Electromagnetic Theory
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Lecture No - 72
Rectangular Waveguide: TM modes

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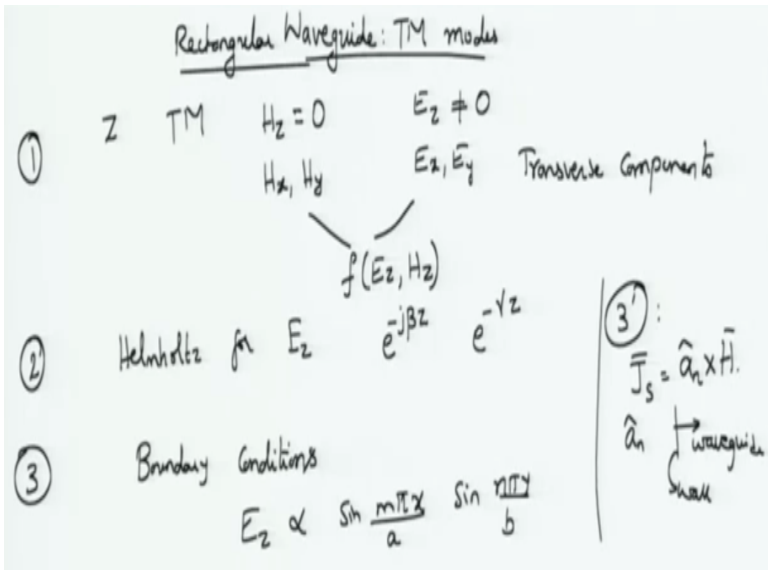

In the last module, we were discussing rectangular waveguides and we discussed how to derive expressions for transverse magnetic modes. If we recall what transverse magnetic modes are, these are the modes for which the magnetic fields lie entirely in the plane that is perpendicular to the direction of propagation. Because, we have chosen Z as the direction of propagation, this statement TM mode simply means that there is no H z component.

So, all the magnetic field components that we would find will be H x and H y. The corresponding longitudinal electric field component along the direction of propagation must not be equal to zero. So, which means that you will have full electric field having components E x, E y and E z, whereas for the transverse magnetic mode, H z component will be equal to zero. By cleverly manipulating Maxwell's equation, we were actually able to write down H x and Hy, E x and E y, which are called as transverse components. Transverse means perpendicular.

So, these transverse components in terms of longitudinal, longitudinal is along the direction of propagation. So, you were able to express everything, in terms of some functions of E z, of course, that function will also include H z, but for the transverse magnetic modes, H z is equal to zero and these components are functions of E z alone. This was our first step, right (()) (01:40) the solution for the transverse magnetic modes.

The second step that we did was to write down Helmholtz equation for E z. So, we wrote down the Helmholtz equation. This Helmholtz equation was solved by a technique called as variable separable method, which allowed us to express E z in terms of X of x, Y of x. Of course, we will assume that all the fields are propagating as e to the power minus j beta z for a loss less waveguide or as in general e power minus gamma z for a lossy waveguide.

So, this was the Helmholtz equation. This is the form of the z dependents that we had assumed and once we wrote down the Helmholtz equation, the third step was to essentially apply, boundary conditions. So, we applied boundary conditions by realizing that the tangential electric field must vanish on the perfectly conducting walls of the waveguide and from there we obtain expressions for E x and E z.

So, from the boundary conditions, we understood how E z must vary and we found that E z must vary as some sin, there is of course some proportionality constants, but that is not really important for us at this point. So, as a function of x and y, it has to be sin of m pi x / a, sin of n pi y / b. Finally, I would call this as some 3prime. This relationship is something that will give us, the relationship between the magnetic field and the surface currents that should exist.

Remember the tangential magnetic fields must be discontinuous by an amount of surface current. So, whenever there is a possibility for surface current to be present, then magnetic field component will be converged to that boundary conditions. So, the surface current that should be there on the walls is given by the normal component times H, where the normal component ‘a n’ is actually directed from the, this the waveguide, from the wall into the waveguide.

So, this is the wall of the waveguide and it would be directed along the direction into the waveguide wall. So, this relationship is necessary when we want to evaluate the losses in the waveguide or just to understand what currents need to be there in order to have the magnetic fields present there. Of course, this H will also be at the surface. That is you have to evaluate the magnetic fields at the surface, whereas to obtain the surface current.

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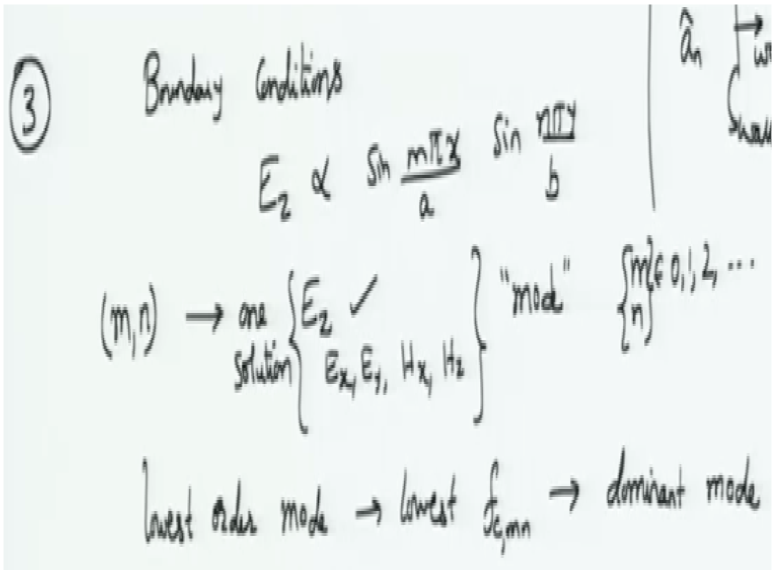

So, what we want to observe here is that given different values of m and n, for every value of m and n combination, there will be one solution that is one form of E z, which would be present. So, one solution for E z and correspondingly for E x, E y, H x and H z. In other words, given any combinations for m and n, you would be able to find the electric fields and the magnetic fields, which will satisfy Helmholtz equation, which will satisfy boundary condition.

But their shapes would be slightly different depending on m and n. This set of components for every value of m and n, which has a particular shape because of x, y and z dependence is called as a mode. Mode is nothing but solution of Maxwell's equation for the waveguide, along with boundary conditions. The solutions must satisfy boundary conditions in addition to satisfying the appropriate wave equation for the waveguide.

So, this is what a mode is and clearly you can see that if you start giving infinite number of values for m and n, there exist an infinite number of modes. So, what distinguishes one mode from another mode is what is the components m and n. Because, the form of all modes will be essentially the same. So, if we leave out the constants out there, electric field, the z component of the electric field will go as sin m pi x / a, sin n pi y / b.

And giving different values of m and n, will produce different modes. So, the question is what is the lowest value of m and n? Remember m and n must be integers. So, m can go from 0, 1, 2 or so on, similarly n as well. So, these m and n are integers and the lowest value of m and n for which the corresponding wave solution or corresponding mode exist is called as the lowest order mode. It is the lowest order mode that will propagate also with a lowest cut off frequency.

So, in other words this lowest order mode is what we call as, this has the lowest cut off frequency, f c, mn and this mode because the moment you exceed f, the generator frequency exceeds f c, this particular mode would always be conducting or this mode would appear inside the waveguide and because this is the one which conducts at the least possible cut off frequency and would always be present whenever f is greater than f c. This is called as a dominant mode.

So, given different values of m and n, the lowest values of m and n together, for which E z and all the other corresponding components are non-zero is called the lowest order mode or the dominant mode. So, this is called as the dominant mode. Let us try to find, what is the dominant mode for the TM mode case?

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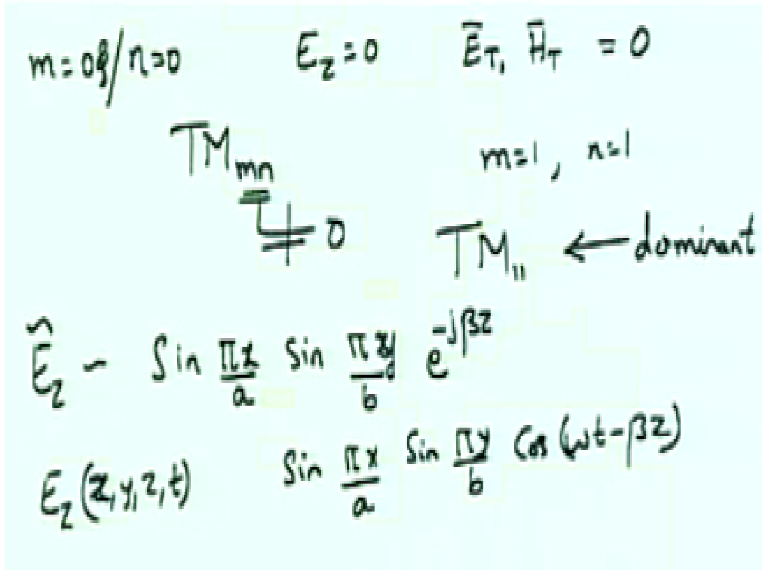

In the TM mode case, the functional forms are ‘sin m’ and ‘sin n’, if I put ‘m’ is equal to zero, ‘n’ is equal to zero, in the hope of obtaining the lowest order mode, I will not be able to obtain anything. Because for ‘m’ is equal to zero and ‘n’ is equal to zero, ‘E z’ will be zero, which also makes the total electric field, the transverse components and that magnetic field transverse components also go to zero.

In fact, this condition would be true, no matter whether ‘m’ is equal to zero and or ‘n’ is equal to zero. Thus the modes, which are designated as ‘TM mn’ can never have any of these subscripts equal to zero. So, these two subscripts cannot be equal to zero. Either one can be equal to zero. So what is the lowest order mode in this case? Well, clearly ‘m’ is equal to one, ‘n’ is equal to one is the lowest order mode for ‘TM’ case.

So this is the lowest order mode ‘TM 11, also called as dominant mode. So, dominant mode for the transverse magnetic case happens to have the mode index of 1 and 1. For this mode, the electric field component ‘E z’ goes as ‘sin pi x / a, sin pi y / b’ and there is ‘e to the power minus j beta z’. This is of course the phasor form of the electric field that we have written. So I should technically have written down the phasor notation for this.

I have not included just to simplify the notation. To obtain the correct expression for ‘E z’, which is a function of x, y, z as well as time, you need to multiply this phasor by ‘e to the power j omega t' and then take the corresponding real part. So when you do that one, what you get is, 'x' and 'y' are unaffected. You get 'sin pi x / a, sin pi y / b' and here you get 'cos of (omega t minus beta z)'. This is somewhat comforting for us, because there is a cosine wave form, but that is travelling.

Similarly, you have 'sin of pi x / a, sin pi y / b' describing how the wave would change along 'x' and 'y'.

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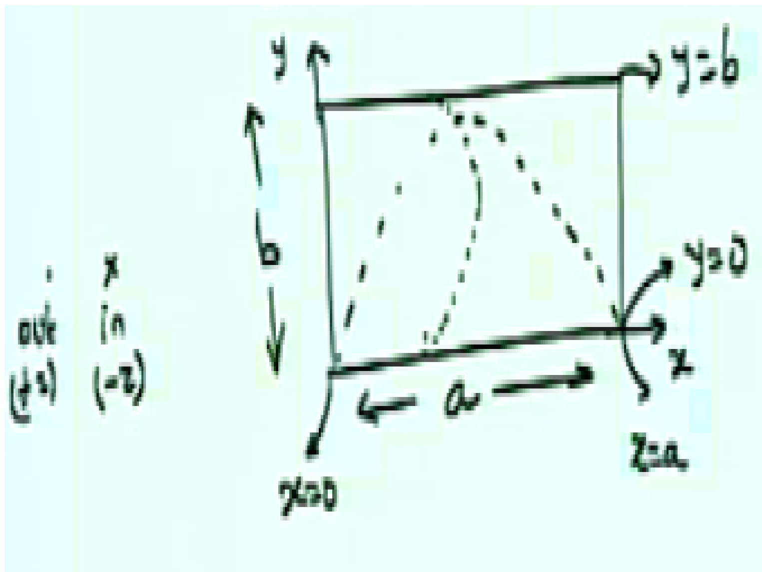

So if you have to sketch this solution, how it would look as a function of say along 'x' and along 'y', so let us say this is 'x' and this is 'y', the width of the waveguide is 'a' and the height of the waveguide is 'b'. What would be the electric field component 'E z'? 'E z' would unfortunately be going away from the page. So, it would be coming away from the page. Use your right hand rule to check that 'x cross y' would be coming towards, out of this particular page.

So, we will not be able to just draw a line or something, we will have to mention them by dots and crosses. So, if you have dot, that would correspond to field coming out and that would be along the 'plus z' direction. And a cross would be, in my notation that would be going in, so which means that it would be 'minus z'. What would be at the walls? See, look at the walls. The walls are at 'x equal to zero', 'x equal to a', 'y equal to zero' and 'y equal to b'.

These are the four walls of waveguide that we have. Of course the wave actually propagates along ‘z’ direction. So in this particular case, what would be the electric field at ‘x equal to zero’? It has to be zero because this is ‘E z’. And ‘E z’ would be tangential to the wall. And that is also given out by this expression. So, substitute ‘x equal to zero’ or ‘x equal to a’, the electric field component will be zero here.

Similarly, at ‘y equal to zero’ and ‘y equal to b’, the electric field will also be equal to zero. So, we will not have any components here, we will not have any components here. Along ‘x’ you will have a component, which would be going as sin and it could do one half of the variation. This would be essentially ‘sin of pi x / a’. Remember all the field components will have to be directed out of the page.

What would happen to ‘sin pi y / b’? Well in this case, if you go like this, you will see that the field components have to be zero in the middle and then they have to be maximum here at the center.

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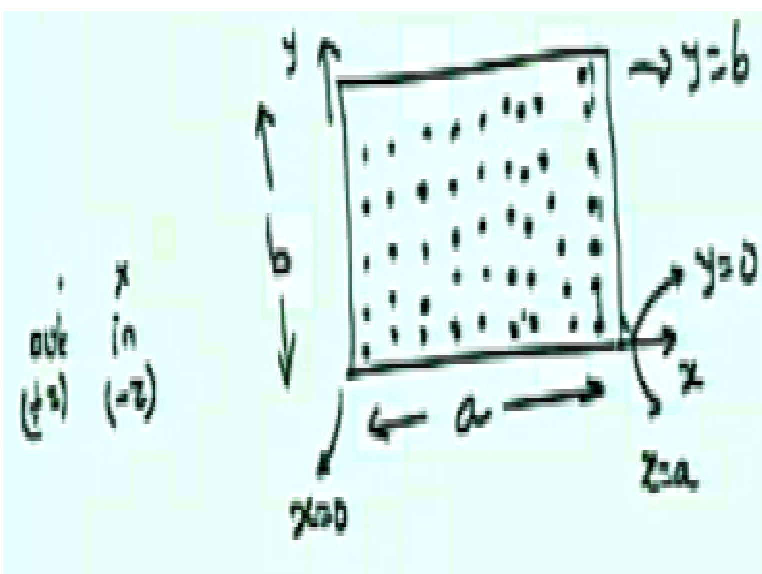

So if you combine these two facts, you would be able to write down the mode, so let me write down the mode here, this is the mode. So, you will have electric field component going to zero there. And then at any point you have electric field ‘E z’ and ‘E y’. We are trying to plot for the

‘TM 11’ case, so it would look like, you know, in the center you would have a maximum electric field and on here you would have a minimum electric field.

So, at the center you will have maximum electric field, at the points here, you would have minimum electric fields. So, it would go to zero here, and at ‘x equal to a’ it would go to zero again. So if you just go slightly beyond here, you will have, you know, density of electric fields that would be increasing. So, here you can see that the density is maximum, not really nicely drawing them, there is a slightly different way of drawing the fields.

This is just the ‘E z’ components, which we have drawn. Of course, if I were to include ‘E x’ and ‘E y’, the picture would look quite different. So, there would be density of the fields, which are maximum.

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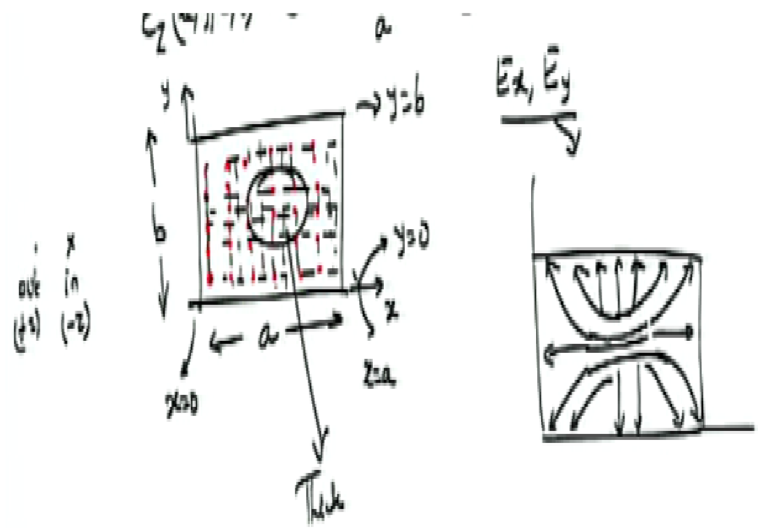

So I should have actually indicated that by, maybe I can draw something like this. So let me draw it in this fashion. And then start putting in the dots here. Again here, I should have maximum dots and then here I should go, see along ‘y’ also I should have minimal dots here and then there is maximum over here and then you go to minimum. So, we should plot the dots here. This is how you would obtain.

These are just the 'E z' plots. So you might actually want to try out this way of plotting by taking a piece of pencil and paper and then nicely rubbing off all the lines that I have drawn here in the center. These are the ones that are not required. But you should get the sense of an idea that electric field lines are crowding at the center and then they would be very thinning out or they are thinning out at the waveguide walls.

So, there is very thick electric field component 'E z' here. Whereas this component keeps on reducing as you go at the outer cases. So, this is for 'E x'. However, if you want to draw 'E x' and 'E y' at a given 'z' plane, the plot would look slightly different. So, given this 'E x' and 'E y' for the 'TM 11' case, the plot would actually look this way. Because you remember also 'E z' actually goes to zero, whereas the components 'E x' and 'E y' are dependent on 'del E z by del x' or 'del E z by del y' and therefore they will not go to zero.

They will actually go to maximum. So because of that reason, the electric field lines would go to their maximum at the waveguide walls, when you are plotting 'E x' and 'E y'. And at the center, they would be minimum. So, at the center they would be minimum and they would look something like this. So, you will actually have a field lines that are going around in this way. So, there would be field lines here like this, slightly on to the curved side.

What would happen to the magnetic field lines? Well, magnetic field lines will always have to form a loop.

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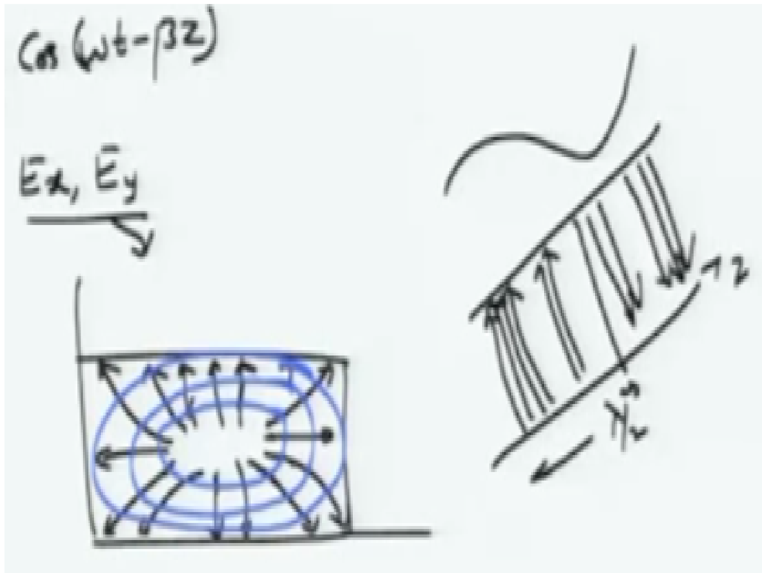

And we can see that the field lines, here there won't be, at the center it would be very small. So, at the center I am actually not plotting like this. So, it would be in this way. So, let me remove this center lines as well. So this we have removed the center. So this is how electric field lines would look. The electric field line does not have any value at the center because if you remember that becomes cos and then in the middle there would be a symmetry.

And therefore there would not be any electric field lines over here. Whereas the magnetic field lines will have to curl around and again the curling would be maximum here and then it would thin out as you go to the waveguide walls. Now at the waveguide walls, they actually do not go to zero, remember because of the continuity of the surface current these 'H' components would actually leave some waveguide wall currents.

So, this is how the electric field patterns would look for a 'TM 11' case. Sometimes it is also necessary to open up along the waveguide wall, along the direction of the wave propagation to see how the waveguides would look. We know that because of 'cos omega t minus beta z' and we are really looking at 'omega t equal to a particular constant', so let us say 'omega t is equal to zero'. So as you go along the 'z' direction, it has to be in the form of a cosine wave.

So it would be maximum here, you know the field lines are all crowding up here. They would start to thin out. So, let us say these are the way in which the field lines are increasing. They would then thin out like this. You know it is a cosine wave form, it would go like that. And then what would happen? After a certain distance, which is 'lambda / two', they would start to reverse their directions. So they would reverse and then they would go to maximum again.

So, this is what a cosine wave form would look along the waveguide direction. So, along the 'z' direction, this is how the waveguide would look, the electric field lines would look as you go along the direction of the waveguide. Similarly, one can plot other modes. We are not going to do that one here. You can look up the pictures of the modes for other higher order modes. Let us not do that one.

But rather, what I am interested at this point would be to try and show you, how the magnetic field lines would give raise to a certain surface current.

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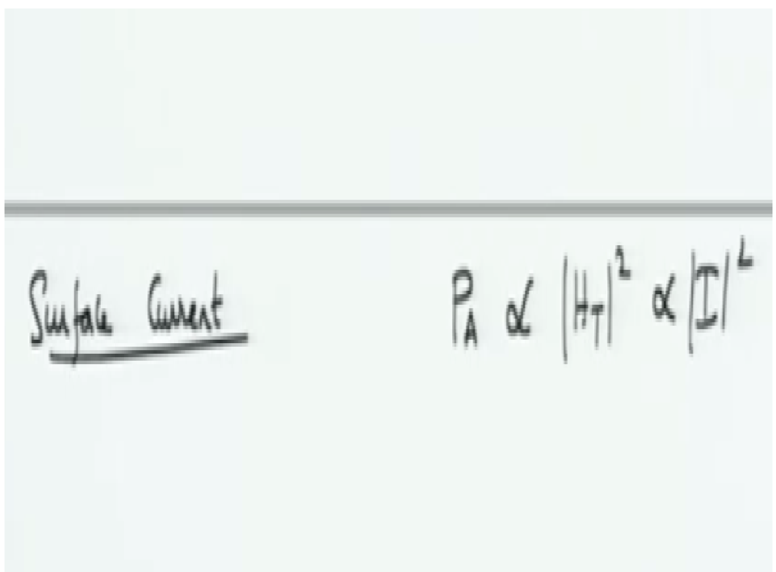

So, let us obtain the surface current because surface current will determine for an imperfect conductor. Surface current is required in order to find out how much power is being lost because of the conduction. If you remember our discussion on skin depth, we wrote something like this, 'the power dissipation per unit area' was something that was proportional to magnitude of 'H T square', where 'H T' was the transverse component for the corresponding conducting surface.

So, it was essentially the power loss being proportional to ‘H T’, which was in turn proportional to the current square that we had. So, once I know the surface current in terms of ‘H T’, then I can find out what would be the power that is being lost or if I know the magnetic field component ‘H’, then directly I can find out what is the power being lost. So, it is because of this imperfect conductor that a certain surface current is induced.

And this surface current that is induced will result in heating up of the wall, in the form of power dissipation.

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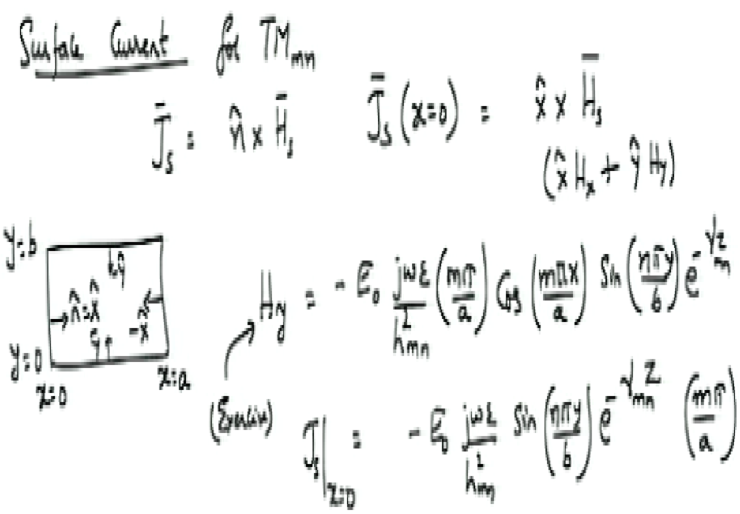

So, let us try to find out the surface current for TM modes. We do not have to specify what is ‘TM of mn’. So let us find surface current for TM modes. And we know the result already, in the sense that, we know the relationship between surface current and the magnetic field component. So, the magnetic field component must be evaluated the surface. And then you take the cross product of ‘n’, the normal to the waveguide wall, pointing inverse into the waveguide, times ‘H s’.

So let us, we have four walls here. So one wall is at ‘x equal to zero’ and the other is at ‘x equal to a’, we have one wall ‘y equal to zero’ and ‘y equal to b’. So where will the directions of normal be pointing? So for the ‘n’, ‘x is equal to zero’ wall, the direction will be along ‘x’ direction. So normal is along ‘x’ direction. Whereas for ‘x equal to a’ wall, the direction of the normal will be along ‘minus x’ direction.

Similarly, for ‘y equal to zero’, it would be along the ‘y’ direction. At ‘y equal to b’, it would be along ‘minus y’ direction. So once you know these direction normals, then you could find out the surface current. Let us do it for the ‘x equal to zero’ wall and because of symmetry, the same expression would be there for ‘x equal to a’ wall. If you do it for the ‘y equal to zero’ wall, if you find out the surface current for ‘y equal to zero’ wall.

It would be essentially be the same for ‘y equal to b’ because of symmetry. So out of four components, you need to evaluate only two components. I will do one more thing. I will evaluate only one component and leave the other component, evaluation of surface current at the wall ‘y equal to zero’ and ‘y equal to b’ as an exercise for you. So, you have ‘J s’ at ‘x equal to zero’ wall given by ‘x hat cross the magnetic field at that point’.

Now what are the magnetic field components? You have ‘x hat H x’, that is you have ‘H x’ and ‘H y’. So you have ‘x hat H x’ + ‘y hat H y’ are the two components of the magnetic fields. There is no ‘H z’ component because of TM mode condition. And when you take ‘x cross x’, the result will be zero. ‘x cross y’, the result will be along the ‘z’ direction. And then you need to substitute for the value of ‘H y’. So, I will not solve this one.

But I will, I mean I will not go through in detail, but I will just recall the value of ‘H y’. You have to probably show this one as an exercise by following the last module, wherein we derived ‘E z’ and also indicated what forms for ‘H y’, ‘H x’, ‘E x’ and ‘E y’ would be. So you have ‘H y’, which is the component of the magnetic field, the transfer component of the magnetic field, which must be evaluated at the surface. First let us look at ‘H y’ itself.

The expression for ‘H y’ as a function of ‘x’ and ‘z’ will be having some constants. Let us just write down the constants. It is your, this one to actually prove that these are the constants that get multiplied to this one. And because of the differentiation of ‘E z’, with respect to ‘y’, there would be ‘npi / p’ that is coming out here. The ‘x’ dependency is still ‘cos mpi x / a’, whereas the dependence for ‘y’ will become ‘sin’.

It would be ‘npi y / b’, there is also ‘e to the power minus gamma z’ as the dependence on ‘z’ direction.

Of course gamma must also be written with its subscript ‘gamma mn’. So, this is your ‘H y’ phasor. Now you evaluate the phasor at ‘x equal to zero’ and at ‘y equal to zero’. So if you evaluate this phasor at ‘x equal to zero’, what do you get? You will see here that this is you will have ‘m pi / a’. So you can actually find out this component from the previous method that we discussed in the previous module.

‘H y’ evaluated at ‘x equal to zero’ will give you ‘minus E zero j omega epsilon divided by h square mn’. I do hope that you remember, what is h square? It was related to gamma and omega. And ‘x equal to zero’ cos of zero will be one. So, you just get ‘sin of npi y / b’, ‘e to the power minus gamma mn of z’. This is the surface current that would be present at ‘x equal to zero’.

The surface current shows a dependence along ‘sin’. It shows a dependence on ‘sin’ as well as it shows a dependence along, this needs to be multiplied by ‘mpi / a’, it also shows a dependence along the mode order itself. So ‘m’ and ‘n’ values determine, how much is the surface current? Again you cannot have ‘m’ equal to zero and ‘n’ equal to zero because then there will not be any surface current.

And moreover ‘m equal to zero’. ‘n equal to zero’ is not the correct solution. For the fundamental ‘TM 11’ case, ‘m’ will be equal to 1, ‘n’ will be equal to 1. And whatever the expression that you obtain is the surface current. Let us close our discussion on TM waves and then proceed to what we call as TE modes inside the wave guide.