Electromagnetic Theory

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Lecture - 48
Wave Propagation

In this module, we will discuss propagation of electromagnetic waves inside perfectly homogeneous linear isotropic dielectric materials. An example of a linear isotropic homogeneous dielectric material is vacuum or free space. What is this electromagnetic wave, and why should we be interested in studying its propagation, and how can Maxwell's equation help us tell how the wave would propagate.

Can Maxwell's equation actually say that, okay this particular process is responsible for generation of the waves, and then also describe its propagation. The answer to both questions is yes. Maxwell's equation, if you solve them under appropriate conditions you can actually show that, under the existence of some time varying currents, right. And which results in accelerated charges, there will be electromagnetic waves generated.

This was predicted by Maxwell's equation, and later experimentally first time verified by Hertz, okay. Sometimes that, these waves are also called Hertizian waves, just to honour the fact that Hertz was the first one who generated electromagnetic waves. Now, we have already seen Faraday's law and modified Ampere's law, right. In Faraday's law, we have curve of electric field equal to whatever the time variation of the magnetic field at a particular point.

So, you had this small region of this space and then the curve of electric field, which could be obtained by putting up an imaginary conducting path, a loop of conductor. The source of that EMF was actually the time variation of magnetic field, right. So, how does magnetic field, how is magnetic field generated, it is generated by time varying current. If the current is not time varying, then it is still possible to generate magnetic fields.

But in that case the magnetic field will be static with respect to time. So if the magnetic field is static with respect to time, then there will be no curl of electric field in any given region of space, there will not be any curl of electric field. Remember Faraday's law is actually, matching two things, one on the left hand side is the space variation of the electric field, in the form of the curl. And the time variation of the magnetic field, right.

So if the magnetic field is not varying with time, because the current was steady and hence the magnetic field was also static. There will not be any space variation of the electric field. Electric field continues to be completely delinked with magnetic field, okay. So, the first rule is that, if you want to produce electromagnetic waves, you want to have time varying magnetic field, at least that is what Faraday's law tells us.

Only when you have time varying magnetic field, you will have curl of electric field, a space varying electric field, right. So a time varying magnetic field is generated by time varying current, that is di by dt must be non-zero. If di by dt must be non-zero, that means, that d square q by dt square must be non-zero, why, because i is dq by dt, current is rate of charge.

So, only when charges are accelerated, there will be time varying magnetic field, which will generate a space varying electric field. Now, if you look at the space varying electric field and then, if that space varying electric field is also varying with respect to time, right, because the charges are actually accelerating, changing with respect to time. So the corresponding electric field would also be varying with time.

So this time varying electric field or time varying electric flux vector d vector, would in turn generate a time of space varying h, correct. This is in the form of displacement current. A displacement current, which is varying with respect to time, will generate a space varying magnetic field, right. So this action is actually couples together, you have time varying magnetic field in a region generating space varying electric field.

This time varying electric field or d field that in case where we are assuming d is proportional to e, time varying electric field will generate a space varying magnetic field. And then this would actually couple together and correspondingly you would actually have propagation. But remember, what is the source of all this. The source of all this is still the chargers, or equivalently the currents, right.

The time varying current is responsible for generating time varying magnetic field. Time varying current automatically implies that you are dealing with time acceleration of chargers, okay. Now, this was a very qualitative picture of how waves are generated, and the further propagation is actually sustained by the two self consistent equations, that is Faraday's law and modified Maxwell's Ampere law, or modified Ampere's law or Maxwell Ampere's law.

So how do we capture all this in terms of a certain equation or a set of equations, that would help us understand these propagation of waves. Furthermore, what is the characteristic of these waves, right. Are these waves exhibiting some sort of a known time dependence do they exhibit a particular spatial mode, like for example you might have seen right, you drop a pebble in the water tank, then the waves would actually all, the surfaces would all go nicely as spherical surfaces, right.

You know all these rings or spherical surfaces, would we actually get only those kind of waves or is there any other kind of waves that are possible, we will see all these answers when we develop the wave equation, okay. So the goal here would be to develop wave equation, okay. To do that let us begin by writing Faraday's law, okay. So we know what is Faraday's law, curl of electric field is equal to minus del B by del t, right.

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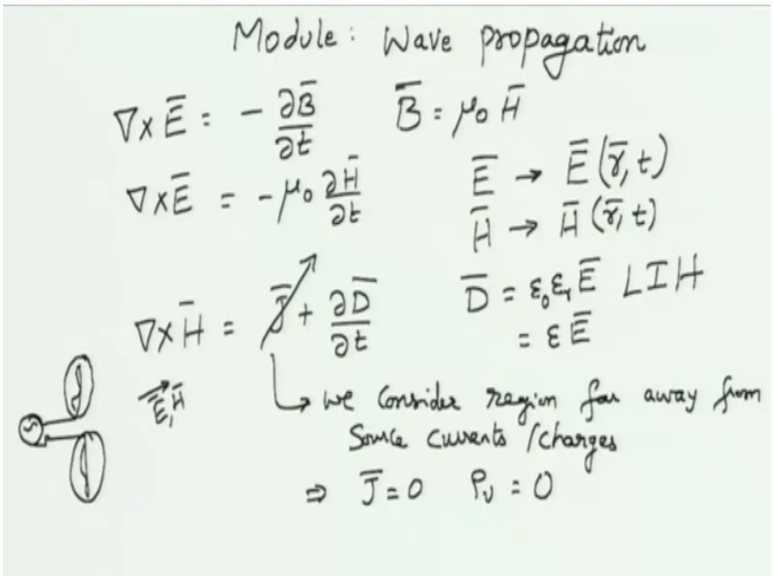

Since we will be considering mostly non-magnetic materials, we have B equals mu zero into H, right. That is the simple relationship between magnetic flux density B and magnetic field H, okay. So we will have this equation, therefore substituting for B equals mu zero H in this Faraday's law, you get minus mu zero del H by del t, which is curl of electric field. What is the nature of electric field and magnetic field, what functions are these?

So electric field in general will be a vector, but it should also be varying with time, right. So we typically denote this part of being a vector, as well as varying with time by saying that electric field is a function of both the space variables, this r if you remember is a position vector, that is the field point that we were considering. So at any point r itself is a position vector, and the result of this position vector is also a vector.

So electric field is actually a time dependant vector field, okay. So this is typically how we denote. If the electric field has to point only along a particular direction and be independent of time, then we will have appropriate substituted expressions for that one, okay. But for general case electric field will be function of both space variables as well as time variable, okay. Similarly, for magnetic field H as well, H will also be function of r and t.

Alright, so this was Faraday's law, what about Maxwell Ampere law. Maxwell Ampere law is curl H is equal to J plus del D by del t, okay. We will assume linear isotropic homogeneous media, for which D will be epsilon zero epsilon r into E. Now, instead of writing epsilon zero into epsilon r, I sometimes shorten this and simply write this as epsilon E, okay. Again remember D is D of rt, and E is E of rt, epsilon is a constant.

It includes the free space permittivity epsilon zero. If the medium that we are considering happens to be a dielectric with epsilon r greater than one, then it would be epsilon zero into epsilon r. What about J here, J is actually the current that we have, okay. So if you, you will have to wait for antennas to really understand how waves are generated. But with an antenna what happens is that, you take a particular conductor, okay.

And then connect a time varying function generator, you know, function generator which will change the voltage here, and then this is an antenna, a typical antenna, and there will be currents which are induced in this antenna. And these time varying currents are responsible for generation of electric and magnetic fields, okay. So this is the wave that are generated, and the J, current density vector J is actually referring to this particular current, okay.

It is actually what is sometimes called as the source current, okay. It is the current at the antenna surface, okay. But what we will be assuming is that, we are very far away from this region, okay. Note out, currents are there, currents are the reason why we have magnetic and electric fields and hence the electromagnetic waves. But, we will be considering the electromagnetic field behaviour or electromagnetic wave behaviour at a very, very far away distance.

Because of that, I can safely turn of J, okay. Only because, we consider region far away from source currents, and any charges that we have. So in the region, we are considering there are no free charges, there are no free currents, okay. This implies that J is equal to zero and for future reference rho v is also equal to zero, okay. So these two equations are sufficient for us to show that electromagnetic waves are generated.

And to do that one let us write these equations once again, and then let me show how waves are generated.

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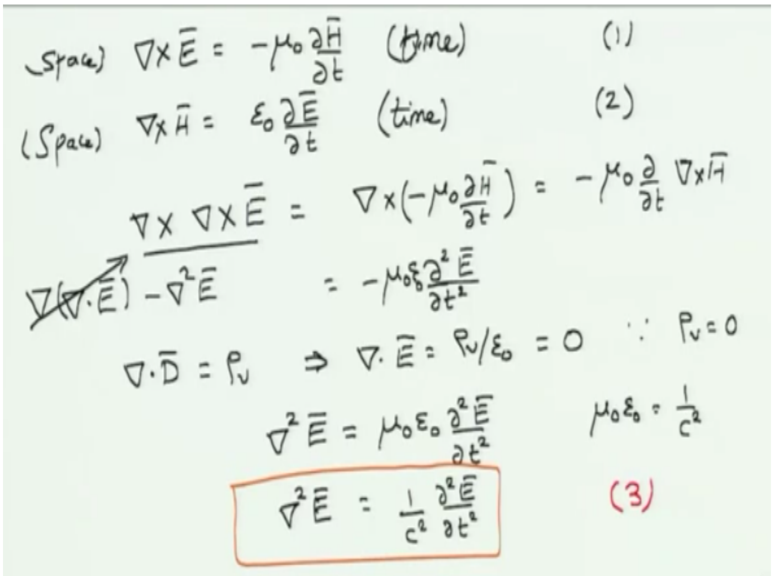

So I have curl of electric field equals minus mu zero del H by del t, curl of H is equal to epsilon del E by del t, okay. If it is free space that we are considering, let us consider free space for now, epsilon becomes simply epsilon zero, okay. In case epsilon is not epsilon zero, then you will have to replace epsilon zero by epsilon zero into epsilon r, no big deal, okay. So you can see hear that on one side I have space variation of a vector field.

In this it is the electric field which is varying with space, and here it is the time variation of a vector field, which is magnetic field H, right. Similarly, you have the space variation of the magnetic field linking to the time variation of the electric field, okay. Now, I can write down a wave equation in electric field E, or magnetic field H, okay. Customary to choose electric field E, okay, we have some sort of an understanding with electric field E.

So we choose electric field E to write down the wave equation. You can follow a similar process to write down the wave equation for magnetic field, okay. So, how do we write down to the wave equation, well I take the curl of this first equation. So let us call this as one, call this as two, okay. One corresponds to Faraday's law, so I have curl of electric field is equal to curl of minus mu zero del H by del t.

Now I can take curl inside this bracket, and interchange del by del t and curl operations. So I get minus mu zero del by del t curl of H. But, I know what is curl of H from equation two, I know what that is. So that is equal to minus mu zero del by del t of epsilon zero, epsilon zero being a constant can be pulled out, so epsilon zero comes out. And then I have del by del t of del E by del t, which becomes del square E by del t square, right.

So this is what I have, this is the right hand side of this expression, right. But, on the left hand side I still have this very weird looking expression, okay. And this weird looking expression can be simplified slightly, if you adopt Cartesian coordinate systems, okay. So by using the vector identity, I can actually write down this left hand side as del of del dot E, okay. This operation makes sense, because del dot E is a scalar, but gradient of a scalar will be a vector.

So we are alright, minus del square E, okay. Now, is there any way to further simplify this equation, we have assumed that there are no free charges in the region we are considering the wave propagation. So del dot D is equal to rho v, and D is equal to epsilon zero into E. So this implies, del dot E is equal to rho v by epsilon zero. But this is actually equal to zero, since rho v is equal to zero.

So we have no charges, no free charges in the region, where waves are propagating. Therefore, divergence of electric field will be equal to zero. So I can put the divergence of electric field equal to zero, and gradient of zero is also zero. So this term can be removed, okay. So I removed the term, so what is that I am left out with is del square E is equal to, after cancelling the negative signs on both left and right hand sides.

I get mu zero epsilon zero del square E by del t square. It turns out that if we actually calculate the value of mu zero into epsilon zero, this will be equal to one by C square, where C is the velocity of light, speed of light, okay, in free space.

So you can write this one as, instead of writing like this you can write this as one by C square del square E by del t square, okay. This expression, which we have just obtained under certain assumptions, which are all very reasonable is called a wave equation, okay, wave equation for electric field, okay. So this is the expression that we were looking for. Let us just quickly recap what really happened over here, okay.

We were considering electromagnetic wave that are generated by time varying current, okay. Time varying currents will generate time varying H field, okay, which in turn generates time varying E field, which in turn generates H field and so on, and then we essentially get a wave. And this wave would propagate, okay, and we are looking at the region, which is very far away from where the sources are located.

It is like these currents, the charge varying with time or the source currents are placed at some far away point in the universe. And wherever I am looking at the propagation of the waves, there are no free charges. I am also assuming that the entire medium is surrounded by or completely filled by linear isotropic and homogeneous dielectric medium. For which, D and E are very easily simply related by a scalar epsilon.

For free space epsilon is equal to epsilon zero. If it is not free space, then it will be slightly different. So we begin with Faraday's law, curl of electric field, okay, and we also wrote down the modified Ampere's law curl of H is equal to J plus del D by del t. We quickly realise that, J was equal to zero, because we are again far away from the region of source currents and charges.

And essentially obtained two equations one and two, which are coupled equations, okay, which couple electric field and magnetic field, okay. These coupled equations, themselves are sufficient to show that waves are generated and propagating, okay. These are the ones which govern the propagation. However, in most cases we are interested in trying to solve these equations. So to solve this equation, we would actually write down them in a slightly different fashion.

And that different fashion is obtained by taking the curl of electric field, okay. And then substituting the second equation into the first equation and then manipulating slightly a bit to arrive at this equation, okay. The one which I have boxed and labelled as three. In this equation the right hand side is quite familiar to us, right. I mean it is just the time variation of electric field, it is just the second order time derivative of electric field, okay.

So there is no surprise out there. The left hand side is still not completely familiar to us, right.

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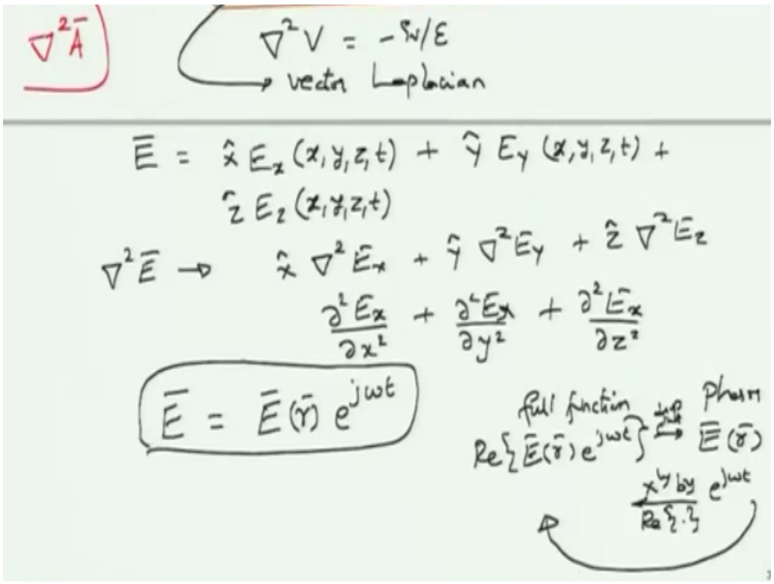

We have seen probably this in one of the examples where we considered del square A, where A was the magnetic vector potential. But other than that we have not really seen what is this del square E is. This del square E is called as the vector Laplacian, because you remember del square should remind you definitely of Laplacian. But in that case the Laplacian was being applied on a scalar, del square V is equal to minus rho V by epsilon.

This was Poisson's equation, where V was a scalar, del square was the Laplacian operator. But in this case I do not have a scalar, I actually have a vector, okay. Therefore, this becomes, or this is what is called as vector Laplacian, okay. Now this is very, very important, this is a vector Laplacian, not the scalar Laplacian. If you expand this vector Laplacian at least in Cartesian coordinate systems.

You will get electric field itself will be X component, which could be varying with respect to x, y, z and t plus the Y component of the electric field, which could be function of x, y, z and t and Z component of the electric field, which would again be varying with respect to x, y, z and t.

This is the electric field E, so when you apply del square E, what you are actually doing is you are applying del square Ex X hat plus Y hat del square Ey plus Z hat del square Ez, okay, where Ex is of course, Ex of x, y, z and t. I just suppressed that x, y, z dependence in this particular expression, okay. So this is the meaning of vector Laplacian, and such a simple expression exists only because we are working with Cartesian coordinate systems.

If it was a different coordinate system, then you could not have written down the vector Laplacian like this. You should actually has gone back to this original definition, curl of curl of E is equal to del of del dot E minus del square E, and then in case you are lucky that del dot E was equal to zero. You could remove this from the equation, and then you would have written del square E.

The vector Laplacian will actually be equal to minus curl of curl of E, okay. This is the case for spherical and cylindrical coordinate systems. Thankfully, you do not have to sit and evaluate all these vector Laplacians. Those expressions are available in most text books, okay, and in internet, okay. So you can just pick of those values from, whenever you are needed, okay. Whenever you require that, you can just pick them off from literature, okay.

And just to emphasize that this is not a simple matter, you have del square of Ex itself giving you del square Ex by del x square plus del square Ex by del y square plus del square Ex by del z square. So in general you have three components of electric field and all these three components of electric field could be functions of x, y, z and t, okay. What you actually have is a very, very complicated thing in your hand, complicated equation in your hand.

So this is what I have wanted to talk to you about the wave equation. Now, I would like to consider some special cases, okay and try to simplify this wave equation, so that we understand how to solve them, okay. One of things that we can say is that well, I know that these equations are linear and one of the great advantages of the linear system that is described by linear set of equations is that.

If I know its response to one frequency or if I know the response to a particular frequency and I can actually build up the response to any other type of excitation, right. This is called Fourier analysis and Fourier synthesis, right. So you take an RC circuit. There are many ways of analyzing an RC circuit to a square input. One of the simplest way is to recognize that RC circuit is a linear system. Therefore, I can take its transfer function.

Transfer function is simply the ratio of output Fourier transform to the input Fourier transform, right, but what the Fourier transform is? Fourier transform is basically how, if you start giving different frequencies, different sinusoidal signals of different frequencies to the RC signal, what would be its response, right.

A linear system would always excite itself, I mean, where linear system when it is excited by a sinusoidal signal of frequency, omega, will also respond with the same frequency, but may be with different amplitude and phase and you can actually put together all these amplitude and phase to form the transfer function. The moment you have transfer function, you can use the transfer function to obtain the response of this linear circuit or linear system to any kind of behaviour, different kind of input function that you want, okay.

So, therefore recognizing that linear system advantage in our hand over here, we have a linear set of equations, it is possible for me to consider that sinusoidal excitation of the waves, that is, I assume that electromagnetic waves are sinusoidally varying with respect to frequency and then start putting them together to form the response of them to any other function of time, okay.

So in view of whatever that I told you just now, we will assume that electric field actually is a function of these variables are, that is perfectly alright, but in terms of its time dependence it would be function of e to the power j omega t. Now there is a reason why I am choosing e power j omega t and not sine omega t, they are actually equivalent to each other, but this kind of expression is quite common when we analyze what is called as phasors.

For that, in fact, we have talked about phasors in Maxwell's equation and we showed how the Maxwell's equation would become, how Maxwell's equation would reduce in phasor form and we are going to follow that idea over here, okay. If you at any particular point of time, want to know what is the full function, just multiply the phasor with e power j omega t and then take the real part of it, right.

So you want to go from a full function to a phasor, okay, so lets us say the full function is basically f r e power j omega t that would be the full function, okay, or the real part of it. The phasor form of this will be efr, okay. You have just dropped e power j omega t and dropped the real part, okay. If I am given a phasor, I need to multiply by e power j omega t and then take the real part of it, okay.

So I can then go from phasor to the full time-dependent function, okay. If I want to go from time-dependent form of expression to phasor form, I need to drop e power j omega t, okay. So phasor, in fact you might have seen, might actually be familiar to you, those are the complex numbers that are used in analysis of steady state circuits, right, so you might have seen those phasors and this is essentially the phasors that we are looking at.

So we have electric field efr being a complex vector, okay. Complex vector does not mean that this is some kind of imaginary vector. What it simply means is that, we have chosen to express using complex numbers, but the actual electric field will be real. To obtain that, we need to multiply the resulting phasor by e power j omega t and then take the real part of it, okay.

So we will assume that time dependence is in the form of a sinusoidal signal e power j omega t and then see what happens to our equation, okay. So what would happen to this del e by del t now.

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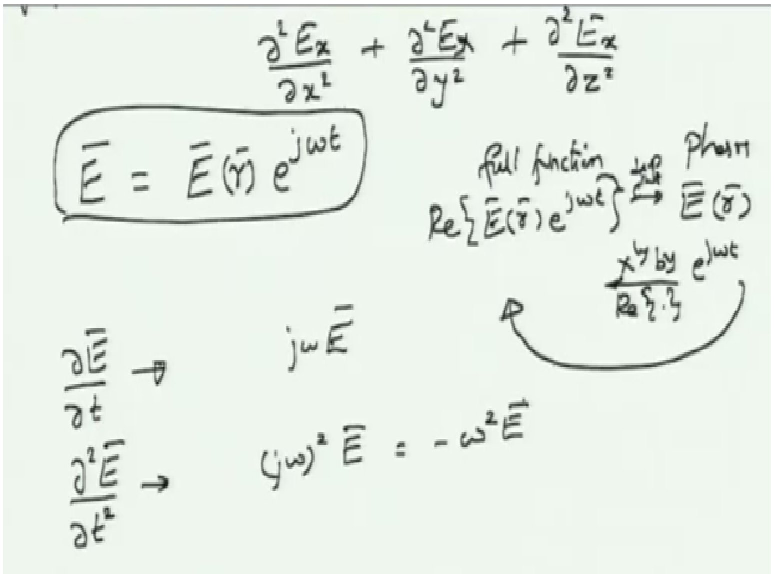

This would actually become, because efr is independent of time, so when you apply del by del t operation on to efr, nothing would happen, but when you apply del by del t to e power j omega t, it will put out j omega into e power j omega t, right. So this becomes j omega into e, correct, alright. So this is what would happen and if you differentiate once more, this is as good as multiplying by j omega twice that j omega square into e, okay, which is minus omega square e.

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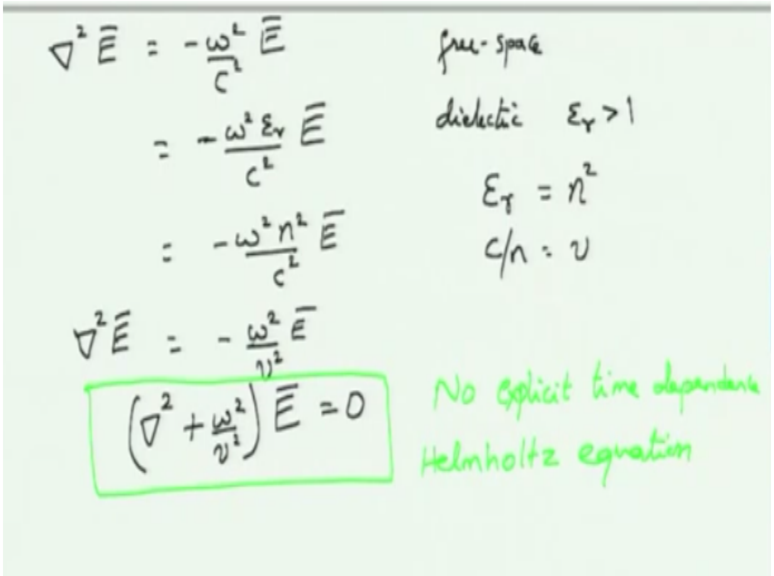

So I can substitute this into the wave equation, so as to write del square E is equal to minus omega square by c square, okay electric field E. This is for free space, for which mu zero and epsilon zero was there. If it is not free space, then you would actually have epsilon r here, c square is still down, right, because epsilon is epsilon zero into epsilon r, but mu zero into epsilon zero is actually one by c square, so I get omega square epsilon r by c square.

This is for dielectric with epsilon r greater than 1, okay. And you would probably not realize this now, we will discuss this sometime later in optics, it turns out that this relative dielectric constant epsilon r is actually related to refractive index, where n is the refractive index and square. Therefore, I can write this as minus omega square n square by c square electric field, but c by n is actually the velocity of wave or light inside a medium of refractive index n.

So if I define c by n as v, okay, I can write this thing as minus omega square by v square into E. These are all different forms of the same equation and a slightly different forms that we have written, because using this becomes easy, okay. So we can say consider a medium, which as a velocity of propagation, which is only 10 percent than that of the free space, right. In that case, I know what is v, v is 0.1 into c.

So I can substitute that and write down this equation appropriately, numerical values appropriately, okay. Let us combine left and right hand side, so move the right hand side to the left hand side, what we get is del square plus omega square. Sorry, I am still in free space, but actually now I am in dielectric medium, times E is equal to zero. This equation that we have contains no explicit time dependence, right.

So it is important to write this as no explicit time dependence. Time dependence is there. Where is the time dependence? Time dependence is sitting in this electric field itself. If I want to obtain what is the actual electric field, I need to multiply this E by e power j omega t and then take the real part. So there is time dependence and time dependence in this case is very simple. It is just a sinusoidal signal of angular frequency omega.

However, because there is no time dependence, explicit time dependence in this equation, this equation is known as Helmholtz equation, okay. This equation is in many cases preferred rather than the original time equation, okay. At least for analytical cases, that is what is typically preferred, but for numerical cases, you might want to go back to the original equation itself, okay.

So this is the Helmholtz equation and we will now look at what possible solutions for this Helmholtz equation can exist.