Electromagnetic Theory
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Lecture - 50
Uniform plane waves

So in this module, we will continue talking about uniform plane wave. In the last class or in the last module, we actually derived an expression for wave equation of a wave, which was propagating along z direction, and then it had no dependency on x and y coordinates, right. So we picked up a particular component called the electric field and we picked up this particular x component of the electric field, and then we wrote down the wave equation.

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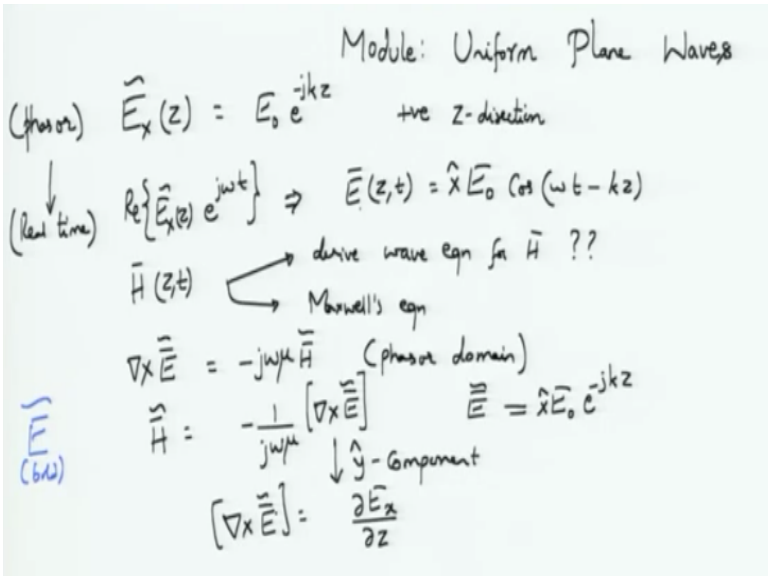

We solved the wave equation and in the process, obtained an expression for electric field, right, which was Ex of z given by some constant E0 e to the power minus jkz. This was the wave which we said was propagating along positive z direction, right and a small tilda over this Ex indicates that this is actually a phasor form.

Because the wave we obtained the solution of this equation or the wave we obtained this particular expression was by solving the Helmholtz equation, and Helmholtz equation is actually given in terms of the phasor, that is the time independent equation. However, to put the time dependence back, you have to go back from phasor to real time notation. And to do that one, you have to multiply this phasor by e to the power j omega t.

And then take the real part of it, right. So when we did this, we actually obtained the electric field in terms of full vector form, the electric field as a function of z and t, z because it is the direction in which the wave is propagating and therefore it depends on the particular coordinate. And obviously, this is a time dependent wave form. To obtain this full expression which actually involves time as well as this direction of propagation.

You have to start with the phasor, multiply by e power j omega t, and then take the real part of it. So when you did that you, you actually saw that this is given by x hat, indicating the direction of the orientation of the electric field, E0 being the amplitude, cos omega t minus kz. So these are the only wave components that we actually saw in the last module. Now, given electric field, alright, can I also find out what is the magnetic field H.

To do this one, should I take an approach in which we derive wave equation. So, do we derive wave equation for H, solve it, and then obtain the corresponding expression for the magnetic field. Actually no, what we do is, we use Maxwell's equation, okay, and then from Maxwell's equation we obtain the expression for H field. So, how can we do that, well we have this Faraday's law, which tells us that the electric field, the curl of electric field is given by minus j omega mu into H.

Of course, the wave we have return this one means E and H are supposed to be phasor. But E and H are also vectors, right. So we have to now indicate that electric field is a vector, as well as a phasor. In print, it might be very easy to do, for example, in print you can actually say that this electric field written in bold, right, would correspond to vector, and on top it you can place this tilda, in order to make it into a phasor.

But in handwriting the notation is little clumsy, but you have to carry out this notation as you work along, because it will remind you that this is actually phasor. What is the phasor and what is the vector and the inter relationship between that. So with that in mind let me write the electric, let me indicate that this is actually phasor by writing a tilda on top of it, okay. As I said this is slightly clumsy notation.

But this is important to keep in here, because this expression is valid only in the phasor domain, okay, because this is valid only in the phasor domain, I have to indicate the field quantities as phasors. And my notations of field quantities as phasor is to indicate them by a tilda, okay. Alright, I have this Faraday's law, and I know what is the electric field phasor, which is E0 e power minus jkz.

And I know what is omega, I know what is mu, and therefore I can easily use this expression to find out what is H, right. So H is given by one by j omega mu minus curl of electric field, right. So electric field phasor, if you look at it, it is actually this x hat E0 e power minus jkz, right. So this electric field phasor is actually x hat indicating the vector direction for the electric field, amplitude E0 and the phase dependent part or the space dependent part, which is e power minus jkz, right.

So, what is the curl of this expression. I have x component of the electric field varying only as the function of z variable, which means, that the corresponding component for the curl will have only y component, right. So, you can actually convince yourself by looking at the expression for the curl. So what you will essentially see is that this will have only the y component.

And for the y component the curl of electric field is actually given by del Ex by del z, okay. So this is the corresponding curl, y component of the curl.

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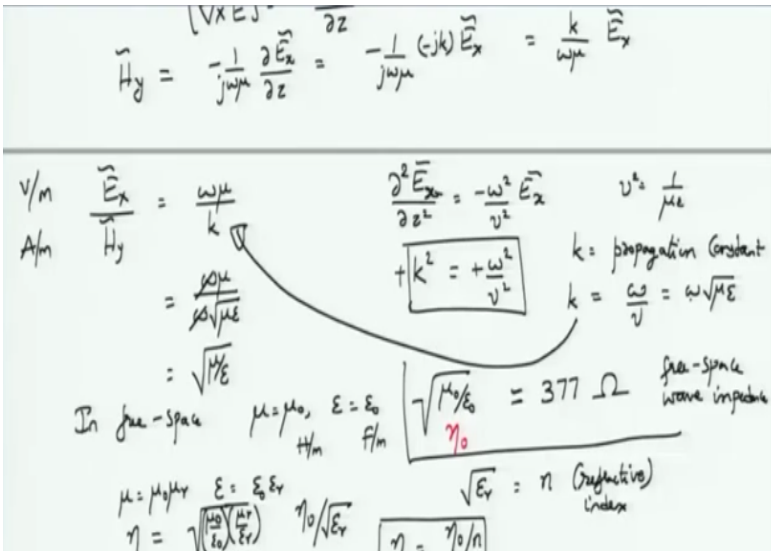

So, you can actually substitute that one into this expression to, into this expression to see that H will have only the y component. So, Hy phasor is given by minus one by j omega mu del Ex phasor divided by del z. Now del Ex phasor divided by del z can be evaluated, which will simply pull out, minus jk and then leave everything else as it is. So what you get here is minus j and the minus j from the numerator and denominator cancel with each other and get k by omega mu Ex phasor.

So if you look at the ratio of the electric field phasor Ex to the magnetic field phasor Hy, you will see that this ratio is given by, so you can actually use this expression, So Hy comes down, so this fellow goes up and this is given by omega mu by k. Now, at this point we can proceed further if we remember what k omega and mu are related to. See, in the wave equation that we had for the electric field Ex, you had something like this.

Electric field del square Ex by del z square was equal to minus omega square by v square Ex, right, this was the phasor, where v square was given by one by mu epsilon. And we know that Ex will have e to the power minus jkz dependence, so when you differentiate it twice you are going to get k square, because you are going to get minus jz into minus jk, which will turn out to be minus k square.

And this should be equal to minus omega square by v square, Ex being the same. So cancelling of minus signs on the both side, we see that k square is related to omega square by v square, and we said that this k is actually the propagation constant, okay. So, this k is the propagation constant, it tells you as you go along the z axis, as you go, as the wave proceeds along the z axis, what is the rate at which its phase is changing, or rather what is the phase variable that is changing, okay.

So this k is propagation constant and is actually given by omega by v, and we already know that v is one by square root mu epsilon. Therefore, this is nothing but omega into square root mu epsilon, right. So v is one by square root mu epsilon. Therefore, this one is equal to omega into square root mu epsilon. Now, coming back, why did we go to this one, because we wanted to use this expression for k omega and mu.

And then see happens to this expression omega mu by k, which gives you the ratio of the amplitude of the x component of the electric field to the y component of the magnetic field. So substitute for k as omega square root mu epsilon, so you have omega mu in the numerator, you have omega square root epsilon in the denominator, omega cancels out. This is for the same frequency wave we are considering, and there is mu on top and mu here, so this can be written as square root of mu by epsilon, okay.

In free space, mu will become mu zero, right, mu is mu zero, and epsilon is nothing but epsilon zero. Therefore, this ratio of square root mu zero by epsilon zero, which will tell you the ratio of the amplitude of electric field, x component of the electric field to the magnetic field y component is given by square root of mu zero by epsilon zero, and when you plug in the numbers you will actually see that this is around 377.

Now, mu zero has units of Henry per meter, and epsilon zero has units of Farad per meter. And when you actually put in the ratio here, and then, if you remember that E itself has a ratio of volt per meter, and H has a ratio of ampere per meter. The ratio of these two should be volt per ampere, right. Now, volt per ampere is nothing but, voltage by current and voltage by current will actually tell you that this is nothing but impedance.

And impedances are measured in terms of ohms, right. So, this ratio of square root mu zero by epsilon zero under root is actually given by, is approximately given by 377 ohms. And this is called as free space wave impedance, okay. So one actually imagines that free space or vacuum is actually an impedance kind of a medium, you know it is kind of a resistance. But it is kind of a generalized resistance in the form of an impedance.

And you are looking at, or you are imagining that free space itself kind of an impedance, but an impedance for electric field to magnetic field ratios, because electric field is volt per meter magnetic field is ampere per meter. Their ratios would turn out to be in the form of an impedance quantity. And this is precisely what you are getting now here. So this square root of mu zero by epsilon zero is indicated by a special symbol called eta zero, that zero stands for free space.

And this eta zero is given by 377 ohms. What if your medium is not free space, right. So, if the medium is not free space, I know that mu can be written as mu zero mu r, epsilon can be written as epsilon zero epsilon r. So, substituting this square root of mu by epsilon, and calling that entire thing as impedance eta, you are going to get square root of mu zero by epsilon zero into mu r by epsilon r under root, correct.

But I already know what is square root of mu zero by epsilon zero, which is nothing but eta zero. And most cases that we are going to consider in this course will have mu r is equal to one. We are dealing with non magnetic materials. So, mu r is equal to one and then epsilon r is whatever the value of the dielectric constant or the permittivity of medium. So, this is eta zero by square root epsilon r, okay.

And sometime in one of the earlier modules we remarked that square root of epsilon r is nothing but refractive index n, okay, this is the refractive index. So, I can equally write this eta as eta zero divided by small n, okay. I know this is getting little confused, because you have n here, you have eta here. But I do hope that once you go through this equation several times, you will really able to appreciate the difference between eta and n, okay, n is the refractive index.

And what you can see is that, the medium impedance, not in free space but any other medium impedance, is actually inversely proportional to the refractive index. So the larger the refractive index, the medium impedance will be smaller, okay. So this is about the magnetic field. Well, we have not yet completed the solution for the magnetic field, let us do that one.

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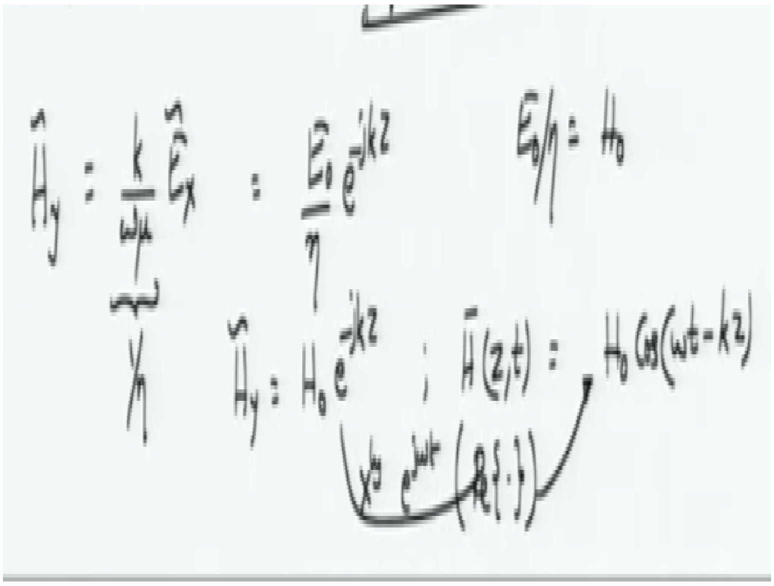

What we have just found out is that, the phasor for Hy actually is given by this k by omega mu, right. This is what we find out, if you are not sure, you can actually go back and take a look at this part here. So, Hy phasor is equal to k by omega mu times Ex phasor. And k y omega mu is a real quantity, which is actually one by eta, right. So this is nothing but one by eta, and then you have Ex phasor, right.

But what is Ex phasor, Ex phasor is E0 e power minus jkz, right. So this is your phasor for Hy written in terms of the amplitude for electric field. If you want you can redefine E0 by eta as some H0, and then write down the corresponding expression. So you will have Hy phasor is given by some amplitude H0, what is actually related to the electric field amplitude E0 as E0 by eta e to the power minus jkz.

As before, we are interested in finding the actual electric field, right, as a function of z as well as time. To do that one we need to multiply this phasor by e power j omega t, and then take the real part of it, right. So multiply by e power j omega t, and then take the real part of it, in order to obtain H0, cos omega t minus kz.

Thus, what you see here is that electric field x component, magnetic field y component, both are in phase with each other, both vary as cos omega t minus kz as a function of z and t. But at the same time, the amplitude for the magnetic field is reduced compared to the amplitude of the electric field by a factor medium impedance, eta, okay.

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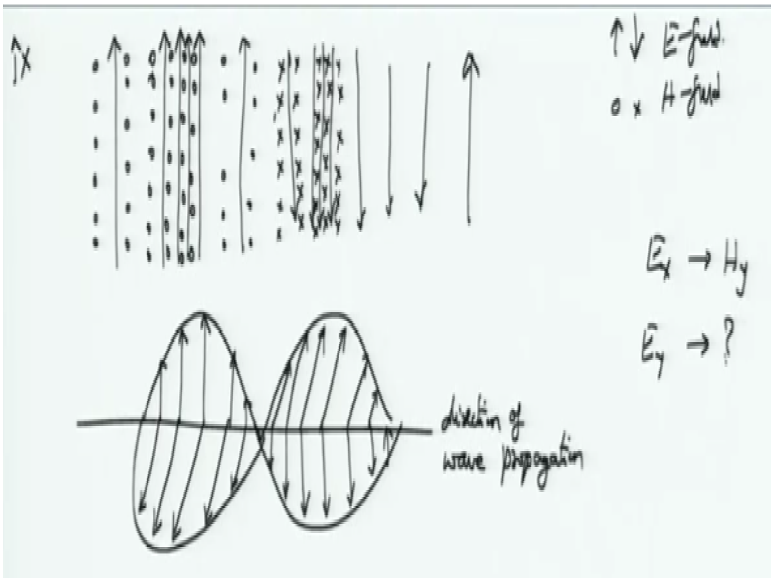

Alright so this what I wanted to talk about, the electric field and magnetic field. Now, at this point, you might say that well what we have done is quite arbitrary, right. Let us actually try and understand how the fields themselves vary, so if I am looking at the electric field lines, this electric field lines will all be very uniform. They do not change their values, so they start getting crowded together, so it is kind of a sinusoidal wave form, right.

So they get crowded and then the field lines will get slightly away from each other, right. After a certain time, then they will have to start becoming negative, right. At this point, they start to become negative, so the field lines actually go to zero over here and then they start to become negative. They go in the opposite direction, okay and then the density of field lines continue to decrease. Again reversing back once the wave hits positive direction, right.

This is how the electric field is. They are all bunched together. This bunching happens at the nodes of the wave and then they are spread out at the trough of the wave. So these are the electric field lines, so this is how I am picturing the electric field lines. As you can see, the electric field lines are all uniform, right. They do not change their orientation. They are all oriented along either plus x or minus x, does not matter, right.

So it could be the plus x or minus x, but in general they are oriented along x direction. What about the magnetic field lines. Well, magnetic field lines will have to curl around them or they have to come out of the page, into the page, out of the page, into the page, right. So that is what actually happens over here, so you have the magnetic field lines over here, right. So they also getting crowded at the node points just as the electric fields lines would curl, okay.

And then they start to thin out again, then they start to thin out again. Then, they eventually become more in the opposite direction. So for example, if you consider this open dots as the field lengths, which are coming out, along the y direction, then this cross would consist of electric field as going in the minus y direction. So this is your h field lines, okay. So this is how the magnetic field lines would look like. They would crowd and then they would separate.

They would change their orientation just as electric field lines would change their orientations. So this kind of behavior is sometimes captured by writing the electric field lines and the magnetic field lines in this fashion. So you have the electric field lines here. These are all the electric field lines. At the same time, you have the magnetic field lines also coming up here. You might have seen this one in many, many textbooks, right.

And this is how the electric field lines would look, okay. So, these are the electric field lines and then these are the magnetic field lines, right and this is the direction of wave propagation, okay. This is the direction of the wave propagation.

As I was saying, we kind of seem to be a little arbitrary in our selection of electric and magnetic field lines. Why do I say that this is arbitrary because, we pick ex, we found out the corresponding hy, so we picked ex, solved the wave equation for that one. We found some way of deriving a sinusoidal wave of a particular frequency and for this ex, there was a corresponding hy component.

Now, you might ask, can I actually have an ey component to begin with, not ex component, but ey component to being with and what will be corresponding magnetic field component and you will be right. You can actually begin with ey component and then write down an expression in terms of, by simplifying Helmholtz equations, so you will actually will have to go back to the steps. I am not going to derive them over here.

I assume that you can actually do this by yourselves after you look at the Helmholtz equation, but nevertheless, again assuming that ey is a function only of z and t and with t, we are assuming them to be in the form of e power j omega t, therefore we can drop that particular thing.

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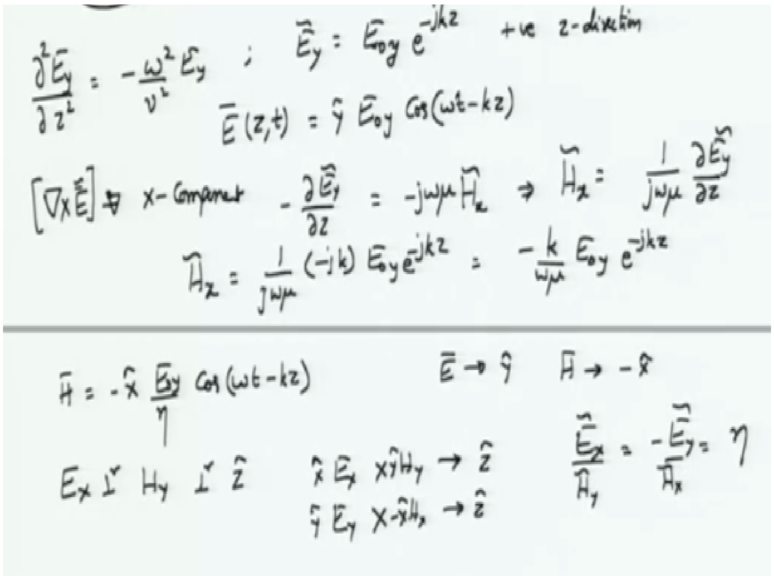

So I have del square ey by del z square given by minus omega square by v square ey. Again the solution for this equation will be ey phasor will be given by some constant E0 and now I have already used up this constant E0 once, that is for x component of electric field. Therefore, let me be slightly more explicit in saying that this is the y component constant. So this is E0y e to the power minus jkz.

This is again for the wave, which is propagating along positive z direction, okay. And then the electric field ey phasor is given by the amplitude E0y e power minus jkz. As before, you can actually obtain the proper electric field component by multiplying it by e to the power j omega t and then taking the real time and also appending the vector notation. Because this electric field is pointing in the y direction, so I have to write down the y component for this one.

So I have to write down the y component for this one, so I have y E0y, which is the constant cos omega t minus Kz. Now what will happen to the magnetic field h, well, we will not have to rederive the wave equation for this, but we can simply use the fact that there is Maxwell's equation available to us and that ey is a function only of z component.

So if you actually look at the expression for the phasor electric field, you will see that this particular thing will only have the x component. Why would it have the x component, because you have ey component varying as a function of z. ey component is a function of z and the x component of this is actually given by minus del ey by del z, okay. So this is the component, which must be equal to minus j omega mu h, right.

Since electric field is along y direction, the magnetic field will obviously be a phasor along x direction, so let me write down this as minus j omega mu h x. From this, I can actually write down what is hx, hx is nothing but 1 by j omega mu del ey by del z and you can see that with this del ey by del z as ey is given by the constant E0y e power minus jkz, all I am going to get here is hx phasor given by 1 by j omega mu minus jk E0y e power minus jkz, right.

So, this is given by minus k by omega mu E0y e power minus jkz. Therefore, the magnetic field component h can be written as x directed component or rather minus x directed component. Amplitude is reduced by a factor of eta, so you have E0y by eta cos omega t minus kz. So you can clearly see that if the electric field is along y direction, right as we have seen in this case, the magnetic field for the wave, which is propagating along plus z direction must be along minus x direction.

Now you already see a pattern emerging from all these. So you had electric field along x, magnetic field along y. These two are perpendicular to each other and furthermore these two are perpendicular to z axis itself, so they are perpendicular to the direction of propagation as well. Moreover, the direction of electric field times, you know, when you take the cross product, the direction of electric field when you curl towards the magnetic field must point in the z direction, okay.

So the electric field times or cross times the magnetic field, the cross product of this must point in the z direction. So based on this, it is very easy to see that if e is actually along y direction, so you have y cross ex, sorry ey, right. So the cross product of term something must point in the plus z direction. What should that be, that should minus x, right, because if you just have plus x, then it would be y cross x and that would be along minus z.

But what you want is a plus z wave propagating, therefore you write down this as minus x hat, right. So we have phasor ratio E0x or ex by hy phasor is equal to minus ey by hx phasor is equal to eta. Eta being the medium of impedance, right. Now there is another thing that you might ask at this point, well Maxwell's equation and wave equation is a linear equation.

Now for a linear equation, if I have one solution and if I have another solution, any linear combination of these two solutions must also be a solution. Why is not this case? Actually it is true. In this particular case, you have electric field along x component, magnetic field along y component, that forms one pair of solution. We have electric field on the y component and magnetic field along minus x component, these two again form one pair of solutions.

So you actually have two pairs of solutions and the linear combination of these pairs are also solutions of Helmholtz wave equation, right, Helmholtz equation.

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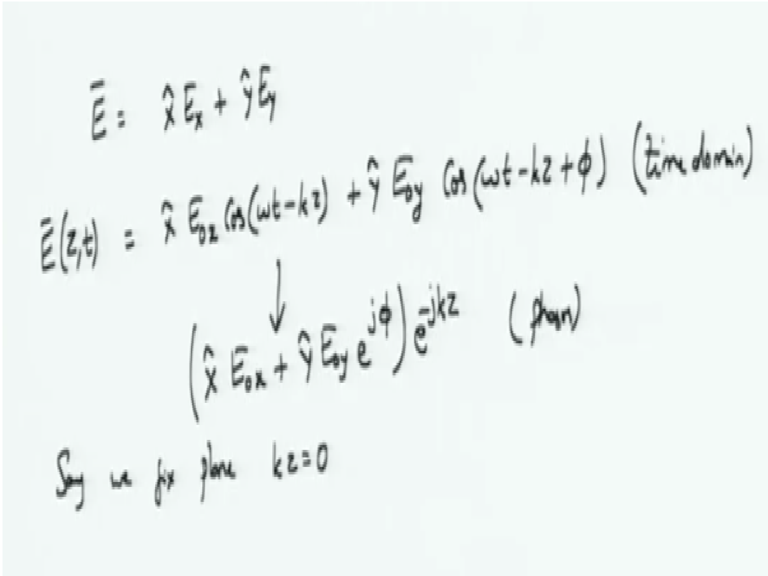

Thus, if your electric field can be written as component for x, right as well as component of y, then both terms can be non-zero at any given point of time, okay. So both can actually be non-zero and the sum of these two in any proportional factor also, you can put in, but that is not really important. The sum or the linear combination of these two will also be a wave, okay. In that case, we actually end up having an interesting concept appearing to us.

To see that one, let us write down the x phasor, so let us write down initially not in the phasor, let us write down the complete solutions up here. So electric field can be written as x hat, just for the purposes that electric field along x and y can have different component amplitudes, I am going to write this as E0x and E0y, okay. so I have E0x cos omega t minus kz. Please note that these two components must be of the same frequency.

You cannot have one at 1 Hz and the other at 2 Hz, and then you say that these two sum are the solution, although they are, but in the concept that we are going to introduce now, we assume that they both have to be the same frequency, okay. So in this case they are same frequency except that they might actually have some amount of phase difference between the two, so there can be some relative phase between x and y components.

To introduce that relative phase, what we will do is, we will replace this omega t minus kz by adding a certain phase phi. This would be the electric field that I am looking for, which is the function of both z as well as t. This is electric field z as well as t and you can see that this as two components x as well as y component. Now, it is interesting. I can actually write down the phasor form of this, right. What would be the phasor form of this.

I have to take this drop this cos, and drop this omega t and then replace this cos omega t minus kz by e to the power minus jkz, so I will actually have x hat E0x plus y hat, E0y e to the power j phi times e power minus jkz. This would be the corresponding phasor form. This is the time domain form and then this is the phasor form. You can actually verify. I think that you should definitely verify that you are comfortable going from time domain to phasor domain like this.

And now, we ask this question, suppose I am fixing myself in a particular plane, so say we fix plane such that kz is equal to 0, okay and then we look at the tip of the electric field, right, the wave actually has electric field and magnetic field, so we look at the tip of the electric field as a function of time. In which way would this change as time progresses, right and that progression of the tip of the electric field as a function of time is called polarization.

Now we have already used the term polarization to mean something else. When we discussed electrostatics, we said that polarization is the net dipole moment per unit volume, right or that was a dipole density that we were considering. This polarization has nothing to do with that polarization and unfortunately the two terms are used, which are the same, but they do mean two different things.

Usually from the context, it is clear, which polarization I am talking about, okay. This is an unfortunate thing that you have to keep in mind, polarization means two things in electromagnetic fields. One, it actually means dipole density and other means the tip of the electric field, okay. Usually the context will be unambiguous and the context will tell you whether you are talking of polarization in the matter or you are talking of polarization of electromagnetic wave, okay.

So with that small caution that I wanted to given, let us get back to what is called as polarization.