Principles of Communication Systems - Part II
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture – 07
Digital Communication Receiver, Noise Properties, Output Noise Power

Hello. Welcome to another module in this massive open online course. So, we are looking at the signal processing or how to process the receive signal at the receiver of the digital communication system and we have seen basically the signal component and the noise component of the output signal after it is passed through the received filter. Now let us characterize the noise power what is the power of the noise at the output of this filter, filter with impulse response h t.

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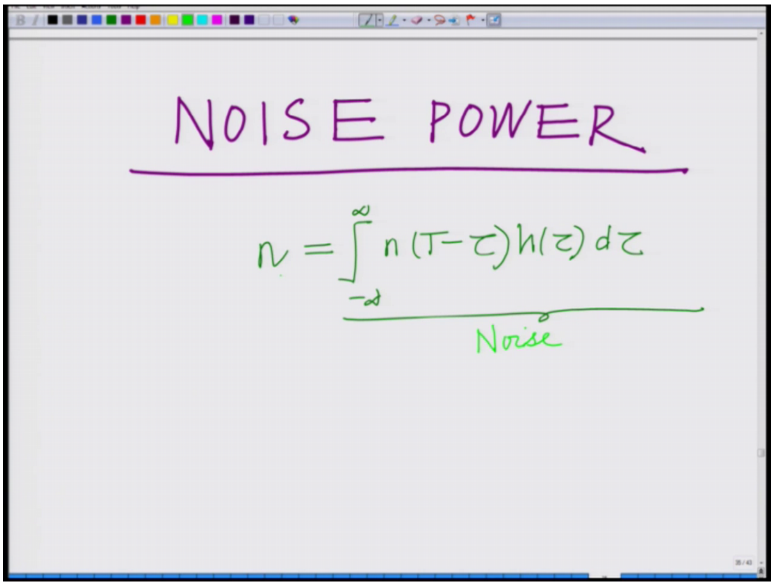

So, in this module let us start looking at how to characterize the noise power. So, we have characterized the signal power. Now naturally I have to also look at the noise power because remember ultimately I want to look at the signal to noise power ratio, which is which we said is an important metric to characterize the performance of the digital communication system.

And in this regard what we have seen is that the noise first let us look at the noise the noise n is given as integral minus infinity to infinity; N T minus 1 let me just write it exactly using the notation that we have used N T minus tau h tau small n t minus tau and this we said is the noise this is the noise component at the output, this is the noise which I am denoting by small n let us not denote this by small n let us simply denote this by let us call this as n tilde.

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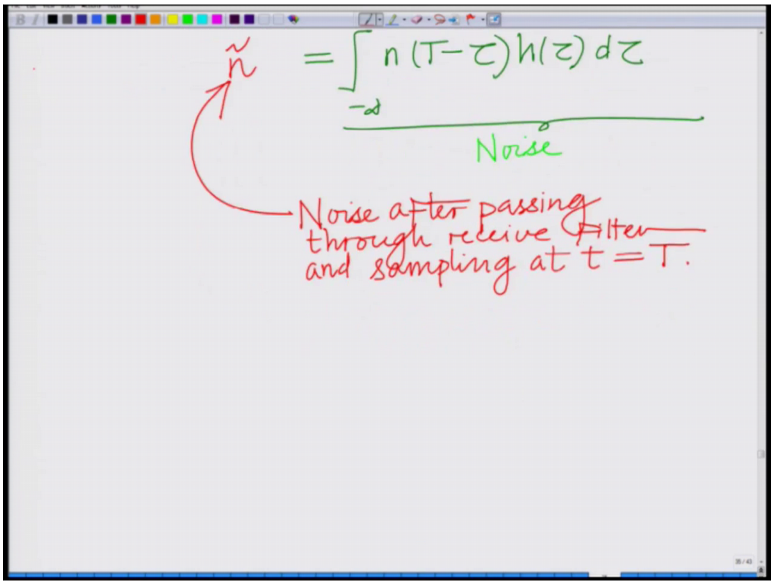

So, this is the output noise. So, this n tilde is noise after. So, noise. So, this is basically your noise after passing through the received filter and sampling at t equal to capital P. So, this is noise after and sampling at t equal to capital T. Now, what we want to do is you want to calculate the noise power remember the noise power is assuming all real quantities noise power is expected value.

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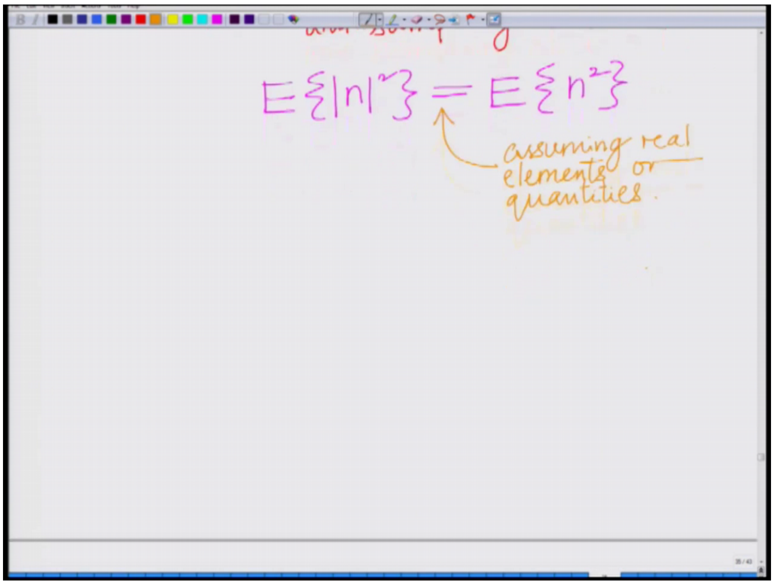

So, technically I have to write expected value of magnitude n square if the noise is a complex quantity, but assuming all real quantities here for again once again for simplicity of analysis, I am going to simply write this as n square. So, this basically is assuming; again I am just hint I am just describing each aspect carefully. So, that you understand that this is for a special scenario with real elements or real quantities real elements or quantities, assuming all the quantities are here. Now in order to simplify this I am going to make a simplifying assumption we are going to now go back to the most popular instead of the most one of the most relevant assumptions that can be made regarding the noise, that is the noise as we already seen typical noise random process is Gaussian it is additive white Gaussian.

So, we are going to assume that this noise n t is additive. So, at this point what we have done until up until this point does not depend on the specific nature of the noise. The noise which is additive right pass through c filter and then sampled, but now to further simplify the noise power I am assume your way to assume we are going to invoke the popular assumption that the noise is white and Gaussian in nature which means.

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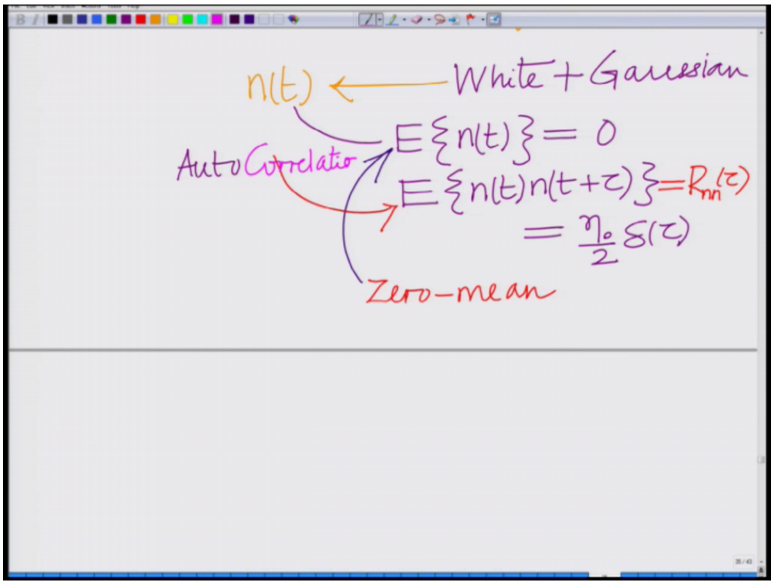

So, we are going to assume now invoke the assumption that noise is white and Gaussian we already see noise is additive. So, this is additive white Gaussian. So, noise is white plus Gaussian implies now with and also we are going to assume that the noise expected value of n t, the mean is 0 and expected value since the noise is white we are going to assume that expected value of n t into n t plus tau is eta not by 2 delta at with the spare we have seen that if them is the autocorrelation is eta naught by 2 delta tau there is an impulse also therefore, the power spectral density which is the Fourier transform the autocorrelation is flat or the entire frequency domain and hence this is known as white noise because the power is uniformly distributed across all the frequency.

So, we have already seen this in the previous modules. So, now, this noise has zero-means. So, this is the zero-mean also something to keep in mind we are assuming that the noise is zero-mean that is a typical assumption noise is zero-mean, and it has basically a autocorrelation this is basically or nothing but your autocorrelation function remember and for this remember we are assuming the noise to be wide sense stationary, since the noise is Gaussian wide sense stationary directly implies strict sense stationary also, but anyway for the purpose of this wide sense stationary is purpose of this analysis wide sense stationarity is sufficient.

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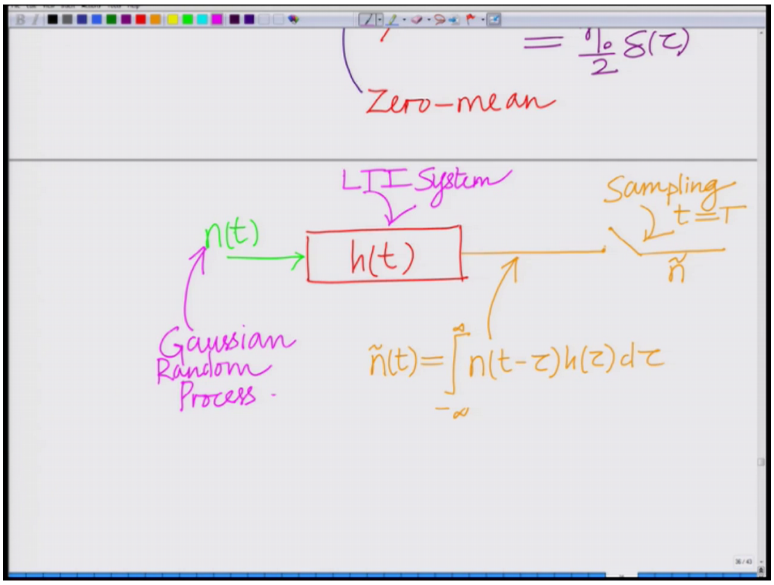

Now if you can look at this noise let us go back a little bit and look at how this noise is obtained; let us just recap a little bit I have noise correct in the noise at the receiver the thermal noise at the receiver which is passed through this, so that at this point you have minus infinity and t minus tau, h of tau d tau followed by sampling correct followed by. So, this is your let us call this as n tilde t followed by sampling. Therefore, this will be n tilde of tau sampling at t equal to tau. So, this is your n tilde of capital T which is calling as n tilde. Now the first thing that you have I d that you would realize is well of course, we have said that this noise is input noise process is Gaussian, this is an LTI system that is linear time invariant system.

So, it performs a linear operation. So, this is in LTI system, and we have assume then the noise input noise is Gaussian or it is a Gaussian random process or it is a automatically implies that noise is a when I say Gaussian it automatically implies that the noise is the Gaussian random process. Now the first important thing that you have to realize is you have a Gaussian random process which is pass through a linear system; a linear time invariant system when a Gaussian random process is passed through a linear system the output is also a Gaussian random process this is a very important. Any linear transformation on a Gaussian random process results in a output Gaussian random process that is the important thing.

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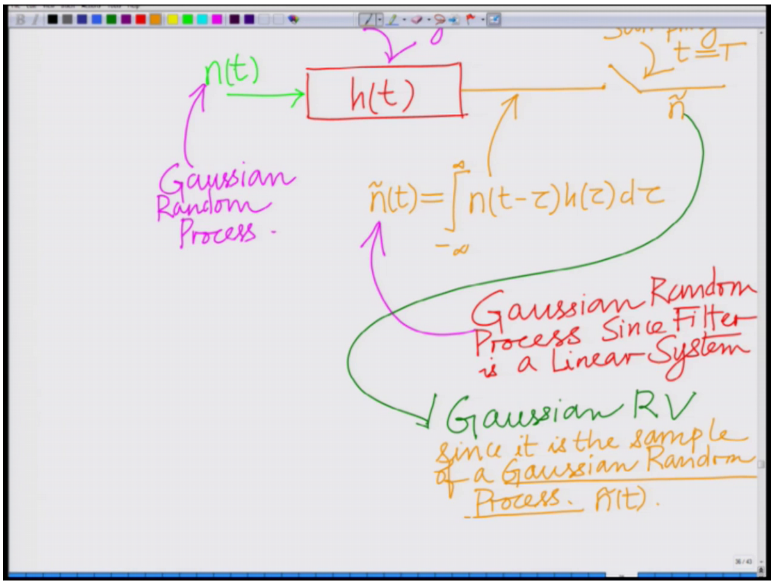

So, therefore, this noise process n tilde t is also a Gaussian random process that is the most important thing this is also a Gaussian random process, silt since the filter is a linear system. So, input random process is Gaussian pass it through a linear system output is Gaussian. Remember we said if the random process is Gaussian that is when output random is gross the process Gaussian the meaning of that is statistics right joint statistics at all times are basically Gaussian, and therefore if you are sampling at this time t equal to capital T. So, joint statistics at all times are Gaussian in particular at any time instant if you consider the noise sample that is Gaussian in particular if you come to consider the sample at t equal to capital T that is also Gaussian, and it is a single sample therefore, it is a Gaussian random variable.

So, you take a Gaussian noise process you sample it at a single time instant; what you get is a Gaussian random a random variable. So, what we are saying is input noise process is Gaussian it is passed through a linear system therefore, output noise process is also Gaussian, you are sampling it at t equal to capital T you are considering a symbol single sample and therefore, the sample is a Gaussian random variable that is the most important thing. So, this sample n tilde in fact, we have justified that this sample n tilde this is Gaussian random variable, since it is a sample of a Gaussian random process. Since it is the sample of a Gaussian random process n tilde t we have already argued that n tilde t is a Gaussian random process.

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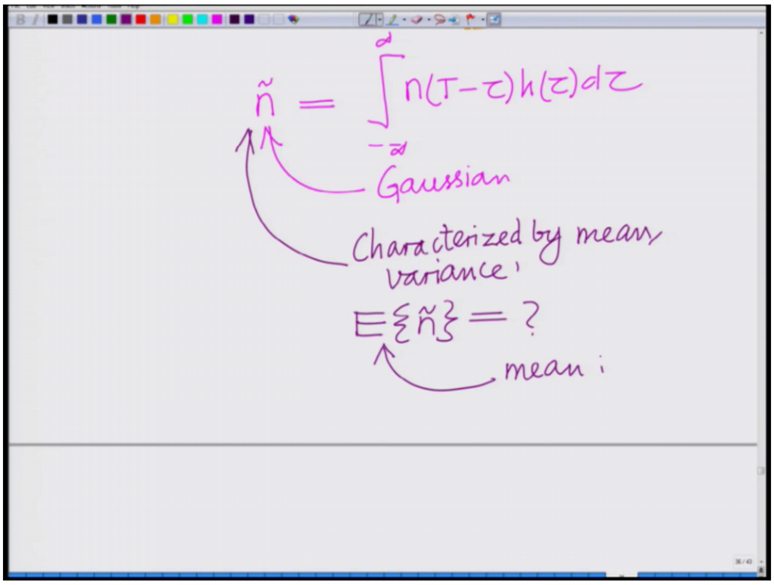

Now therefore, n T or n tilde which is minus infinity to infinity n T minus tau, h of tau d tau this is Gaussian in nature. For a Gaussian random variable remember it is characterized by the mean and variance any Gaussian random variable is characterized completely by mean and variance; now let us see let us ask this question what is the expected value of n tilde, what is the mean that is expected, expected value is nothing but the average or basically mean what is the expected value of n tilde.

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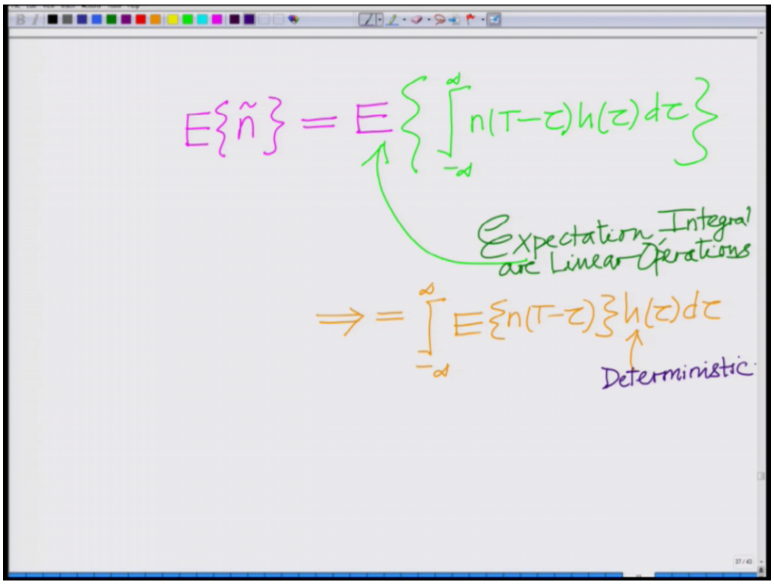

Now, to answer that look at this the expected value of n tilde equals the expected value of minus infinity to infinity, n t minus tau h tau d tau. Now expectation operator is linear, so I can take it inside the integral since the expectation operator is linear integral comma integral, expectation and integral are linear operation.

So, this implies basically this is equal to. So, can take the expectation operator inside the integral and then you will see something very interesting you will have expected infinity to minus infinity, expected value of n t h tau d tau now you realize that I am not using the expected value for h or expectation for h of t, because this is a deterministic quantity because, the receive filter is a fixed quantity it is a deterministic quantity, deterministic means basically it is a fixed quantity.

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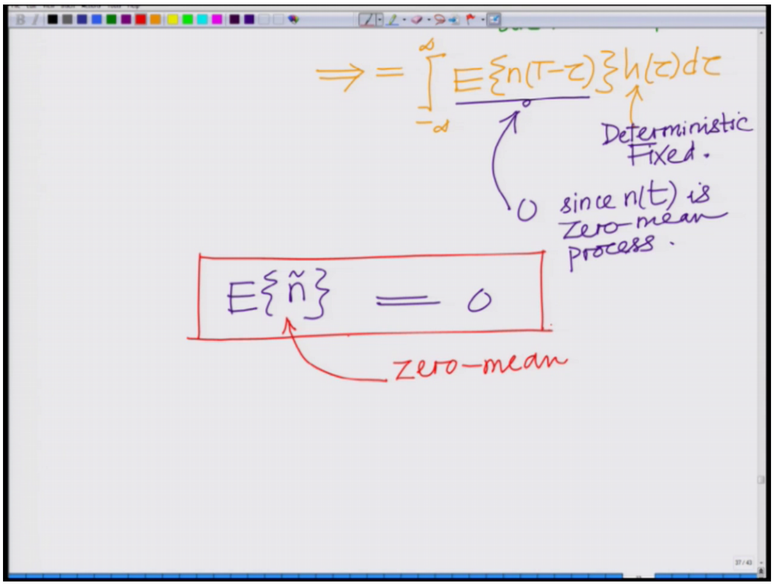

Now, we have seen that expected value of n t is 0, the noise process zero-means. So, this is 0.

Since n t is a and therefore, this is 0 integral of 0 is basically 0 and therefore, what you get is basically this quantity which is basically your mean of n tilde is very simple, although you might have realized that even without this elaborate derivation, but just for the sake of rigor mathematical rigor and formality we have shown that this expected value of zero-mean expected value of n tilde is also 0; so x. So, n tilde is a zero-mean random variable.

So, naturally if you take a zero-mean random process correct that is a random process which has an average value 0 pass it through a linear system or apply linear transformation on that, then the output will also be a zero-mean output will be a zero-mean random process, you are sampling it at t equal to capital T and that will be a zero-mean random variable. So, this follows naturally although we have also proved it verified it rigorously since the expectation operator is basically linear.

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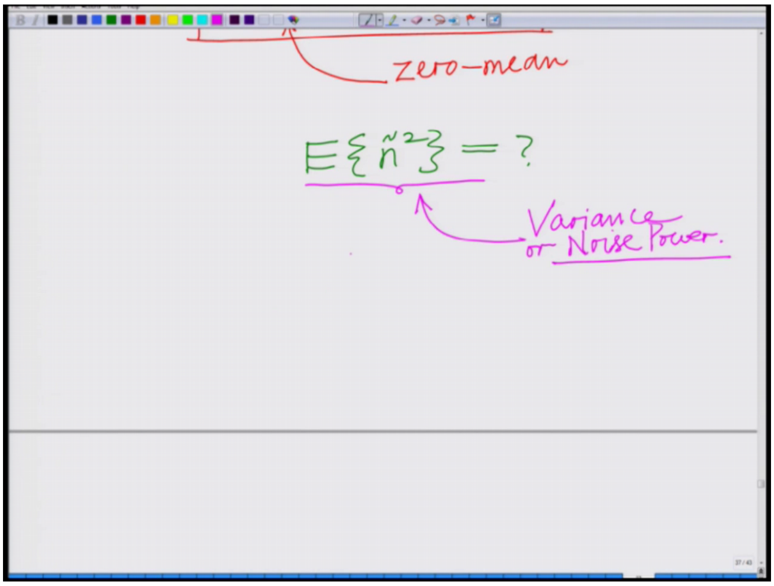

Now, the second aspect the more important point is the variance which is the noise power that is the more important and more interesting thing that is, how do we find out what is the variance. The variance or basically this gives us the noise power; what is the variance or noise power to do that let us compute expected value of n tilde square.

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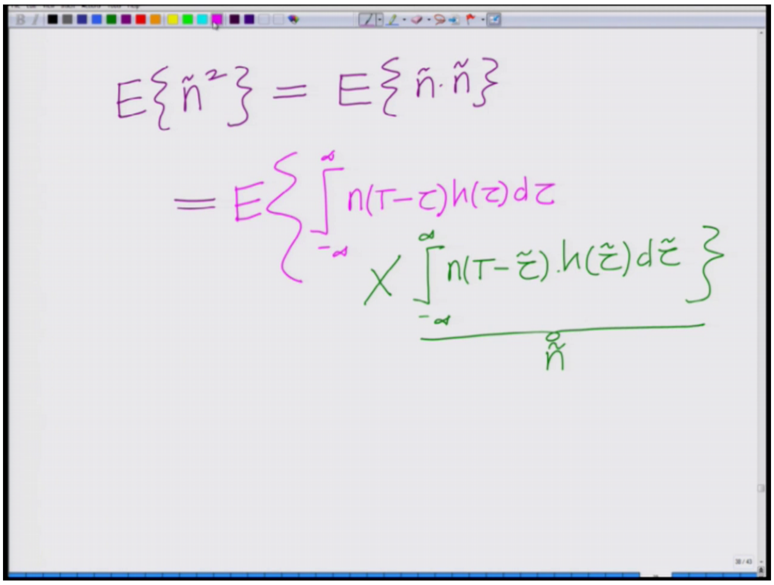

So, we have expected value of n tilde square I am going to write it as expected value of n tilde square, I am going to write this as expected value of n tilde times n tilde, just for the sake of convenience because it helps me now I am going to write it expected value of n tilde that is integral minus infinity to infinity, n of t minus tau h tau d tau this is 1 n tilde times I am going to just simply change the variable of integration.

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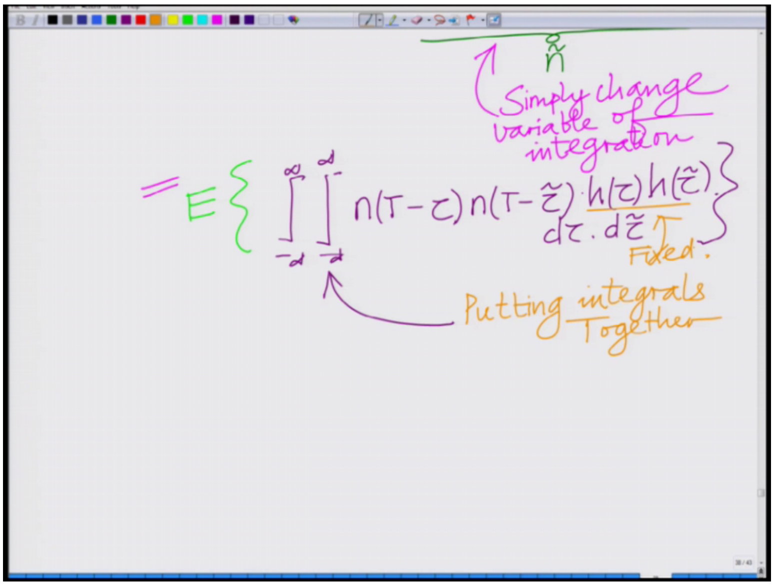

The integral remains unchanged, n t minus tau tilde h tau tilde n tilde. You will realize that this is also n tilde, which have obtained by simply change of integration variables.

So, that does not change the quantity. So, simply I can call it tau or tau tilde I will yield the same integral, but now let us now simplify this product.

Now, I can simplify this product as now I can write it as a double integral that that is very interesting, I can write this as the double integral expected value of minus infinity to infinity minus infinity to infinity well n of t minus tau, n of t minus tau tilde into h of tau into h of tau tilde, d tau, d tau tilde, this is what I can write it as I am bringing both the integrals together I am clubbing the integrals. So, evaluated as a double integrals putting the integrals together, and now you will observe something interesting now again we are going to use the properties that the integration integral is a linear operator there can so I can take the expected value inside the integral.

And again realize that n t minus tau and n t minus tau tilde both are random, but h tau and h tau tilde these are deterministic quantity. So, the expectation operator does not apply to them, because these are fixed quantities h of tau and h of tau tilde. So, naturally I can take the integration operator. So, h of tau and h of tau tilde realized and it goes without saying that these are fixed quantities.

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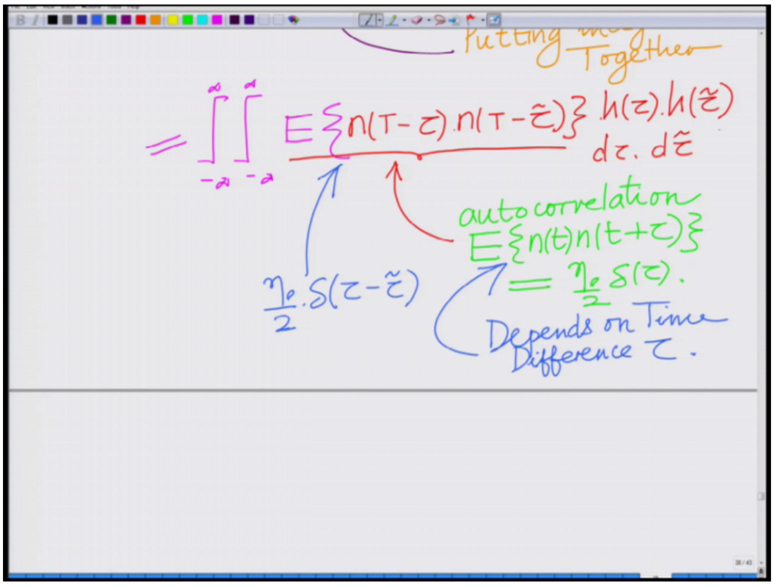

Therefore I have integral minus infinity to infinity applying it only to n of t minus tau, n of t minus tau tilde h of tau tilde d tau, d tau tilde. Now you will realize something very interesting here this is nothing but the autocorrelation correct, this is basically nothing but the autocorrelation n t minus tau n t minus tau tilde, and you realize that expected value of n t, n t plus tau is equal to eta over 2 delta tau, that is it depends only on the that is what we have seen, and we have already said this many times before the autocorrelation depends only on the time difference tau on time difference tau that is the property of the wide sense stationary process depends only on the time.

So therefore, this if you look at this expected n t minus tau into n t minus tau tilde this is nothing but eta or 2 times delta difference between both that is delta of tau tilde delta of tau minus tau tilde or tau tilde minus tau both of them are one and the same, delta of tau tau minus tau tilde ok that is the interesting.

So, we have used the property of the wide sense stationarity of the noise process n to simplify this quantity expected value of n capital T minus tau into n capital T minus tau tilde that is simply nothing but by eta over eta over two times delta the time difference that is t minus tau that is t minus tau tilde minus t minus which is basically tau it tau minu tau tau minus tau tilde.

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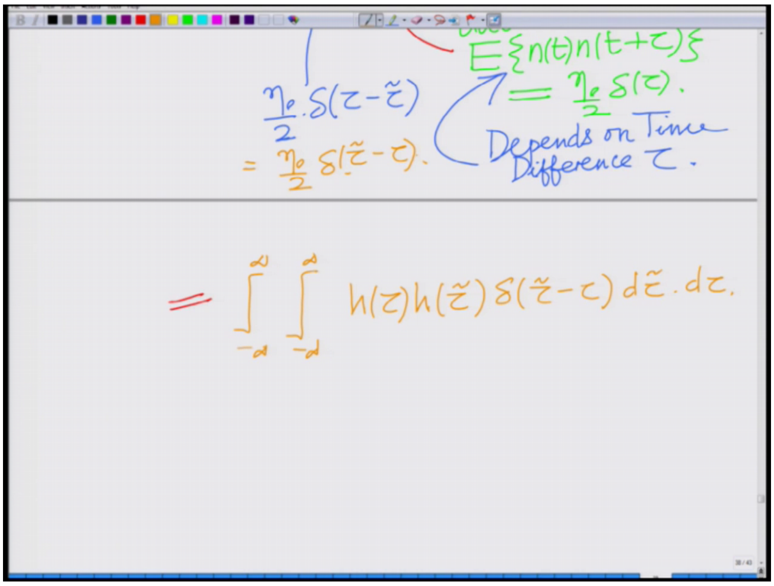

Basically, it depends only on the time difference and that gives us something very interesting now when you substitute that you will immediately see that you get something very interesting. So, what you get is well it tremendously simplifies the integrals whenever we have a delta function, you can you will realize. So, h tau h tau tilde delta well tau minus tau tilde, delta tau minus tau tilde all you can also write this as delta tau tilde minus tau because the delta function is symmetric.

So, you can also write it as delta tau tilde minus tau, d tau tilde d tau, I can also write this as delta tau tilde minus tau delta tau minus tau tilde both of them are one and the same does not matter.

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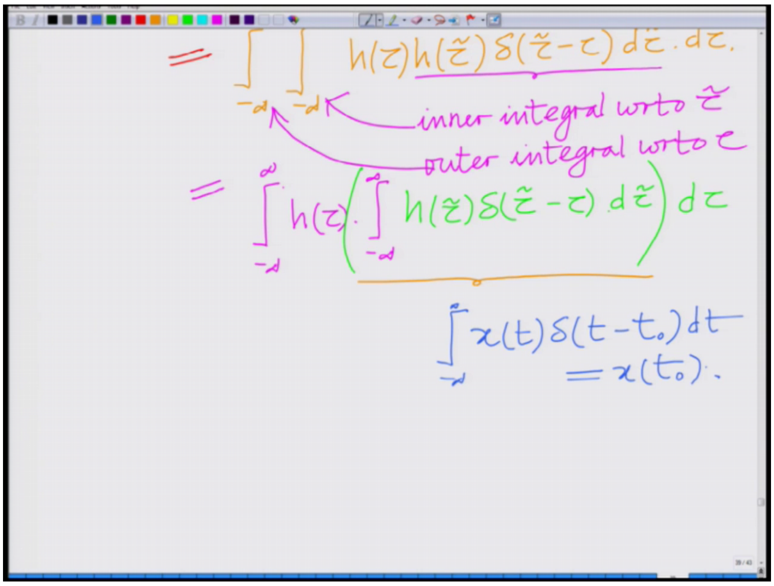

Now, if you look at this now if you look at the inner integral is with respect to tau tilde. So, let us say inner integral is with respect to tau tilde, outer integral is with respect to tau. So, I can move h tau into the outer integral minus infinity to infinity. Now inner integral will be h tau tilde delta tau tilde minus tau, d tau tilde whole thing into d tau; so two integrals separating it into inner integral outer integral.

Now if you can look at this h tau tilde delta tau tilde minus tau this is basically can be simplified by using the sifting property that is if you have a function the sifting property states that if you have a signal x of t minus t naught sorry x of t times delta t minus t naught d t that is equal to x of t naught; this is what the sixth sifting property tells me. So, this is known as the sifting property of the dirac delta function.

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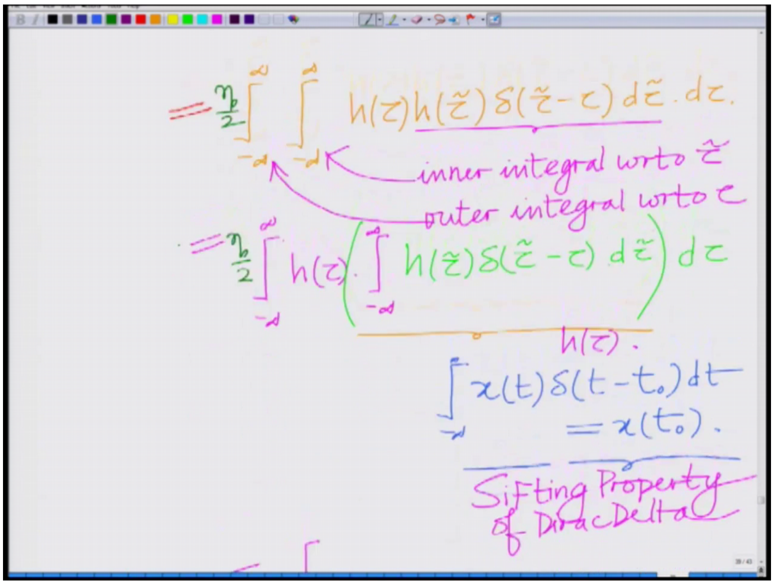

From your knowledge of the Fourier transforms Fourier transform theory you must know that this is the sifting property of the delta dirac delta. Therefore, integral minus infinity to infinity h of tau tilde delta tau tilde minus tau d tau tilde this is nothing but h of tau this is follows from the sifting property. And therefore, now if we can look at it h of tau into h of tau that is nothing but h of tau square. So, what happens is inner integral is also h of tau by virtue of the function delta of course, there is going to be an eta naught by 2 I am sorry I have missed the factor, but eta naught by 2 is a constant.

So, that will come outside of the integral no need to worry about that eta naught by 2 is a constant. So, this is eta naught by 2 minus infinity to infinity and what you have is something very interesting; h of mod h of tau square or since all the quantities are real you can also write it as h tau square, but we leave it as mod h tau square just two also account for the general scenario and now this is nothing but you can see eta naught by 2 mod h of tau square there is nothing but energy of the filter.

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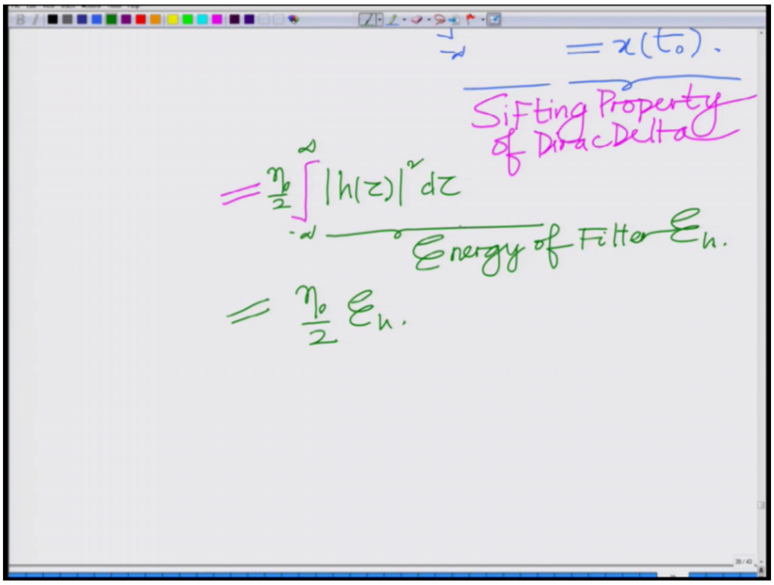

That is I can denote this by E of h right this is energy of the filter energy of the filter e of h energy of the filter corresponding that is the LTI system corresponding to the impulse response h of t. So, we have a very beautiful relation the noise power is eta naught by 2 times the energy of the filter that is eta naught by 2 times minus integral minus infinity to infinity magnitude h tau square d tau, which is also magnitude h t square d t because the integral the variable of integration does not really matter.

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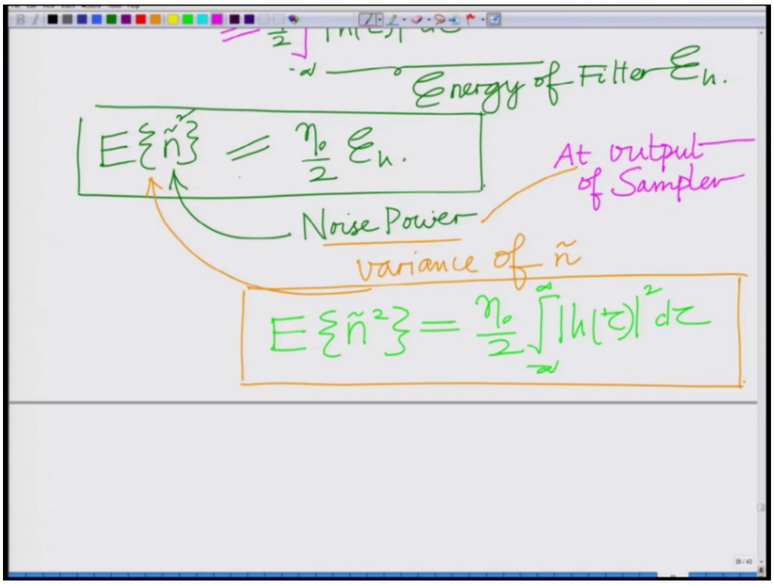

So, this is basically a very beautiful relation; what we have is you have this remember this is the quantity expected value of n tilde square. So in fact, we can say expected value n tilde is a Gaussian random variable this is the variance or the power, this is the noise power also you can say this is the variance of the since this is a zero-mean Gaussian random variable expected n tilde square itself is also the this is the variance of n tilde therefore, the noise power at the output.

We have very beautiful relation for that that is simply eta naught by 2 integral minus infinity to infinity magnitude either you can say h tau square d tau or magnitude h t square d t integral because tau or t is simply a variable of integration, it does not really matter that is simply a variable of integration that does not really matter. So, therefore, we have derived the noise power, that is noise power remember noise power at output of the sampler, that is after sampling remember this is the noise power at the output of the sample.

So, we have carried out perform this analysis and now what we have derived previously we had derived the signal power now we have derived the noise power and to derive the noise power we have invoked a very important assumption that is this analysis remember is not valid for any noise process the assumption here is that the noise n t is a zero-mean additive of course, additive has already been assumed it is a zero-mean white Gaussian noise.

And this kind of noise as I have already said is one of the most relevant and one of the most practical models for the thermal noise occurring in systems and is also a popular very popular model noise model to be used which is used to model the noise at the receiver industry in a digital communication system. So, this model is a very relevant model and this gives us the noise power at the output of the sample.

So, previously we have calculated signal power we have the noise power we can calculate the signal to noise power ratio; and from that we will find the filter h of t which maximizes the signal to noise power ratio that is known as the optimal filter optimal filter in the sense something is optimal in the sense something it is which is the most suitable something optimizes something; in the sense something that is gives you the maximum benefit or maximizes some parameter.

In this case we are looking at the optimal receive filter h t which maximizes the signal to noise power ratio, that is what we are looking here. So, we will stop here and continue with this analysis in the next module.

Thank you very much.