Electromagnetic Theory
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Lecture No 43
Inductor and calculation of Inductance for different shapes (contd)

So, one can also find similar examples of inductance calculations for toroid. We are not going to do that one.

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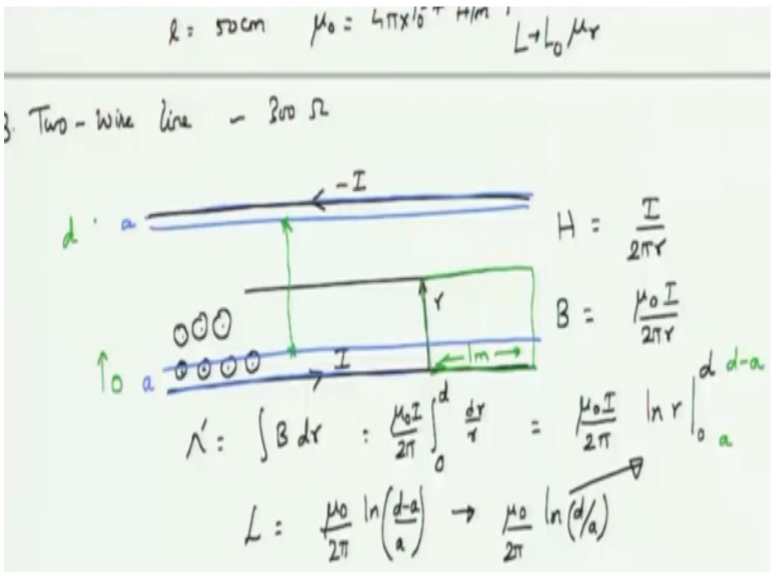

Instead go to a different example, which is very interesting because this was one of the commonly used transmission line or a cable that about 15, 20 years ago people were using everywhere to connect their television sets to the antenna and this called a two-wire line. This was around 300 Ohm line that was normally used. With two-wire line we actually have two wires, long wires which are carrying certain currents.

So, you have a current 'I' and then you have a current '-I'. That is current in one path and the return current in the other path. Now I want to find out the magnetic flux linking. So, I take this current which is along positive 'I' to calculate the magnetic field because of that and then I will do the appropriate integration to find out what is the flux. First let us write down the magnetic field, magnetic field is of course along the Phi direction.

But for this plane that I am considering I do not really have to talk about Phi. Because this distance is r and then the magnetic field at this particular point is given by H is I / 2 Pi r where r is this plane. Although we know that, these are actually along the Phi direction. But for this plane that I have cut away, this is enough for me I / 2 Pi r. If you really want to talk about the magnetic field you actually have to insert this Phi direction everywhere.

So, the magnetic field would be around at all points coming out of this particular page or particular screen. So magnetic field is I / 2 Pi r and I can find out what is B field. B is Mu 0 I / 2 Pi r, assuming that the Two - wire line is actually kept in air. So, this B will be Mu 0 I / 2 Pi r and the magnetic flux will be integral of B along this radial direction. I need to again consider the similar thing.

So, I have to consider one meter around the Z axis if I am considering the current to be along the Z axis and then whatever the value of r that can go from here. So, if I take this as the origin, then this would be zero and let us say the separation here is d. So my r would seem to indicate to go from 0 to d. So, (()) (02.44) those values B dr. This would be again the flux linkage per unit length. I am removing the integration with respect to Z here.

So, this will be given by Mu 0 I / 2 Pi is constant, so let me pull this outside and put the integration limits of 0 to d 1 / r and dr. Now clearly we run into this problem. See, we will run into a problem here because if I integrate this, I know that integration of 1 / R is log r and if I try to apply the limit of log to 0. Try to find out what is log 0 that is undefined. So, it turns out that I am actually running into problem if I take r is equal to 0.

So what is actually happening? Is the mathematics incorrect? Did we actually set up the problem wrongly or mathematics is correct? It is giving you what it is supposed to give except that we have forgotten a very important characteristic of wires. No wire will have a zero cross section. Wires actually have a finite cross section. So, because of that the proper way to have written down this two - wire line would have to be like this.

With let us say a cross sectional area of ‘a’ and this line will also have an equal cross section let us assume that and therefore my integration should have begun from this inner conductor to this inner conductor. Between that two conductors is what I am supposed to find the flux through. So, the integration limits must be changed from 0 to ‘a’ and from d to d - a. And if I do that I will get the exact expression.

So, and if I also take this current ‘I’ and divide it on to the left hand side I get the inductance per unit length as Mu 0 / 2 Pi log of (d - a / a). And in case ‘d’ happens to be much larger than ‘a’, the separation happens to be much larger than the cross section, which is almost what happens in a practical two - wire line. This can be approximately written as Mu 0 / 2 Pi into log of d / a. A slightly better result can be obtained if I replace this (d – a) by the geometric mean of these two distances.

So, I can do that but I will get a small improvement not really worth for it. Now you might have for the coaxial cable that we consider, we did not actually talk about finite cross-section at all. There was a finite cross section for the inner conductor. The inner conductor was having a radius ‘a’ and that was the finite conductor that we actually had to take into account. Similarly, the outer conductor of the coaxial cables will also not be at 0.

It will also zero cross section, but it will also have a certain finite cross section. Then why have we neglected those effects in the coaxial cables. Strictly speaking whatever we derived for the coaxial cable is wrong, because we have neglected the cross section. And I should not have been neglecting those cross sections. However, there is a very good reason why we have done so. When you look at this coaxial cable, this is normally operated although at low frequencies it is operated.

At high frequencies you know a few megahertz or few hundreds of megahertz when you are operating, current does not flow in the entire conducting surface. Current actually gets concentrated or pushed to the walls. And a very very thin layer called as skin depth. This skin depth actually starts to become very small as the frequency increases and if you know operate this coaxial cable at around 100 or 200 megahertz you will see that the current is actually confined to a thin walled area around the inner conductor.

And similarly the current will be confined to outer conductor on a very thin layer. This layer is around few micro meters and therefore there is not much that this result can be affected. So for coaxial cable we did not really include that term. But strictly speaking one has to include, but when you try to include that you will run into a problem of partial fluxes, just as you have run in to a problem of partial fluxes here. But we have not really talked about it.

There will be partial flux problem and that would be mathematically slightly tricky to take up into account. And that is the reason why we neglected for a coaxial cable. What happens for that one? Whereas a two wire line is normally not used at those high frequencies. Alright, so we have talked about finite cross sections and partial flux has been mentioned. But we have not really talked about what a partial flux is.

It is perhaps interesting for us to take that partial flux and discuss it a little bit more in the context of two-wire line. So, we are going to do that now. So, we will discuss partial flux now. To do that one we need to go back and understand that filaments that we have assumed or conductors that we have assumed which had zero cross section cannot really be true. And we have to take into account this finite cross section of the conductors. How do we do that one?

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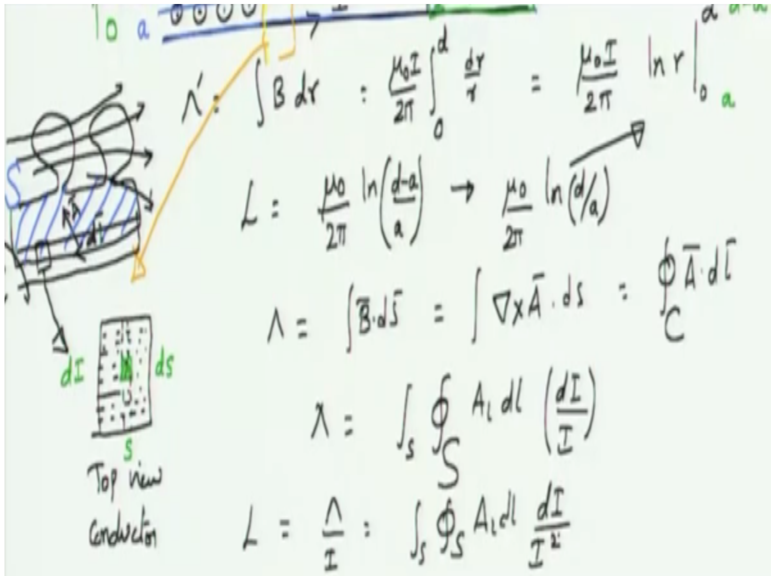

Well, if you take open one of the filaments over here. So, let me try an isolate filament over here. So, if I try to take the filament over here and then look at it in a microscope or in a very exaggerated way what I would actually see is that this filament actually has a finite cross section. So, this is what from the top view for example. This is the top view of the conductor that we are looking at and I am assuming square cross section just for simplicity.

Although this is actually a circular cross section that I should have assumed. But nevertheless if you zoom in on to a circle with a very large zooming factor it would essentially resemble a square. And that is what I am looking at. Now within this, one can actually imagine this entire thing is a conductor. //ok//. This is not hollow space. This entire thing is a conductor. But I can actually //kind of// imagine this to be composed of multiple filaments.

And over a particularly small segment that I am interested in, the amount of current that is carried by this would be just a small amount of the current compared to the overall current ‘I’. See if the entire cross section ‘s’, a small ‘s’ carries the current then the current carried by this differential element will be dI. It could be only carrying a fraction of the total current. And this small section within this cross-section ‘s’ has a cross-sectional area of ds.

And then if you want to obtain the total current and the total cross-section and to do that one you want to simply sum all the contributions or essentially integrate the contributions. So, if you can find a way to take into account the fact that conductor has a finite cross section ‘s’ and this finite cross-section ‘s’ can be composed of multiple smaller, smaller segments of cross section ds, each carrying a current dI then we will be able to talk about partial flux or flux that is linking with conductors of finite cross-section.

To do that one, let us actually review what magnetic flux linkage was, the magnetic flux linkage was integral of B.ds. The magnetic flux integrated over ds. Now B can be related to A, the magnetic vector potential and I can replace B with Del cross A and replace this Del cross A. ds, the surface integral with a line integral over vector potential A.

This is one of the places, where vector potential is really helpful in dealing with calculations rather than the direct magnetic field itself. So this A dot dl is what I have, but where is this A. dl getting integrated into it. It is actually getting integrated over the entire circuit. So, if for example I have this circuit. So, in between I have to make some space. If I have a circuit with very small segment here, call this as dl and there will be a magnetic vector potential A.

And this is the filament that I have actually considered and expanded over here. So for this loop, this loop actually forms a certain surface as well. So, the B field that would be linking has to be found over this surface. This gaps being very small I can neglect them. So, I can also define this entire surface as capital S. This is the region over which I have to integrate the magnetic field or this would be the periphery over, which I have to integrate the line integral of magnetic vector potential A.

So, why have I written like this? So let us say there is some current flowing then this current would actually be linking the second circuit. So, this would be the current that would be linking. There could be some magnetic field that would be linking; there would be some magnetic field that could be linking over here. And this is this magnetic field that we are actually interested in. So, if we want to calculate the inductance of this loop, the square loop or the rectangular loop I have to find out how many flux lines are actually linking to this loop.

So for that reason, I have to consider this big surface S and I have to consider the corresponding contour C. So, this is the integration that I am actually performing. Now this integration that we have, which I have shown is actually applicable to only one filament. So you imagine that the filament actually has a cross section that is not 0. And there are a lot of filaments inside. Each filament there will be a line integral A dot dl which is going over the entire closed circuit C.

And then you are now summing up those over the cross section 's'. So there are two integrations going to happen. One integration will be integration over the big surface S or equivalently over the big contour C. And then there will be a second integration over the cross-section itself. Because you have to sum all the contributions of the filaments inside there. Therefore, the magnetic flux properly for wires with finite cross section will involve two integrations.

One integration over the smaller 's', which is the cross section of the wire and then the integral of A dot dl. Now instead of writing this over the contour C, I am going to write this one over the big surface S just to indicate that this surface S is the Periphery surface. The surface bounded by the peripheries of the wire that I am considering. Remember that this capital S is actually bounded by this capital C (()) (14:06).

And I can write this as A L. Where A L is the component of the vector potential along the line integral dl. So I can write this as A L and there is dl and there will be cross sectional area also. So, the cross-sectional area actually carries a current of dI, a fractional current of dI / I. Now I can write down what is L here. The inductance, inductance will be Lambda / I. So, I multiply by 'I' here.

And then I get two integrations smaller integration over cross section, larger integration over the loop and then A L dl dI / I square. So, it could be dI / I square. So, this is the expression for inductance that we are looking for.

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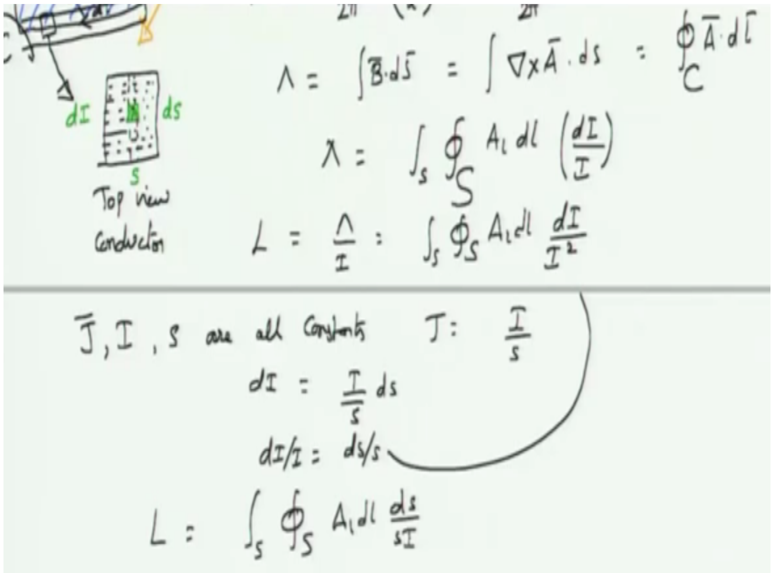

However, this expression can be simplified slightly, if you assume that the current density ‘J’ and the current ‘I’ as well as the cross section are all constants. J of course is the one that is coming out of this particular conductor and J is given by I / s. The current density is given by I / s. Therefore, dI by, the fractional current that is carried by the element of cross section ds will be dI which is I / s the current density times the cross section ds.

Therefore, I can rewrite dI / I as ds /s. And substitute that into this expression. Into the expression for L. And when you do that you are going to get L is equal to integral of s integral of the capital S A L dl and ds / s into I. This is the expression that we are looking for the inductance. And we can use this expression for the two-wire line.

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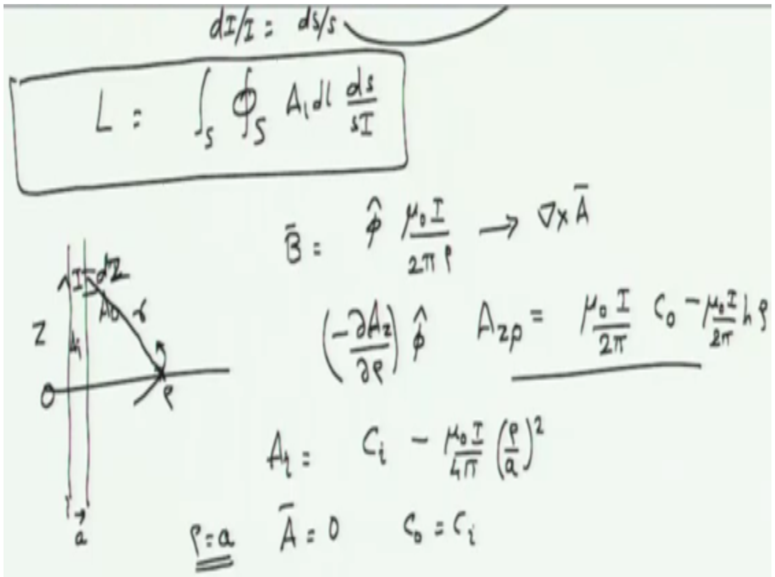

We will do that one. But before that I need to consider the vector potential for a line carrying a current ‘I’ of finite cross section. And if I know that one I can use that value for the vector potential and calculate the inductance of a two-wire line. So, the first job would be to try and find the vector potential. This vector potential clearly will have two components. Because this would be the conducting wire that we are considering of cross section are of radius ‘a’.

There will be magnetic vector potential inside and magnetic vector potential outside. So, outside the current enclosed is ‘I’ and you can actually write down the expression for magnetic vector potential A 0 or you can start with the magnetic field B and then calculate what is A 0. So, outside B is actually given by Mu 0 I / 2 Pi r. If you are at a particular distance r over here or let us say I want to use r for a different reason, so I am going to write this as Mu I / 2 Pi Rho.

So, I am actually at a distance of Rho here. And I am actually considering the current distribution at this particular point. Sorry, the line element at this particular point at a height Z above the horizontal plane. And the distance between the current elements to this field point will be small ‘r’. This is my notation for this particular problem. So, the magnetic field around this will actually be along Phi direction and this would be given by Mu 0 I / 2 Pi Rho.

From this you can actually calculate what is curl A. Curl A should obviously have only the Phi component. So, it does not have any other component and you will actually see that the Phi component is - Del A z / Del Rho along Phi. So, from this you can actually integrate for z and you get A z of Mu 0 I / 2 Pi C 0 some constant - Mu 0 I / 2 Pi log of Rho.

So let me not show the derivation over here, because as such we want to finish the calculation for the inductance but I will leave this as an exercise and I guide you how to do this step by step in the exercise. You can actually show that the magnetic vector potential outside is actually given by this Mu 0 I / 2 Pi some constant C 0 - Mu 0 I / 2 Pi log of R. So, clearly this constant C 0 one has to choose carefully, so that there is a proper zero reference for the potential.

And you cannot choose infinity as the potential because, if you choose infinity as the potential, potential everywhere will also become infinity. That is infinity point cannot be chosen as a reference for zero potential here. So, this is one result you keep in mind. Similarly, you can actually find out what is the internal A i that is the vector potential inside the region and this turns out to be some constant C i - Mu 0 I / 4 Pi Rho / a whole square.

So, if you are considering inside then this would be Rho / a whole square. The reason why you get a Rho / a square is because the current density inside the conductor is actually proportional to Rho square / a square. Again this will be developed during the exercises, so do not worry if you are not able to get this one at this point. You will have the opportunity to derive this expression for yourself.

All we need to do is to select an appropriate value for reference potential and you can choose any circle value or any value of rho as reference potential. And if you choose rho equals to A, that would give you the one possible solution, that is a simplified solution, where I am setting the reference for A to zero. So if I said rho equal to A and take that as a reference for A is equal to zero, I have to choose C zero is equal to C I, so that I actually maintain the continuity of vector potential.

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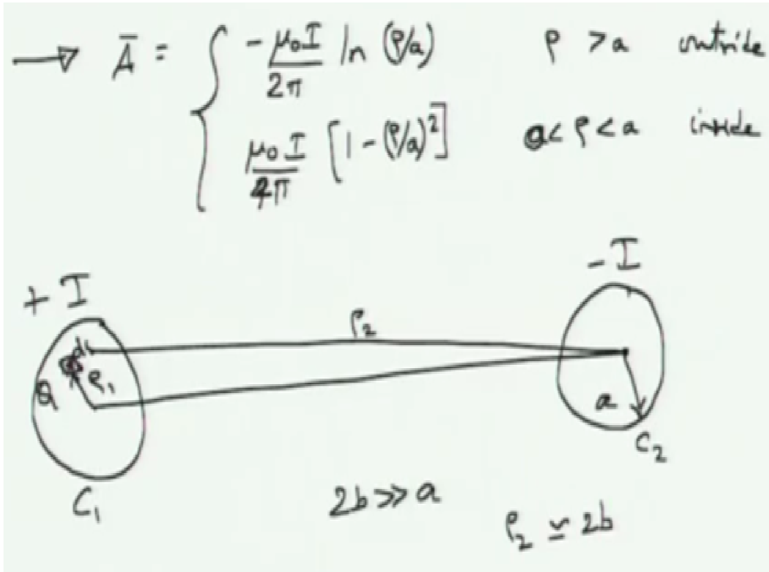

So if I do that one, then if I remove this one, you will see that the magnetic vector potential can be written as minus mu zero I/2pi log of rho/a, for rho outside a. So this is the side potential. And for inside, you have mu zero I/4pi, sorry not 2pi this is 4pi, one minus rho by ‘a’ square. This would be for ‘a’ less, sorry zero less than ‘a’, that is inside. So, inside the vector potential is positive and then goes to zero at rho equal to ‘a’ and then goes to negative value.

Now with these expressions, we are ready to calculate the inductance of a two-wire line. For simplicity, let us assume that their center to center spacing 2b, is much larger than their radius ‘a’. Each of them have a same radius ‘a’. And now I want to calculate, and this left cylinder carries a current + I and the other current carries - I because one current is going and other current is returning.

Now, the flux that is linking to this left one, which we call as C 1 and the right one, which we call as C 2, actually has two components. One because there is an internal ‘a’ field internal vector potential A I and there is an external potential A 0 because of the circuit C 2. So the contribution at any point or any cross sectional area of this one that you consider, will have two components, one internal A field and an external A field.

So this fellow letters located at a distance of rho two. But, because we have assumed that center to center spacing is much larger than ‘a’, rho two is approximately 2b. And to find out what is the vector potential that is appearing at this point, where we call this as point Q. So at this point Q, rho two is 2b and you can actually obtain this by substituting for 2b in this expression for rho. For the vector potential A, if you see there is factor of rho there and you substitute rho is equal to 2b and take I to minus I.

So, that will give you the vector potential A acting at this particular point. Now, what is the direction for line element? Line element direction is along dz. Again in the previous for that two-wire line case, where we assumed that integration along the z axis is one meters, we will do the same thing here and therefore we do not really have to integrate over the surface 's' the larger surface 's', because I am considering flux linkage per meter.

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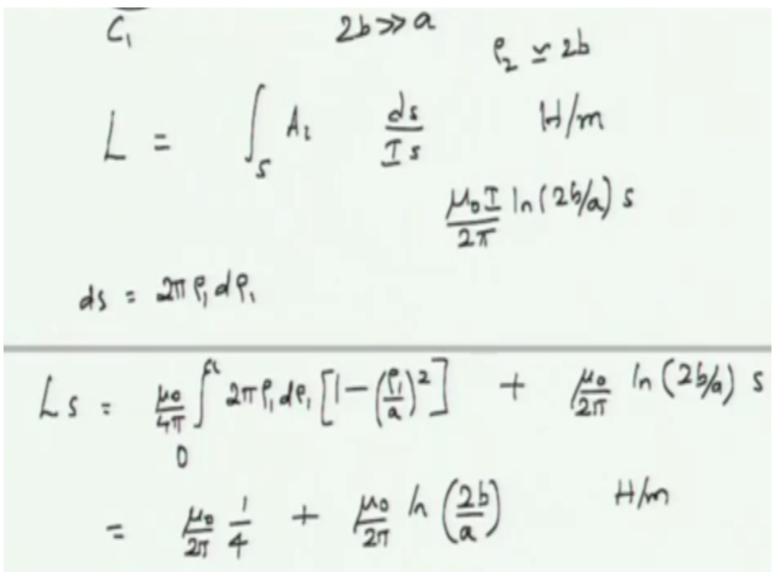

So therefore, the integral that I have to evaluate for L will be small integral only. Because you can imagine that this is one filament, one filament, one filament, one filament and these are all different filaments that I can imagine each having a cross section of ds, kept at these different points. Then integrate this one A L dL, d small s/Is is what I have. So this is the integral that I am going to evaluate.

So as I said, I have removed the outer integral because that would be only integration along z axis and we are going to consider integration for one meters. So, if I do that one and then push this, when I remove that integral, the dL will also go, so only I am left with AL ds/Is. And this inductance will be in henry/meter. So this is the induction that I need to calculate. If I substitute for AL, this integral splits into two parts. For the second term that is contribution from the external, from the circuit C 2, the contribution will be mu zero I/2pi log of 2b/a and this would essentially be constant.

Multiplied by the cross sectional area ‘s’ will be the value for this second term. As whereas the first term is concerned, I have to consider A internal that would be given by this expression mu zero I/4pi, one minus rho/a square and then integrate over that cross section. And for that, the cross sectional is actually proportional or given by, so this is for the cross section ds, is given by 2pi rho 1 d rho 1. Because this is the inside integration that I am going to do.

And this, here I cannot neglect by taking rho one is equal to A. So I carry out this integration over the first term, so that I obtain the total integral. I take this small ‘s’ to the left hand side and multiply by that one. So I am left out by just integrating AL ds/I. And since there is a denominator ‘I’ and numerator ‘I’ in both the terms that I will cancel out each other. So I actually, I am left with zero to A, there is 2pi rho 1 d rho 1.

What is the field inside? Field is mu zero/4pi minus rho 1/a square, plus the second term. The second term is simply mu zero/2pi, ‘I’ cancels out, log of 2b/a * s. That is the area, cross sectional area that I have already integrated. So, if you evaluate this left hand integral, you will actually see that, this will be equal to one by four mu zero/2pi. From mu zero/4pi, it becomes mu zero/2pi.

And then there is one by four, plus the second term, which is mu zero/2pi log of 2b/a, henry/meter. This is the inductance of this parallel two-wire line with finite cross sectional areas.
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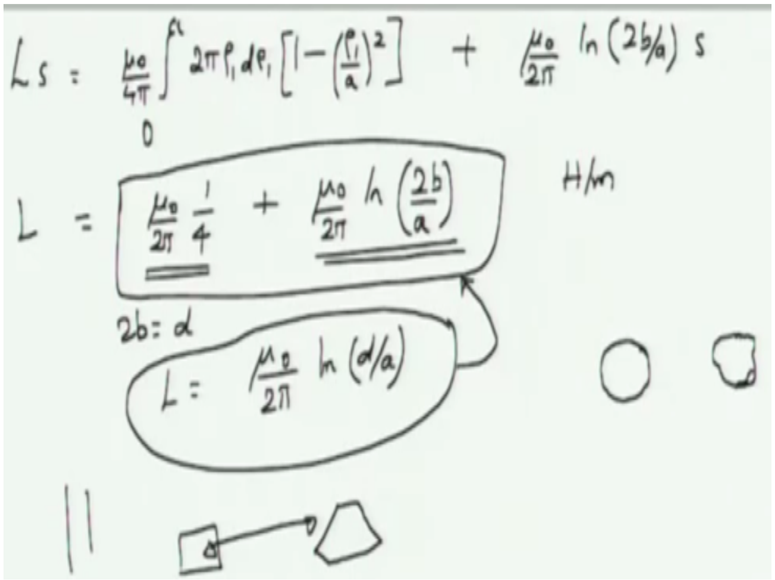

So there are two terms, one term because of the internal A field and the other term because of the external field. And in the two-wire line that we actually calculated, we took 2b is equal to'd' and then this is essentially the expression that we obtained. So you actually had, if you go back and check, the expression that we obtained for the inductance/unit length was actually mu zero/2pi log of d/a, where d was 2b.

So this is how we obtained the inductance for a two-wire line, which is a simpler case than that of the inductance that you can obtain from a slightly more sophisticated value. If, this has actually given you an idea that the analytical expressions are sufficient for us to calculate inductance for all these different geometries. You should be sadly mistaken because if I change the problem from circular conductor to a square conductor or may be a slightly different sized conductor.

You will be stumped, because no amount of analytical calculations can be applied to these weird cross sections, although it may not seem to be so weird. Because, when you fabricate a two-wire line, you would expect a nice circular cross section. But what you would actually get is some sort of a deformed cross section. And if you want to calculate exactly an analytical expression, then you would not be able to do that and people have not been able to do that.

Because that requires a lot of complexity, that is lot of complexity and there is no guarantee that if I change the cross section slightly, that result will hold. So there is no general formula for a general cross section. That is what I am trying to tell you. And secondly, this also is the reason why one has to go to numerical method. We have been talking about numerical methods, but we will introduce numerical methods in different module.

There you will calculate magnetic field numerically, from there calculate inductance numerically. So however, you have to understand where one can put in some approximations, to obtain quickly a value for inductance. And then, if you want to obtain qualitatively better answers, quantitatively better answers, then you have to imply numerical methods or solve the complete electromagnetic problem.