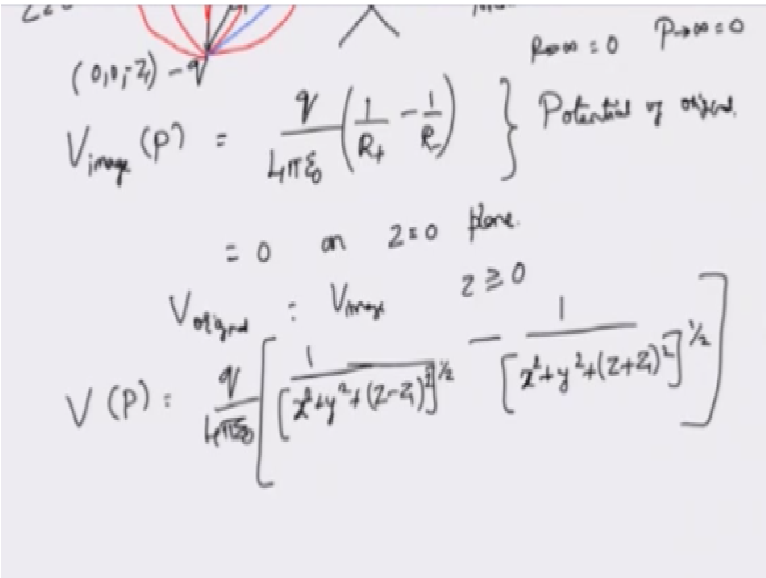

The potential at the point p is given by q by 4 pi epsilon 0 1 by x square plus y square plus z minus z 1 square, this is r plus right minus 1 by x square plus y square plus z plus z 1 square and you can see that this solution satisfies the condition that we are interested in at the z equal to zero plane, this will actually turn out to be equal to zero because z will be equal to zero and the distances will be equal to each other.