Electromagnetic Theory
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Lecture - 49
Solution of Helmholtz equation

In this module, we will be looking at solution of Helmholtz equation, okay.

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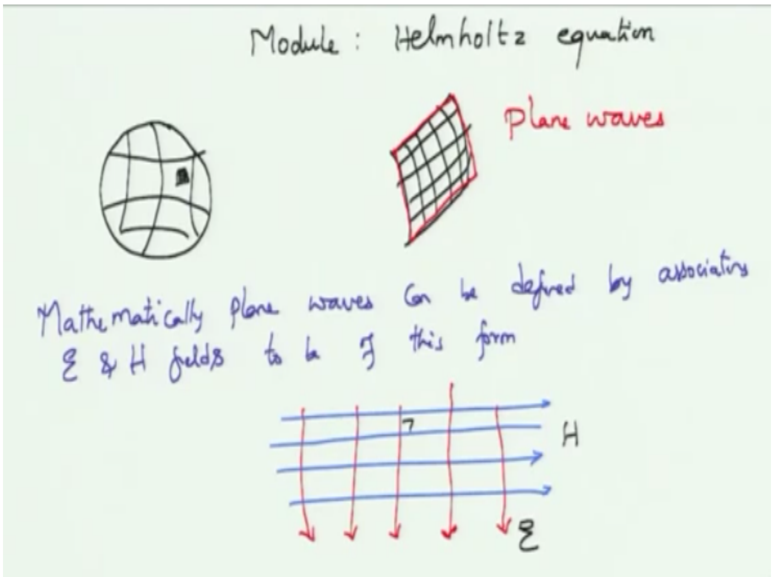

We have defined what is Helmholtz equation, it is actually the one that does not contain explicit time dependence, okay. So we want to find out how to solve this Helmholtz equation, in generally turns out that it is actually very difficult to solve this Helmholtz equation. But it is possible to pick certain, based on some intuition, it is possible to pick certain directions for the electric field, okay.

And then see if our picked solutions or our guessed solutions are actually consistent or not, okay. Let us imagine that these waves, electromagnetic waves, are being generated by some action at sun, okay, which we can assume to be very, very far away from us, right. And let us also assume that these waves are, of a particular frequency only. Although this may not be true, and it is not true in fact in general.

But let us assume for simplicity that, these waves which are generated on sun are of a particular frequency. So these waves are coming to us, okay. The waves are actually spherical, okay. This is, we are not going to prove that one here, we can prove that one later.

But the waves which are generated are all spherical, and then they would start expanding like this.

You must have seen many movies, or you know, you must have seen it if, when you drop a pebble in the pond. That there would be this expansion of spherical surfaces. So this surfaces expand like this but by the time they reach earth, the expansion would be so large, that over the entire earth, or at least over the region that I am considering, let us say I put up my receiving antenna over here to intercepts the sun's rays.

Then over this antenna, which could be say one metre by one metre, for example, that wave which is coming although spherical becomes locally like a plane wave, right. See, it is like imagine a ball, okay. So you imagine a ball and then imagine the radius of the ball being increasing, increasing, increasing. And then you start to pick of a very small region of the ball, okay.

And if you see that region that looks almost like a plane of paper, rather than a curved surface, it looks almost like a piece of plane paper, right. So this is the sphere that I have, okay, and then if I now pick up a very small region, which is what would happen, right. See, this is the small region why because this spherical surface which represents the wave generated from sun, is actually quiet large, right.

So, on that I am putting my small receiving antenna, and seeing what is the variation of electromagnetic field over here, on this antenna. So this is the small region, which I am seeing, so although I would be looking at a curved surface like this, right. This is the curved surface that I am looking at, but for all practical purposes, this can actually be replaced by a plane graph, okay.

So this kind of waves are called as plane waves, okay. And we will be looking at plane waves, okay. And mathematically plane waves are quite easy to specify, because they can be specified easily, I will tell you in a minute now. But, physically plane waves cannot really exist, okay, because mathematical plane waves cannot exist, simply because, we will see why they cannot exist.

But, the idea is that if a wave source is sufficiently, sufficiently far away from the place where you are looking for the waves, then you can actually model that wave as a plane wave, okay.

So, that is the whole idea of using plane waves. This is why sometimes; this is called as plane wave approximation. And that is very widely used in electromagnetic studies and optical studies, okay. So, how do we define mathematical plane waves. Mathematical plane waves can be defined by writing, so mathematically plane waves can be defined by associating E and H fields to be of this form, which form, I will just in a minute, I will draw that.

The form is this. These are the electric field lines, okay. These are the electric field lines of the plane wave, okay. And these are the magnetic field lines. You can see that these electric and magnetic field lines are all crossing at right angle. So, you can see that, these are all crossing at right angles, okay. So, this is like taking up wired mesh and then putting it up in front of you with one axis pointing to the electric field.

This is the electric field, and this is the magnetic field, E and H fields, okay. So, these are called the plane waves, and this plane waves have to be advancing, they have to be propagating somewhere, right. So, if you go back to the spherical wave, you know, which is expanding like this, spherical wave expanding like this, right. It actually means that, I have these plane waves, right, which are expanding all the way, right.

So, these plane waves are all expanding all the way over here, right. And then, this is how the plane waves are expanding. So if I consider the direction in which these plane waves are expanding as z axis, right, okay. Let me consider the direction of expansion of wave surface as z axis, then it is immediately clear that electric field and magnetic field both have to be in plane, that is perpendicular to z axis, correct.

They have to be plane that is perpendicular to z axis. So, for example, this is my axis of propagation, the wave is propagating like this, and this is my magnetic fields, okay, or you can think of this board, that is there at the back here, okay. As consisting of lines which are electric and magnetic field lines, okay. And then the board is advancing towards you, the board is coming towards you in the z direction.

So, this is the direction of wave propagation, and these are the electric and magnetic fields, okay. So, we are free to choose electric and magnetic fields to be lying along x or y direction, but they have to lie only in x and y direction, and they have to be perpendicular, to each other and they have to be perpendicular to the plane that contains both, right.

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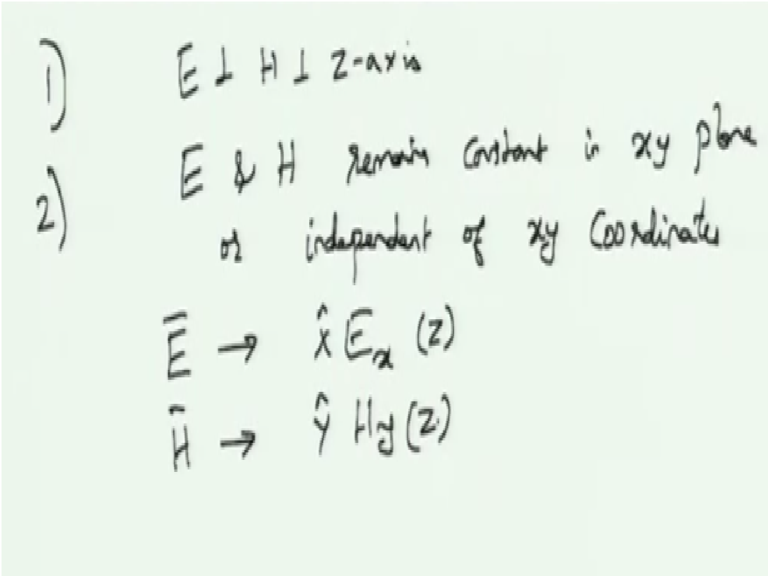

So, E is perpendicular to H, which is perpendicular to z axis, which is the direction of propagation. This is the first rule for a plane wave, which is propagating along z direction. If the plane wave direction is considered to be x axis, then E and H must be perpendicular to each other and they have to be perpendicular to x axis. In that case, electric and magnetic fields will lie in yz plane, okay, it is as simple as that.

So, you have electric fields and magnetic fields. Now in xy plane and perpendicular to z axis. Now, within this xy plane, can the strength of electric field vary. We can rigorously show that the strength of electric field and magnetic field cannot vary, they have to remain constant. This is a plane wave, the strength of electric field and magnetic field remains constant in xy plane or completely independent of xy coordinates.

You can, kind of intuitively think about why is this so. If they were to depend on x and y, then there would be some curl, right, and there would be some divergence. But, if there is divergence, then del dot E will not be equal to zero, right, and del dot B is not equal to zero. Therefore, we cannot have variation, okay. We will not prove them rigorously, it is not important.

But, take this point that, electric and magnetic fields must be independent of x and y coordinates. These conditions are sufficient for us to consider electric field, magnetic field and z axis, as a system of three mutually perpendicular axis, right, so one electric field, magnetic field and then you turn electric field and magnetic field, it would propagate in the direction of z axis.

Now, you might rightfully ask, who stopped you from considering propagation along minus z direction, answer is no one. Actually, there could be exactly a wave which is going along minus z direction also, because the equations actually do not tell you that you have to consider only forward propagating equations. They also tell you that you can consider, in fact mathematically the solution would be backward propagating, or negative z propagating solutions as well, okay.

And, if you go to that electric field and magnetic field should be in such a way that, electric field to magnetic field will point in the direction of z, then you have to appropriately change the electric field and magnetic field orientations, okay. Other than that, there is no problem if you consider negatively propagating wave equation, I mean wave solutions or waves. But, of course, physically you might not have a source, right.

If only there is a source on to the right side, there would be some waves which are going in backward direction. If the source are on the left side, and you consider this as the right side, or the positive z axis, the waves would only propagate to the forward axis, I mean the forward z region, okay. If there are some reflectors or scatterers in between, then it is possible that, at any given region of space there could be propagation of both plus z and minus z waves.

And, those are the some examples that we will see later, okay. So, to recapitulate the solution for Helmholtz equation is in general complicated. To simplify those complications, we will assume a plane wave approximation. Plane wave approximation is very nice, very valid approximation, when the source of waves are quite far away from the region, where you are talking about this.

Mathematically, plane waves can be defined by associating electric and magnetic field such that, they are both perpendicular to the direction of propagation, say z axis, okay. And, they remain constant in the xy plane, or they remain independent of xy coordinates. So, effectively what I have now is electric field being consisting only of say, x component, which is x hat Ex, right. And it should be independent of x and y.

So what can it be dependent on, only z, yes. Only when you keep moving far away from the plane wave, then its amplitude might change, amplitude would decrease, right. Otherwise, the amplitude is independent as long as you are at a constant z plane, okay. So, this is the expression for electric field now. Similarly, magnetic field has to be along y, Hy of z. Now let us see whether, whatever we used some physical justification and intuition actually, is also mathematically valid.

Can we show that, these assumed solutions are actually solutions, if they are what is the nature of this Ex of z and Hy of z, okay.

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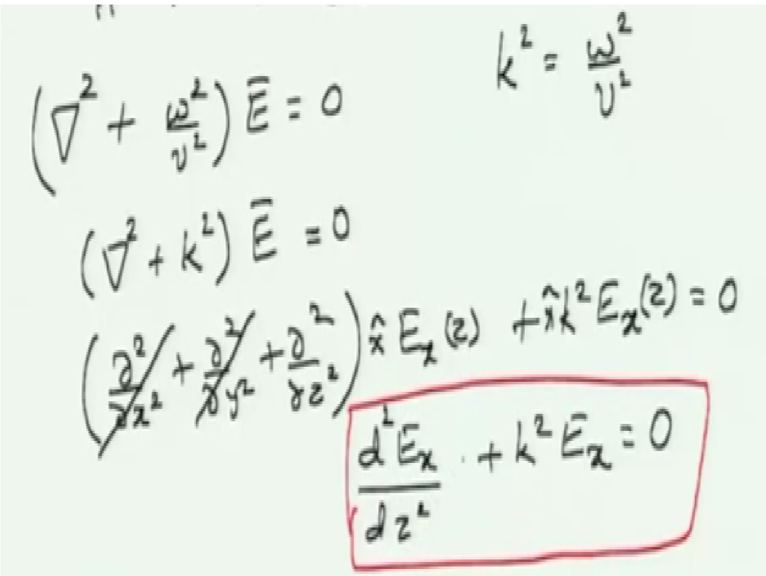

So, today that let us recall Helmholtz equation, that is del square plus omega square by v square into E is equal to zero, and instead of writing omega square by v square every time, let me introduce an expression, okay, which a constant K, which is equal to, in such that K square is equal to omega square by v square.

So let me introduce that, so that this equation becomes del square plus K square, okay, into E is equal to zero. Now, expand this equation, what is del square, del square is del square del x square del square by del y square plus del square by del z square, electric field E, electric field is E is nothing but x hat Ex of z, right, plus K square, that is x hat Ex of z is equal to zero, correct.

Now, Ex is only function of z, it is not a function of x or y, so there is no point writing this del square by del x square and del square by del y square terms. They all cancel each other and then there is x hat, x hat everywhere, so which simply means that I can drop the vector also from this condition and then just replace this with the scalar equation. And I get del square Ex by del z square plus K square Ex is equal to zero, right.

And since, Ex is only function of z, there is no need for me to write down this as del square by del z square. I can simply write this as d square Ex by dz square, okay. This is the equation that is highly simplified from Helmholtz equation, okay, based on our ideas that we have discussed previously. And we want to see what is the solution for electric field Ex.

How do we solve this equation, well I do hope you remember your solutions for differential equations. So the solution that we were looking for was for this equation d square Ex by dz square plus K square Ex equal to zero.

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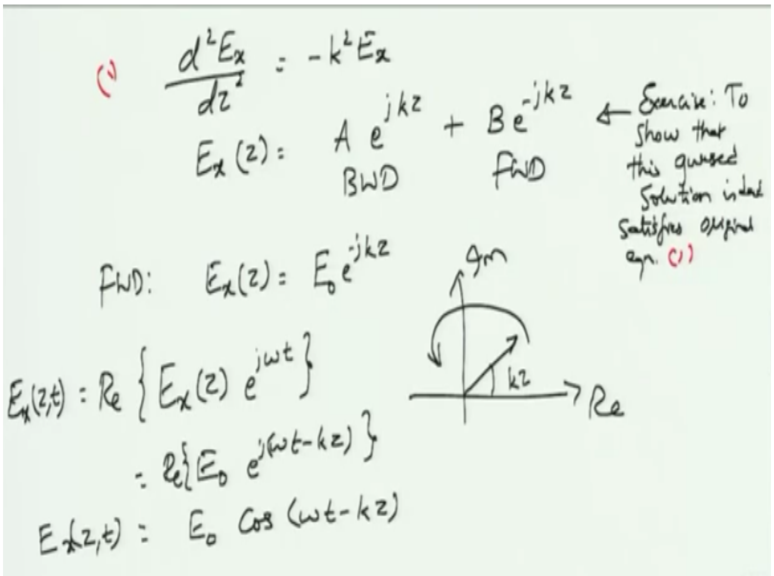

The solution of that equation is, have d square Ex by dz square is equal to minus K square Ex, correct. The solution for this would be Ex of z is equal to some constant Ae to the power jkz plus some constant Be to the power minus jkz. Does it actually get satisfy this equation, can you just verify that this is actually the equation. I leave this as an exercise to you, okay.

This is a small exercise to you to show that this assumed solution, or this guessed solution in deed satisfies original equation, okay, which is this one, will call this as some equation one, okay, equation one. You should verify that, for example, you turn of B, okay, then what would happen, we have Ae power jkz, so differentiating one with respect to z you will get jk, twice you will get jk square, jk square is minus k square.

So minus k square into Ex is equal to minus k square into Ex. Therefore, this is the solution. Similarly, minus jk would also be the solution. So it is up to you now, to whether retain a plus kz solution or a minus kz solution, mathematically both exist. But if you now say that, well you know I do not have any wave, any physical source which would actually push the waves along minus z direction, then I can make this equal to zero, okay.

If not then, I do not have to make it equal to zero. So this is called as the forward wave, and this is called as the backward wave. I am using forward and backward in energy with transmission line, which we are going to study after the wave modules, okay. Based on that, any wave which is going along positive z direction is forward wave for me, any wave which is going along minus z direction is called backward for me, okay.

So that is my simple, short notation that I am going to use, okay. I have Ex of z is equal to considering only the forward wave solution I have Ex of z is equal to A, which is some constant and instead of talking about A which combines no intuition, let us call this as E0, okay, E0 stands for some amplitude of electric field. So, you have E0e power jkz as the solution. Do you recognize what has actually happened over here.

The expression for electric field is depending on z, correct. But it is actually a complex number, right, if you try to sketch this one, you will not able to sketch this. It would actually be a complex number, you will have to sketch the real part first, sketch the imaginary part first, right, or you can interpret this as a phasor, right. So, if you actually think of this in terms of the phasor with real and imaginary axis, then corresponds to a phasor, which is rotating, with an angular velocity of kz.

So, you fix z and fix k, then it would point to a particular direction, but if you keep moving along z, then this would actually be a phasor, which is rotating, okay. We also satisfied that its amplitude is changing only with z, yes because there is no change in x and y direction, right.

So this is just a wave which is propagating along z direction, right. The second thing that you have to notice is the, this is a phasor.

Therefore, then it is not the complete time dependent form. So, if you want to find the time dependent form, you have to multiply this Ex of z by e power j omega t, then take the real part of it, right. So, to obtain the time dependent form, which is Ex of zt, and what do you get if you do this operation, multiply by e power j omega t. You get e power j omega t into, e power j omega t plus kz, okay.

And with that if you go back here, you have electric field here as real part of Ex of z, and substitute for Ex of z here I get E0 e power j omega t minus kz, and real part of this is nothing but E0 is real. Therefore, E0 comes out of the real operation, we have E0 cos of omega t minus kz. This is the expression for electric field, okay. This is in the scalar form.

If you want to write down the expression of the electric field in the vector form, you can attach the appropriate vector direction, right. This electric field would always be directed along the x axis. And if you want to fix z and t and everything, this cos term is fixed, then it would point with an amplitude of E0 time something, whatever the cos value is, something and it would always point along x, okay.

There are two things we can do now, fix t and see what happens with respect to z, and fix z and see what happens with respect to time. This is like, I take an oscilloscope I put it up in to air, okay. and this oscilloscope display should tell you the electric field, okay, that of the form cos omega t minus kz. And I do not, I am satisfied with only one oscilloscope at this particular point insert, have two, three different oscilloscopes, okay, kept at two different or three different points along z direction, okay.

Now, if I were to say, this is z equal to zero and this is z equal to something, this is z equal to something, rather than talking about z, it is easier for me to talk in terms of kz, okay. So, kz equal to zero, kz equals Pi by two, kz equals Pi. I have got three different oscilloscopes at values of kz equals. zero, kz equals Pi by two, and kz equals Pi. This kz equal to zero is my reference, okay.

Now, with that, if I were to see what is the display on the oscilloscope, right. What would be the display that I would see.

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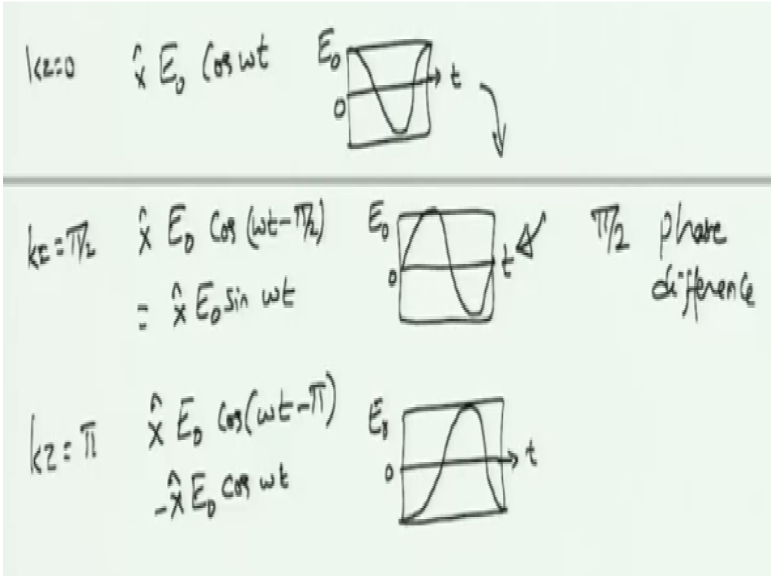

Go back to the expression, take kz is equal to zero with kz equal t zero the expression for electric field will be x hat E0 cos of omega t. So if you were to look at the oscilloscope display, you would actually see, a display that would look like this, correct. This is the cosine wave form, this is the time. This is the axis with respect to time, so this is the time t equal to zero, with an amplitude of E0, right.

This will have an amplitude of E0, and it would go as in this particular fashion, okay. Now, if you try what happens at kz is equal to Pi by two, get x hat E0 cos of omega t minus Pi by two. Now, cos of omega t minus Pi by two is sin omega t, right. So this would be cos omega t minus, so it would be cos omega t, cos Pi by two, which is zero, minus, sorry, plus sin omega t sin Pi by two.

So this would actually be equal to x hat E0 sin omega t, right. So you look at your oscilloscope display, the oscilloscope display would now show here, right, a sin wave, at time t equal to zero with a maximum amplitude of E0 again. But, if you notice at kz equal to zero and kz equal to Pi by two, you will actually see a Pi by two phase difference, right. This wave at kz equal to Pi by two is actually lagging the first wave by a value of Pi by two, right.

Similarly, now when you write kz is equal to Pi, what you get is, x hat E0 cos of omega t minus Pi, right. And this is cos omega t cos Pi, sin will be anyway will be zero. So this will be minus x hat E0 cos omega t. So if you look at the oscilloscope display, the oscilloscope display will show you a wave that would look like this, with an amplitude, initially at t is to zero, as minus E0, right.

So, you can actually see that if you were to put a point of reference, you know some small object over here, right. This small object, as you see is at different times, appearing at different times, right. You do not have to do this one with respect to time. You could actually fix time, okay.