Electromagnetic Theory

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Lecture - 30
Method of images - II

In the last class, we talked about uniqueness theorem and discussed one technique for solving electrostatic boundary value problem, called as method of images. We will talk about method of images now, giving you more examples on that method. And then we will consider solution of Laplace's equation. If you remember the image problem that we considered was that of an infinite plane, conducting plane which was kept at zero potential.

And you had a charge, which was placed at a certain height Z1. The corresponding image problem was that of remove the conducting plane, but keep a negatively charged charge at the same height as that of the positive charge. The distance between the two must be the same with respect to the, where the conducting plane was originally. This guaranteed that the potential on the conducting plane was equal to zero.

And the potential at infinity was equal to zero, satisfying both the required boundary conditions, right? So that was the image problem that we considered earlier, and we found out that the potential in the case where Z is greater than Z1, okay. The solutions have to be such that Z greater than zero, the upper part of the hemisphere or the plane, then the potential was given by, Q by four pi epsilon zero one by x square plus y square plus Z minus Z1 square minus one by x square plus y square plus Z plus Z1 square.

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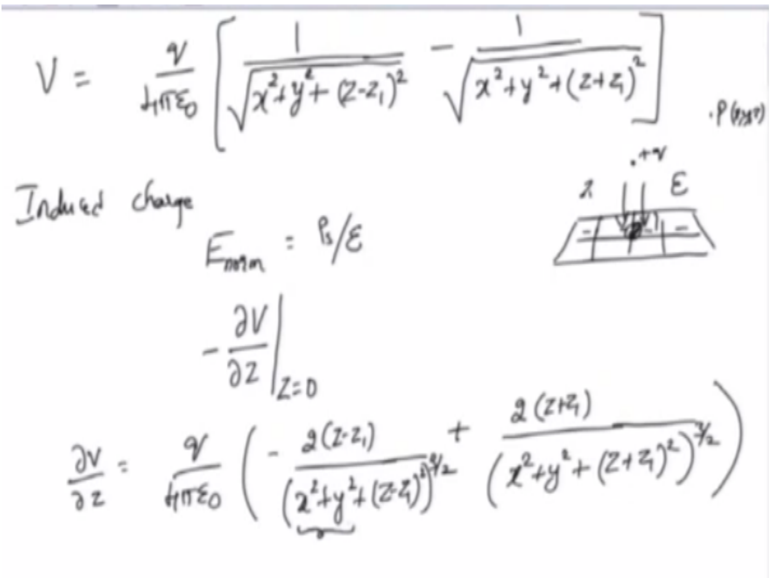

Here is where the positive charge is located, here is the negative charge that was located, okay. Now, if I ask you what is an induced charge, okay. See, now, this is the potential but now the original problem is still the same. I have the plane here, okay, I have the plane here and there is a corresponding charge at height Z1, okay. So there is a charge at this point, I am considering at any point P (x, y, z), what is the potential.

And that potential is given by this expression. So, if you look at this situation, the original problem, and because there are field lines because leaving the positive charge and then, you know, landing on the conducting plane. These will induce negative charges, right. So what is the amount of charge that is induced? Let us calculate that.

To calculate that I need to know what is the charge density, the surface charge density, at any point, and then if I integrate this surface charge density, I should be obtaining the total charge. To obtain the surface charge density, all I have to understand is that this is the conducting plane, therefore for the conducting plane the normal component of d must be equal to rho s, the surface charge density.

Therefore, electric field normal component must be equal to rho s divided by epsilon, where epsilon is actually the epsilon of the medium above the plane, okay, not of the conductor, of course. And what is the normal component of the electric field, this is simply the derivative of the potential or the differential of the potential V with respect to Z direction, because the normal component of the field is perpendicular here.

And the normal to this plane is actually along Z direction. Therefore, E norm is minus dV by dZ, but not at any value of Z. This has to be done, this has to be evaluated at Z equal to zero. So if you differentiate this potential function V, what you get is del V by del Z will be equal to q by four pi epsilon zero on the outside, and then if you differentiate this quantity, with respect to Z, keep x and y as constants, okay.

And this fellow will become two times Z minus Z1 divided by x square plus y square, which has actually remains constant plus Z minus Z1 square to the power three by two. Similarly, you will get two times Z plus Z1 divided by, sorry this becomes minus and this becomes plus. I think this becomes minus, because there is a Z minus Z1 whole square in the denominator under root and the denominator.

Therefore, this becomes minus and I think this becomes plus, you can verify this one, okay. It becomes x square plus y square plus Z plus Z1 square to the power three by two, okay, this would be the electric field. Now, you apply this one at Z equal to zero, okay. So if you apply this one at Z equal to zero, and call this x square plus y square as some zeta square, which is a constant which I am considering.

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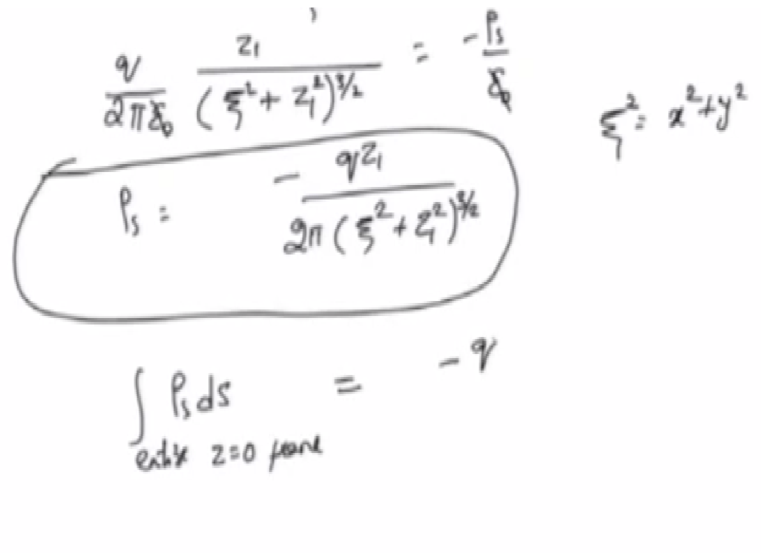

So this is zeta square, right, at Z equal to zero, if you find this out, you will see that this would be q by there is a two in the numerator, four in the denominator, so this becomes q by two pi epsilon zero Z1 divided by zeta square plus Z1 square to the power three by two. Why, because Z equal to zero so minus Z1 square is as good as Z1 square, so you can actually combine this equation and find that Z cancels with each other, and you are left with only Z1.

This is must be equal to minus rho s by epsilon zero, okay. So if you see this, this is minus rho s by epsilon zero, epsilon zero on both sides cancel with each other. So rho s, the surface charge density is given by minus q Z1 divided by two pi zeta square plus Z1 square to the power three by two, where zeta square is actually x square plus y square, right.

This is just a short hand notation that I am using to denote this quantity x square plus y square. So this is the surface charge density and it is interesting to find that the surface charge density is negative, which is what you would expect, right. You have a positive charge on the plane, and because of the positive charge having the field lines, you know, falling on the conducting plane, they would all induce negative charges.

So the surface charge density must be negative, okay. Now, what is the total charge induced. The total charge induced can be obtained by integrating this surface charge density over the entire conducting plane. In the case that we have considered, the conducting plane goes all the way from minus infinity to plus infinity in the x-y plane, correct. So I need to integrate this one, over that plane. I can choose various ways of integrating this one.

You can choose different ways of approaching this problem by integrating this. You can consider circular areas of radius zeta, okay, and then integrate as a function of zeta so that surface, that circular patch will have a surface area of two pi zeta, okay. So you can actually consider this and then start expanding the patch, you know, to all the way from zero to infinity, or you can integrate this one with respect to x and y, which ever method you choose, you are essentially looking at integrating this over a entire Z equal to zero plane, okay.

Over the entire Z equal to zero plane, you can integrate this one and you will see that this will be equal to minus q. Thus, the total charge induced on the ground plane is equal to minus q and it has to be that much, because if it is anything greater o lesser, then there is no charge balancing happening, right. So the total charge that must be there must also be equal to minus q, okay.

So you can solve this I will leave this as a small exercise to you, while you are solving this exercise, you will come across certain integrals, which you have used earlier. So that might be easier to work out, okay, because you have seen this integrals earlier. So you will come across certain integrals.

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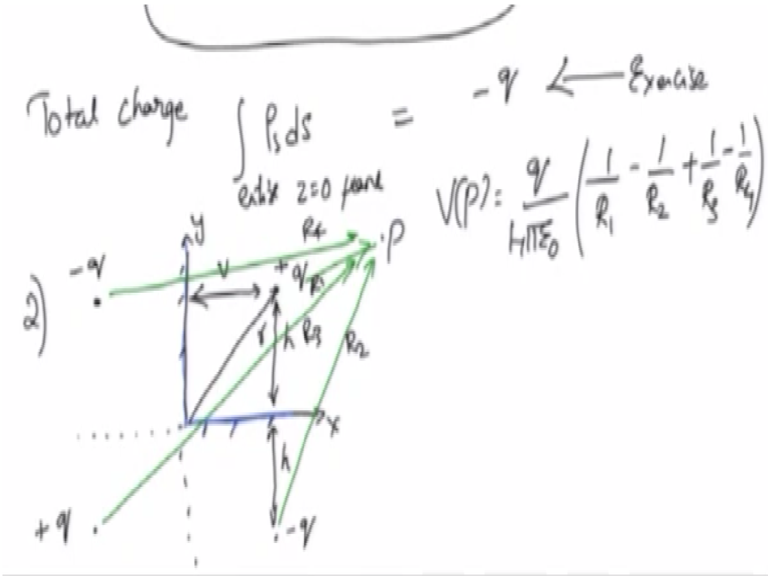

Let us consider a second example of images, we will not be giving you the entire solution here. The idea is to just see how to extend this solution, okay. Consider two conductors which are bent, such that they form right angled corner, okay. They form these corners in the entire quadrant with x greater than zero and y greater than zero, okay. So these are the conductors, okay, and now I place a point charge here, okay.

So I place a point charge q at a distance r or a radius r, in front of this infinitely bent corner, conducting bent corner, let us say, okay. Where should the image charges be located? Now there are two conducting surfaces, so you expect at least two charges be located, and you are right. In order to make the potential zero on the horizontal plate, I have to find out, you know, this height, let us, whatever the height that might be let us not even worry about that.

So this height I have to find out and from this height I have to place another charge, okay, at the same height, but with opposite polarity. If I do that, this is the image charge, so let us call this as q and let us call this as say, yeah, this is minus q, okay. If this height is h, this height must also be h, okay. This is obvious, because only then the combined potential on the horizontal plate will be equal to zero. Is this enough?

No, clearly because there is a potential here, and if I assume only two charges q and minus q, I will not be able to make the potential zero here, because the distance from plus q is smaller compared to the distance from this minus q charge. So the potentials will be unequal and that is not acceptable, because this conductor is an equipotential surface, right. So, because I want to make the potential zero here I have to put one more charge, the image charge.

So, if this height is say v, okay, I should have probably chosen this as v and this as h, that would have made it vertical and horizontal, but anyway, that is not really important out here. So if I chose this at a same distance v to the left of the vertical plate, okay. So I have minus q here, same charge value, then the potential on this charge can be made equal to zero, correct. I can make the potential equal to zero, here on the vertical plate.

Will that be alright, unfortunately it turns out that it is not alright, okay, because the potential of the three charges, okay. If there are, because there are three charges the potential here gets imbalanced. And because of these charges, the potential here gets imbalanced, because of this charge there will be a potential here, which means that the cancellation of q1 minus q is not sufficient.

There has to be some extra cancellation that is required for the horizontal plate. Similarly, this minus q and plus q over here they will compensate if they were themselves. But because of this minus q down here, the potential that it will be carrying on the vertical plate will unbalance the situation. So, I need to actually create one more image charge, which is located at the same height, you know, or the same distance minus q.

But located below, and this must have a positive charge, okay. Why this should be a positive charge, because on the horizontal plate the combined potential is becoming negative, if you did not have this positive charge, if this was not there, on the horizontal charge, on the horizontal plate the potential is becoming negative, because minus q from the left hand side, plus q from the top and minus q from the bottom they would cancel.

But the potential because of this minus q charge on the left hand side in the upper corner would make this potential negative. Now, because there is a potential, because there is a charge plus q down here, that will contribute an equal positive potential. And hence the total potential on this horizontal plate will be equal to zero. So very similarly, to the vertical plate as well.

So, in this image problem, it was necessary to introduce not one, not two, as you would have imagined two conducting surfaces, you would have imagined only two charges are sufficient. Unfortunately, that is not sufficient, you need to introduce the third charge as well, okay. So, the potential at any point p can be obtained by looking at what is the distance of each of these charge, looking at the distance from each of these charged particles.

So you can call them as r1, r2, you know, r3, and r4, and the potential will actually be equal to potential at the point p will be equal to q by four pi epsilon zero one by r1 minus one by r2, r3 is the distance from the positive charge, so one by r3 minus one by r4, okay. So this is r1, this is r2, r3, and r4, okay. So, if you write down the expressions in Cartesian coordinates you will actually be obtaining the potential function, okay. Alright, so this was the second image problem.

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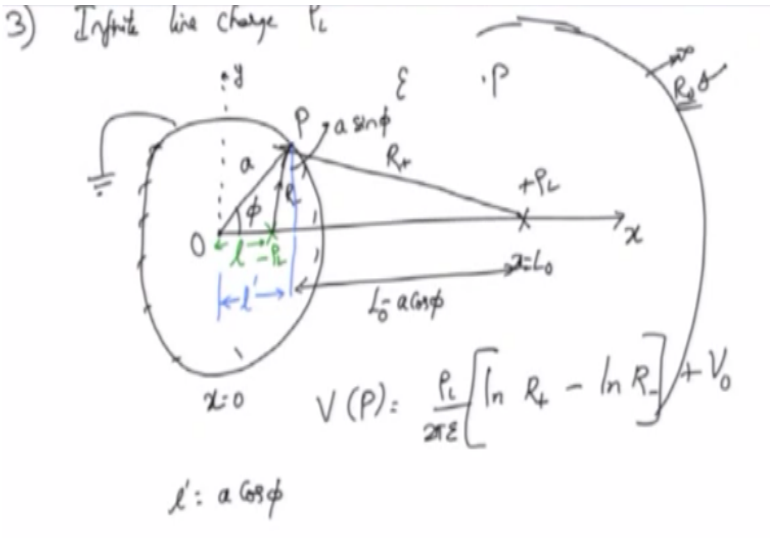

Now, we will solve one more image problem, this is important in the practical sense. They have actually used the results of this, without really telling you where we have used it. But you will soon recognise that one. Suppose I have an infinite line charge, okay, I have an infinite line charge, okay, positively charged with a charge density rho L. And I take grounded conductor, okay, I take a spherical grounded conductor over here.

And I will place this infinite line charge, so this is along x equal to zero, okay. And this is at x equal to L zero. That is the centre here is placed at x equal to zero, having a radius a, okay, and grounded up here, okay. And I am now placing a positive charge density plus rho L at equal to L zero, along the x axis and now, I want to find out the corresponding image problem, okay. So what would be the corresponding image problem.

Well you suspect that, because there is a infinite line charge here, outside the cylinder, there must also be an infinite line charge inside a cylinder. Where should that cylinder be posted? Well let us call the distance from the origin, where the infinitely line charge but with minus rho L, okay, it can also be minus rho L prime, and but for simplicity, let us assume that rho L prime is equal to rho L, and turns out that the simplified assumption is correct.

Because uniqueness theorem tells me that if this assumption is correct and I satisfy all the boundary conditions, then everything should be alright, okay. So, I consider minus rho L at a distance, let us say L, okay, at a distance L from the origin. Now, I want to find out what is this length L, okay or distance L, at which I have to place minus rho L, okay.

Now, we first face an important problem, if you start surrounding this with a spherical surface, okay, a spherical surface and start letting the radius of the spherical surface go to infinity, you will actually see that the potential because of the infinite line charge does not go to zero.

So I cannot take zero as the reference potential, but if I take any other point, you know, if I take any other distance, arbitrary distance as the reference potential. So, with that reference potential, the potential at any point P in the region will actually be equal to rho L by two pi epsilon log of R plus, there will be this R0, where R0 is the radius of the reference point that I am considering, not infinity but at some other point, okay.

Now, because there are two charges here, one charge with plus rho L, there is another charge with minus rho L. One of them is at x equal to L zero, the other one is at x equal to L. The actual potential at this point P will be given by, point P on the cylinder, right, because on the cylinder I want the potential to be equal to zero. So, on the cylinder if I place the point P on the cylinder, then the potential at the point will actually be equal to, so this is R minus.

So the potential at the point P will be equal to rho L by two pi epsilon, epsilon of course being the dielectric that is sitting out here, okay. So I have rho L by two pi epsilon log of R plus minus log of R minus, okay, plus some constant V0, okay. This constant V0 is necessary, because the potentials are to be, with respect to a certain reference point R0. They cannot be at infinity; therefore, I have t include this constant V0.

The constant has to be included, so that the potential on the grounded point will be equal to zero, as does the potential at this reference point R0. So if I take any other point R0 as the reference, or the sphere of radius R0 as the reference, then I have to include a potential here. So that the potential on the grounded cylinder is also equal to zero, okay, so this is the potential.

Now, I want to find out the relationship between L and L0, right. I want to find out where the point L lies, so in order to do that one, let me draw a perpendicular point P on the x axis. And call that length, from the origin, as L prime, okay. So, let me call that length as L prime. This is obtained by dropping the perpendicular from point P on the grounded cylinder, on to the x axis, okay.

Now, with that I can relate R plus. What could be the horizontal distance L prime? If this radius a, which joins the origin and the point p, makes an angel phi with respect to the x axis, then this length L prime is simply a times cos phi, correct. So L prime is equal to a cos phi, this is the point where I have the x axis. And this length on the x axis where plus rho L is situated is L0, right.

Therefore, this length difference along the x axis will be L minus a, L0 minus a cos phi. What about the component for y? The y component will be equal to a sin phi, right. So this component is equal to a sin phi, okay. So R plus square will be equal to L0 minus a cos phi square plus a square sin square phi.

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V(P) on Cylinder independent ϕ

$$\frac{R_+^2}{R_-^2} = k^2 \qquad \underset{(\text{Cylinder})}{V(P)} = 0 = \frac{\rho_L}{2\pi\varepsilon} \ln\left(\frac{R_+}{R_-}\right) + V_0$$

$$V_0 = -\frac{\rho_L}{2\pi\varepsilon} \ln k$$

$$a^2 \sin^2\phi + L_0^2 + a^2 \cos^2\phi - 2aL_0 \cos\phi = k^2 a^2 \sin^2\phi + k^2 a^2 \cos^2\phi + k^2 l^2 - 2a k^2 l \cos\phi$$

$$a^2 + L_0^2 - 2aL_0 \cos\phi = k^2(a^2 + l^2) - 2a k^2 l \cos\phi$$

Of course you could have easily seen this one, because this is the circle of radius a and the point at, any point on this radius a is given by a cos phi and a sin phi in the Cartesian coordinate systems, right, a is the radius, phi is the angle from x axis, so there is no surprise up there. So similarly, R minus square will be equal to the vertical component remains the same a square sin square phi. But the horizontal component is not the same.

So horizontal component is in fact given by this fellow, which is L prime minus L. So horizontal component is given by L prime minus L, L prime is a cos phi and L is L itself, the small length l. So this is R minus square. Now, on the cylinder the potential V of P, on the cylinder, must be independent of the angle phi, of the angle phi must be independent, because the potential on the grounded cylinder is equal to zero.

And it say equipotential surface, and if I start making the potential dependent on phi, then at different values of phi the potential might be different. Therefore, this must be the case, the potential on the cylinder must be independent of phi. So, if I want to have this condition, then I can, it is possible if I makes this ratio R plus by R minus to be some constant, right.

So, if I make that one as some constant, and call the constant, say kappa, then R plus minus R minus square will be equal to some constant kappa square. Why is this necessary? Because if you look at the potential on the grounded cylinder V of P, which must be equal to zero, this is equal to rho L by two Pi epsilon ln of R plus by R minus, right, plus some V0. So if R plus to R minus ratio is some constant kappa, then V0 must be chosen such that, the potential on the grounded cylinder must be equal to zero.

So this is the constant which you have to chose in such a way that potential on the cylinder must be equal to zero, at other points it is not zero, but on the cylinder it must be zero, okay. And if you do not include this constant, you would not be able to satisfy this condition. That is the reason why you had to include this constant V0, okay. So coming back to this, the ratio of R plus square to R minus square must equal to some constant kappa square.

Now substitute for what s R plus and R minus, you will end up with two equations, okay. I mean you will end up with one equation, left-hand side and the right-hand side, that would be a square sin square phi plus, there is L0 minus a cos phi, I am opening the bracket for that one, so I will get L0 square plus a square cos square phi minus two a L0 cos phi equals kappa square a square sin square phi plus kappa square a square cos square phi.

Let me not spell out loudly you can show that this is actually equal. So combining sin square and cos square and simplify this equation gives me a slightly simplified expression a square plus L0 square minus two a L0 cos phi equals, okay. Now, I want to make this left hand side and right hand side equal to each other, for all values of phi. In other words, if I am able to choose this value of kappa in such a way that this expression becomes independent of phi.

Then I am alright, because if that becomes independent of phi, then V of P on grounded cylinder will not depend on phi, which is what I want to find out. To do that one, let us go ahead and equate the coefficient of this cos phi term, okay. So if I equate the coefficient of the cos phi term.

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$$2aL_0 = 2a\kappa^2 l \qquad \kappa = \sqrt{L_0/l}$$

$$a^2/L_0 = l \Rightarrow l = a^2/L_0$$ distance from 0

$-\rho_L$

What I get is two a L0 equals two a kappa square l, I might have made, there is a a here, so there is a a here. So two a cancels with each other, there is already an a here, so sorry, to a kappa square L. So two a cancels from both sides, leaving behind kappa as square root of L0 by l, okay. Now, kappa is square root of L0 by L, I also know, or can actually find out what is the relationship for a, by substituting this expression for kappa in the previous expression.

So if substitute the expression in the previous expression, what I get is a square L0 equals l, you can show this one, which implies that l is equal to a square by L0. This is the distance that we wanted, this is the distance from the origin, okay, where you will keep your infinite line charge of value minus rho L, that is of charge density minus rho L.

So this was what we were looking for, and at any other point you can find out what is the total potential, okay. So we have solved the problem in the sense that we know where to place the infinite line charge of charge density minus rho L, okay. So we can place that charge and potential at any point in the cylindrical coordinate system R phi and Z, okay is obtained by substituting the appropriate values for the length.

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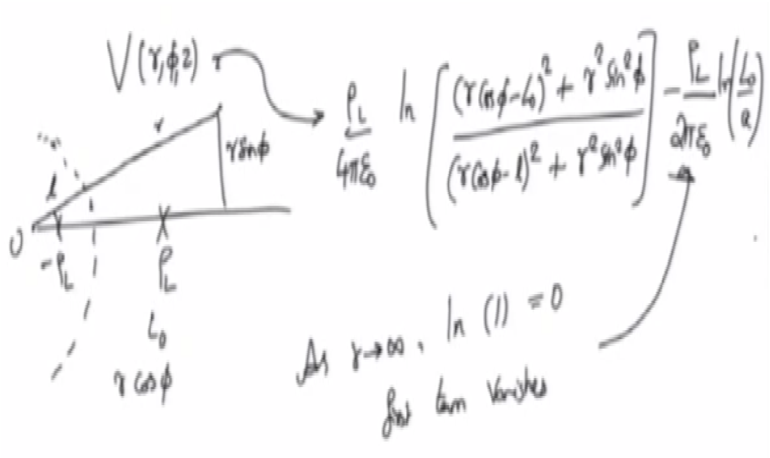

So if the positive line charge is at L0 on the x axis, now the negative line charge is at minus rho L, note that I have removed the conducting cylinder that is not really required now, but my solution has to be extended or valid only for the region greater than a, okay. So this is where the original cylinder was located, the grounded cylinder was located, okay. This is the origin.

This length L also I know, so at any point, if I consider r phi Z, I can actually find out what is the potential, okay. So from the origin this distance is r, so if you drop the perpendicular you will find that the vertical distance is r sin phi, and the horizontal distance is r cos phi, and therefore, the potential at r phi Z, which is this fellow is equal to, or it is proportional to log of r cos phi minus L0, because in this case r cos phi is greater than L0.

But it is really, it does not matter, if you bring the point close to the cylinder also, right. So the expressions would remain essentially the same, divided by r minus, right, so r minus is r cos phi minus L square plus r square sin square phi, this entire thing there is a square root and square root. There was a rho L by two pi epsilon here, so this was rho L by two pi epsilon, but because there is the square root for r plus and r minus.

That square root can be brought outside of the log, so this becomes rho L by four pi epsilon plus the constant V0. Remember what the constant V0 was, constant Vo was rho L by two pi epsilon zero, or yeah we have assumed epsilon zero here. So log of L0 by a, why L0 by a, that is because it was the constant kappa. And kappa is related to L0 and a in this expression, right, kappa is given by L0 by l.

So L itself is given by a square by L0, so the square root cancels with each other, and you will get kappa as L0 by a. And because of that square root factor you are left with this rho L by two pi epsilon zero log of L0 by a, okay. Now, this is interesting, because as R goes to infinity, what happens to this. Second term of course is just a constant, the second term remains finite, it does not go to zero.

What happens to the term, the first term over here, as R goes to infinity both the numerator and the denominator will approach infinity, and the ratio approaches one, and we know log of one, right, is equal to zero. And therefore, the first term vanishes, okay. The first term vanishes, but the second term is there, which will give the constant potential. So this is the uniqueness theorem with a restriction that the two solutions earlier were not differing at all.

But, now the two solutions, they are different in terms of only the constant. However, if you want to find out the potential, that constant does not really matter, right, because potentials, by themselves do not matter physically. What matters is only the potential difference. So even if you include this overall constant, that does not really change anything, because when you try to.

When you take the gradient of this potential, the derivative of the constant will actually goes to zero and for electro static case, the fields can be obtained without regard to this constant. So whether you include the constant or do not include the constant, depends on what further action you are going to take. If it is for the potential, you need to include the constant so that the numerical values turn out, alright.

However, for the electro static field calculation, the constant value of the potential any way just drops out, okay. So this is as far as we go with method of images, there are lots of other interesting method of images, which can be applied, but we will not go into those details.