Principles of Communication Systems - Part II
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Lecture - 10
Probability of Error in Digital Communication, Optimal Decision Rule, Gaussian Q function

Hello. Welcome to another module in this massive open online course. So, we are looking at the performance of a digital communication system and we have said that the output statistic after sampling can be described as follows.

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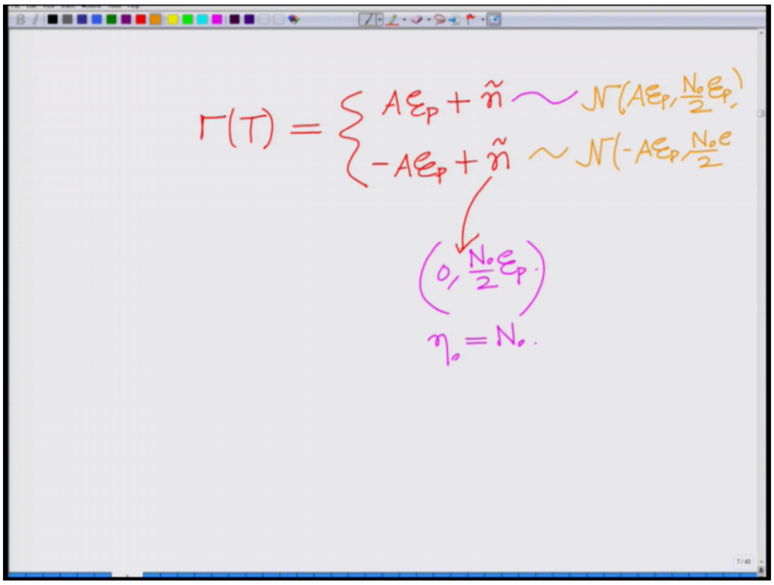

That is we have r of T corresponding to the transmission of plus A, we have A E p plus n tilde corresponding to the transmission of minus A we have minus A E p plus n tilde where n tilde is Gaussian noise correct, with mean 0 variance that is eta naught by 2 times E p or n naught by 2 times E p; so n naught equal to eta naught.

So, corresponding to this we have the Gaussian distribution n tilde plus A E p which is now Gaussian distributed because addition of A E p shift this is a mean to A E p and the variance remains the same. So, the variance is simply n naught by 2 times E p in this case you will have a Gaussian which is mean minus A E p and variance remains the same n naught by 2 A E p.

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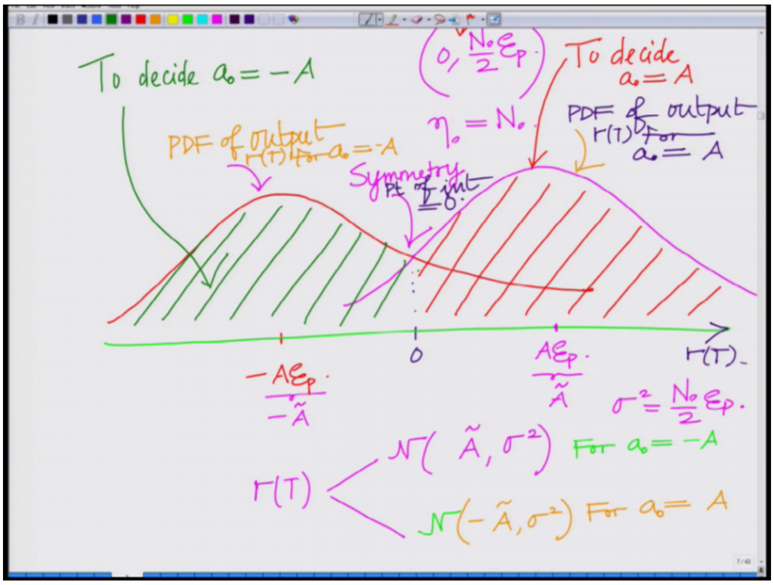

And we had drawn a figure to describe that that is I can represent corresponding to minus A I have a Gaussian, corresponding minus A E p that is the mean corresponding to the mean minus A E p and corresponding to A I have a Gaussian with mean A E p. Variance of both is there and by symmetry point of intersection is 0. This is the PDF, remember PDF this is the probability density function of output that is r of T, PDF of output r of T for your a naught equals minus A, this is PDF of output r of T for a naught is equal to plus A. And by symmetry this point is 0, by symmetry this point of intersection, point of intersection is 0 and this you can say is the (Refer Time: 03:34).

So, corresponding to both the transmission of minus A we have a Gauss the probability density function of the output r of T is minus A E p all right. it is a Gaussian probability density function with mean minus A E p variance and naught by 2 E p and corresponding to the transmission of plus a its a Gaussian the probability density function of the output is Gaussian with mean plus A e p and variance N naught by 2 times E p.

And now, therefore, one simple way, now, let us first denote this A E p by A tilde for convenience. So, this is you can write this as minus A tilde and A tilde. So, corresponding to, so r T has either the probability density function Gaussian with mean minus A tilde or A tilde comma let us write sigma square let us denote sigma square equals N naught by 2 E p. So, in my A tilde corresponding to for a naught, so this corresponds to a naught equal to for a naught equals minus A and this is n Gaussian distributed with mean minus A tilde variance sigma square for a naught is equal to plus A.

Now, therefore, now if you look at it by symmetry therefore, now why lies now, r T lies in the range minus infinity to plus infinity. By symmetry look at this, by symmetry I can decide this region that is this region corresponds to the transmission. So, by symmetry now if you think about it we can say this region corresponds to the transmission of or this region can be used to decide a naught equals well plus A and if you look at this region which is to the left of 0 where the probability density function corresponding to a naught equal to minus A is higher this can be used to decide a naught is equal to.

Now if you observe this probability density functions you will see that for r T greater than 0 probability density function corresponding to the transmission of a naught equal to A is higher and for r T less than 0 the probability density or the probability corresponding to the transmission of a naught equals minus A is higher for instance.

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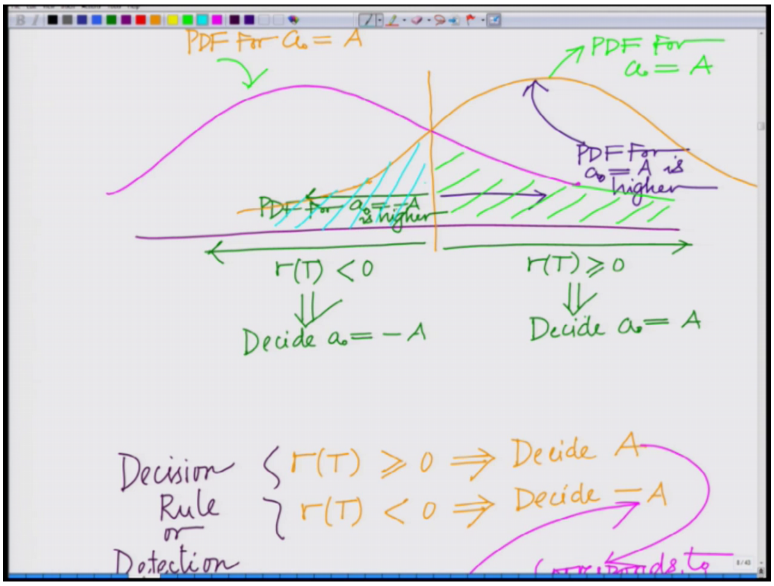

Let me just again illustrate this, if you look at this we have already said that by symmetry mid point is 0. So, now if you look at this, this is the PDF for a naught equals A and this is the PDF for a naught equals minus A, and here if you can look at this, if you can look at this region you can see the PDF for a naught equals A is higher. And if you look at this region the PDF for a naught equals minus A is higher, PDF for a naught equals A is higher, PDF for a naught equals minus.

Therefore, I can say that if you look at this region that is r T greater than or equal to 0 I can decide a naught equals a or detector is this is known as the decision rule or the detection rule one can decide a naught equals A and corresponding to this region r T less than 0 decide a naught equals a naught equals minus A. So, corresponding to these two, so corresponding to these two regions one can make an appropriate decision this is termed as the decision rule.

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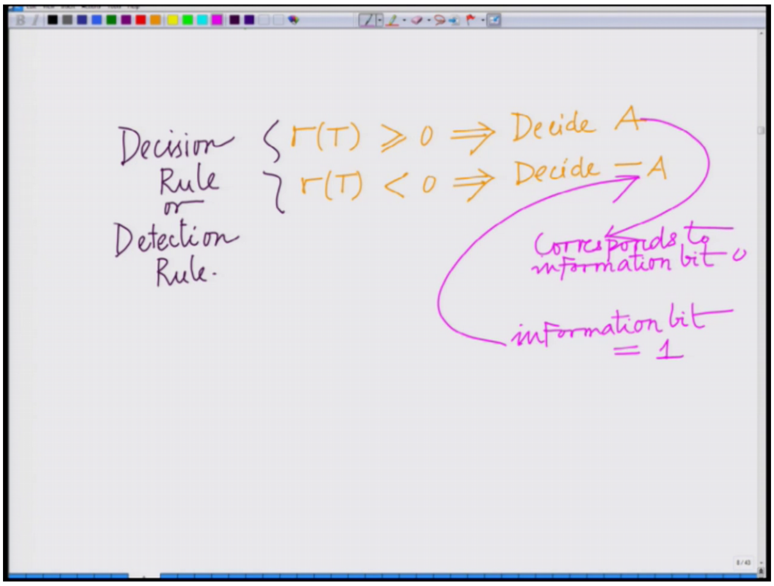

So, our decision rule is r T greater than or equal to 0 implies decide A, r T less than 0 implies decide and as we said this is basically the decision rule or the detection rule or also known as the detection rule, simply known as the decision rule. So, it is a very simple decision rule or detection rule that is of the sample at t is greater than or equal to 0 we decide or we take a decision that a naught is equal to plus A which corresponds to the transmission corresponds to the information bits 0 if r T is less than 0 then we decide that a naught is equal to that is a naught equals minus A has been transmitted corresponding to the information symbol one. So, that is our simple decision rule. And it is also very intuitive by symmetry. So, this corresponds to information bit 0 and this corresponds to information bit 1, corresponds to the information bit 1. So, that is our simple decision.

Now, when does error. So, now, we have to calculate; what is the probability of error. So, we want to calculate; now we want to calculate when does, so we want to calculate the probability of error.

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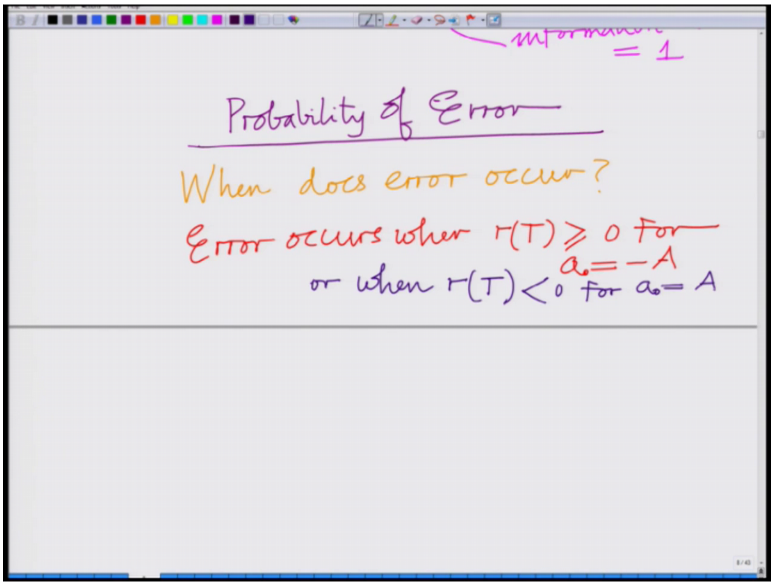

We want to ask the question; when does error occur. Now if you can look at this error occurs either when r T is greater than 0 corresponding to the transmission of a naught t equal to minus a that is basically if you look at this it corresponds to this probability or error occurs you can see if r T is less than 0 corresponding to the transmission of a naught equal to minus A that is this shaded region. So, error occurs when r T is greater than 0 or greater than equal to 0 for a naught because if a naught equals less; if r T is less than 0 then we choose a naught equals minus A. So, if r T equals greater than equal to 0 then error occurs.

So, error occurs or when r T is less than 0 for a naught equals A. So, error occurs if the sample r T is either greater than or equal to 0 when the actual transmitted symbol is a naught equals minus A or when r T is less than 0 when the actual transmitted symbol a naught equals A.

So, now let us look at when error occurs.

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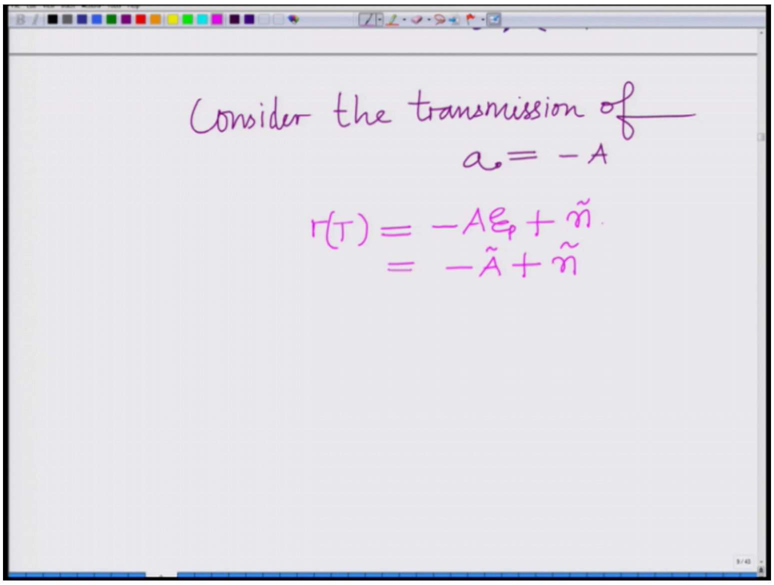

So, error occurs for instance, let us consider a naught equals, consider the transmission consider transmission of a naught equals minus a we have r T equals, well corresponding to a transmission of minus a we have r T equals minus A E p plus n tilde that is we have called this as minus A tilde plus n tilde.

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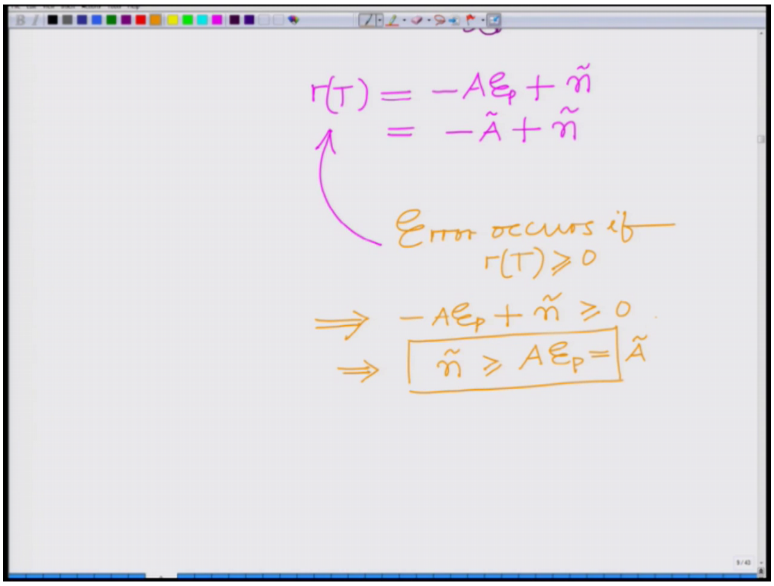

Now, error occurs if r T is greater than or equal to 0, if r T is greater than or equal to 0 which implies minus A E p plus n tilde is greater than equal to 0 which implies n tilde is greater than or equal to A E p. So, error occurs if n tilde is greater than equal to A E p or n tilde is greater than or equal to A tilde which means it pushes although what we are expected to get is minus A tilde, the noise addition of the noise pushes this minus a tilde to be that is adds when noise n tilde adds to this minus A tilde it pushes it to a level that is greater than or equal to 0.

And what is the probability of this? Therefore, the probability of error is simply the probability that n tilde is greater than or equal to A tilde therefore, probability of A tilde probability of error is simply the probability that n tilde is greater than or equal to A tilde.

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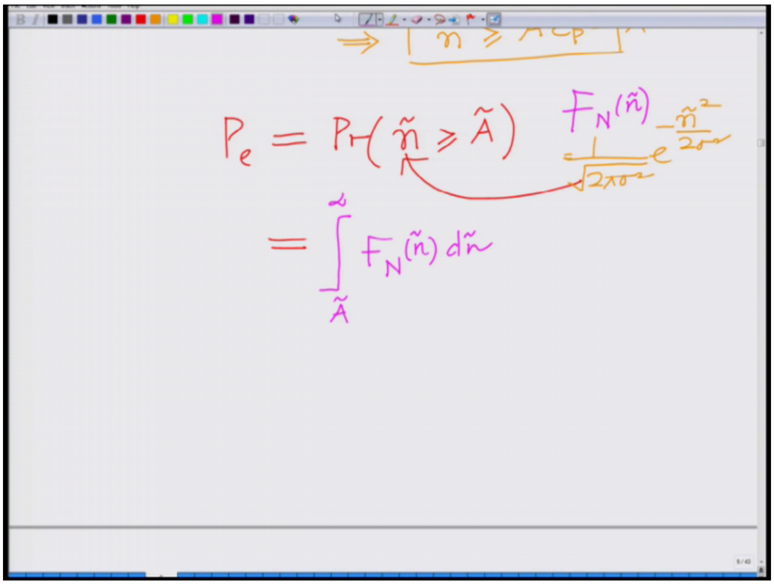

We know that the probability, probability density function of the noise is noise is Gaussian in nature we know that. So, this is 1 over 2 pi sigma square e raised to minus n tilde square divided by 2 sigma square that is what we have correct, we have the noise and therefore, this is the probability density function and therefore, to get the probability that n tilde is greater than equal to A tilde I simply have to integrate the probability density function of the noise from A tilde to infinity.

That is this is the probability density function f of Gaussian probability density function. So, I have to integrate this probability density function from A tilde to infinity. And this is given as well A tilde to infinity 1 over square root 2 pi sigma square e raised to minus n tilde square divided by 2 sigma square d n tilde.

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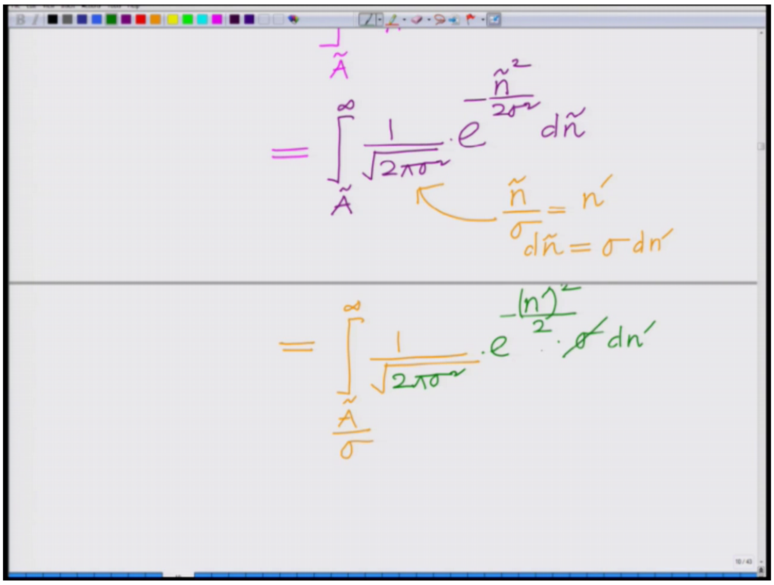

So, basically integrating this probability is nothing, but integration of the Gaussian probability density function from a tilde to infinity. And now that is the probability that n tilde is greater than a tilde that is it contagion value from A tilde to infinity that is the corresponding probability of error.

And if I integrate this now I will make a simple substitution I am going to substitute n tilde by sigma is equal to let say we call this as n prime, then d n tilde equals sigma d n prime then this will become well the limits will become a tilde divided by sigma to infinity 1 over square root 2 pi sigma square e raise to minus e raised to minus n tilde square divided by sigma square is n prime, n prime square divided by n tilde square divided sigma square is n prime square. So, n prime square divided by 2 d n tilde is sigma d n prime, so this sigmas cancel.

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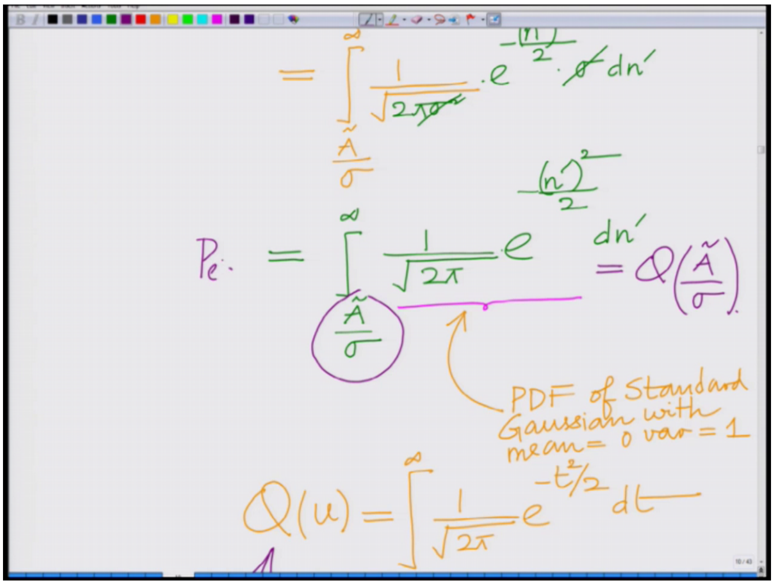

And what we have is interestingly if you observe this you will have a tilde divided by sigma 1 over square root of 2 pi e raised to minus n Prime Square divided by 2 d n prime and if you look at this, this is the probability density function. If you look at this, this is the probability density function of the standard Gaussian random variable with mean 0 and variance unity we already seen this. Standard Gaussian or standard normal random variable is a Gaussian random variable with mean 0, with mean 0 and variance, variance equals unity or variance equal to 1. And this probability the probability that standard Gaussian random variable with mean 0 this is known as, this is defined as a standard function that is.

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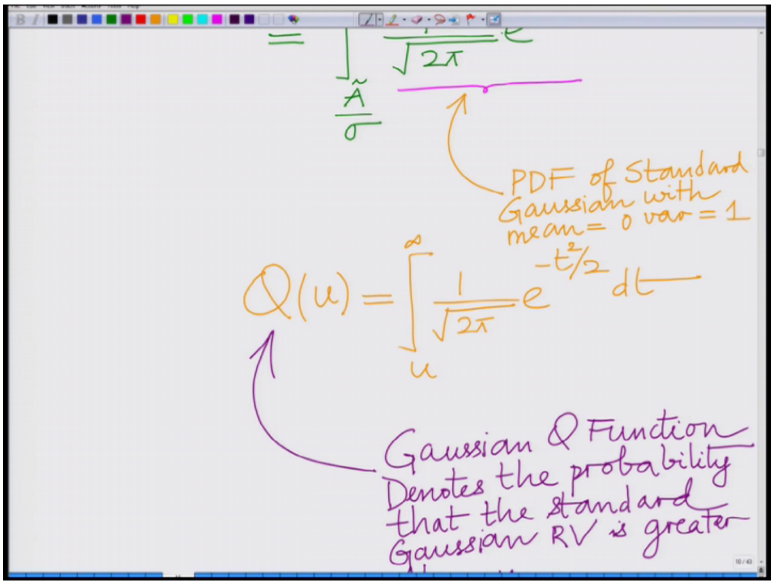

Q of u is equal to u to infinity 1 over square root of 2 pi e to the power of minus t square by 2 d t this is known as the Gaussian Q function, this is known as the Q function or the Gaussian Q function. And this denotes this denotes the probability that the standard Gaussian random variable with mean 0 and variance one that is Gaussian standard Gaussian RV is greater than u.

So, Q of u is basically the integral of the standard Gaussian probability the standard Gaussian probability density function mean 0 and variance from u to infinity. That this is denoted this is known as the Q function. And if you look at this the probability of error that is nothing, but basically this is the integral of the standard Gaussian random variable with mean 0 variance from A tilde by sigma to infinity and therefore, this is equal to Q of A tilde divided by sigma and that is the probability of error and therefore, what we have been able to show is basically this probability of error.

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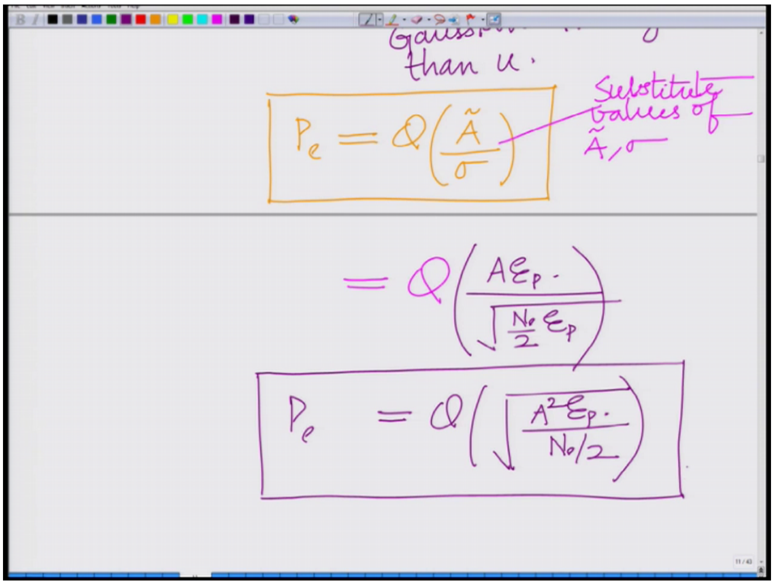

This is an important result this is equal to A of A tilde by sigma. And now we all we have to do is substitute the values of a tilde and sigma. So, now substituting the values of A tilde and sigma what we have is Q of A tilde is A E p sigma square is n naught E P by 2. So, sigma is square root of n naught by 2 over E p and therefore, simplifying this is Qs Q of A square E p square divided by E p is E p divided by N naught by this is the probability of error where Q is the Gaussian Q function this is the probability of error for our system. So, this is the probability of error.

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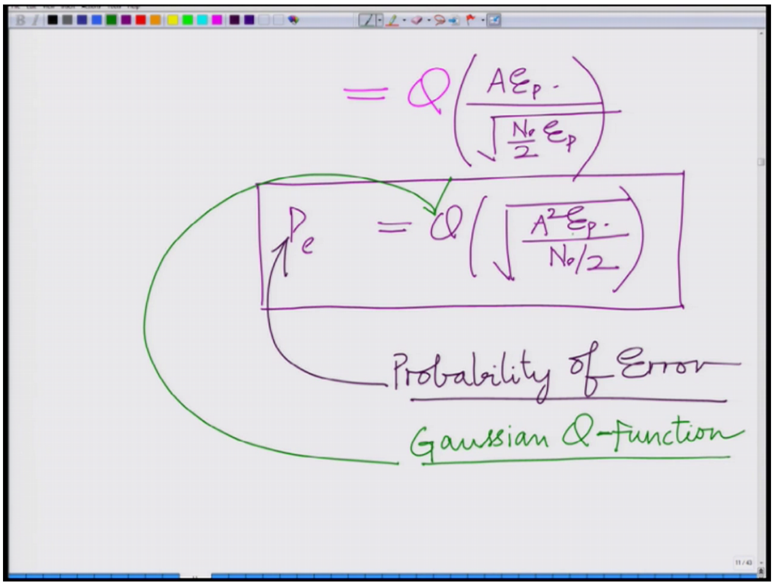

So, this is the probability of error corresponding to all digital modulation digital communication system, this is the probability of error which is given in terms of the Gaussian Q function. So, probability of error is Q under root a square E p divided by Q under root A square E p divided by eta naught by 2 of course, we have seen a is basically the amplitude, E p is the energy of the pulse eta no it naught by two is basically well n naught by 2 denotes basically the denotes additive white Gaussian noise with autocorrelation function N naught by 2 times delta tau that is expected value of n t into n t plus tau is N naught by 2 times delta tau.

So, basically what we have done in this module and its important because this lays the foundation one of the fundamental aspects of a digital communication system that is what is the receiver employing the matched filter how do you compute the signal to noise power ratio and for a given modulation scheme correct, what is the resulting probability density function corresponding to the transmission of different, corresponding the transmission of different symbols belonging to the digital constellation correct, at the output the probability density function at the output corresponding to the samples; corresponding to the samples for the various symbols all right and from that how do we come up with an optimal decision rule and what is the resulting probability of error. So, we will stop here and continue with other aspects in the subsequent modules.

Thank you.