Principles of Communication Systems - Part II
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Lecture - 09
Probability of Error in Digital Communication, Probability Density Functions of Output

Hello. Welcome to another module in this massive open online course. So, we are looking at the receiver of a digital communication system, in today's module this module let us start looking at.

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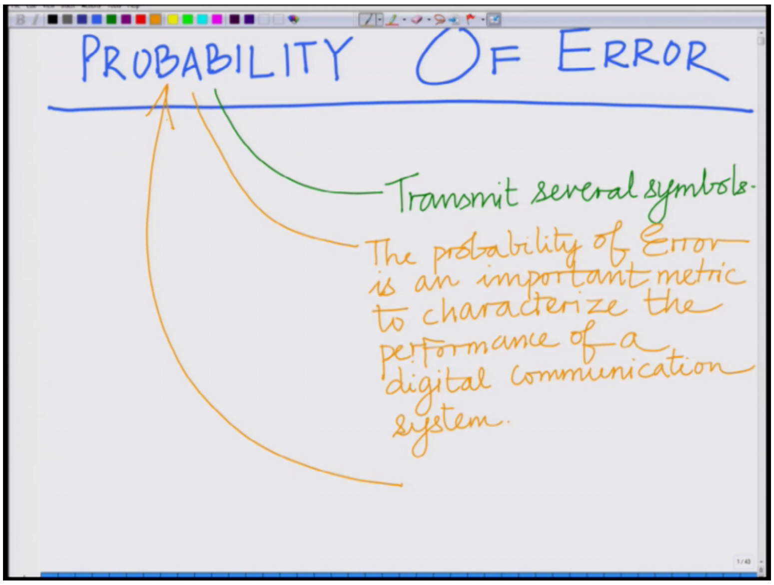

The probability of error how to characterize or how to or basically what is the probability of error corresponding to a particular scheme at the digital communication receiver. So, we would like to look at the probability of error. Now the probability of error remember in a digital communication system we are transmitting several symbols, the probability of error is the probability that a certain symbol is received in error, and this is a very important metric to characterize the performance of a digital communication system and of course, naturally the performance is better if the probability of error is lower alright.

So, we always try to achieve the lowest possible probability of error alright. So, the aim is to design a reliable communication system which means the probability of error that is we want the symbols to be received accurately at the receiver therefore, the probability of error that the probability that the same received symbol is in error must be as low as possible. So, the probabilities of error describe it we transmit. So, probability of error now the probability of error this is an important metric; is an important metric correct to characterize the performance of a digital communication system. Characterize or quantify of a digital communication system performance of a digital communication system.

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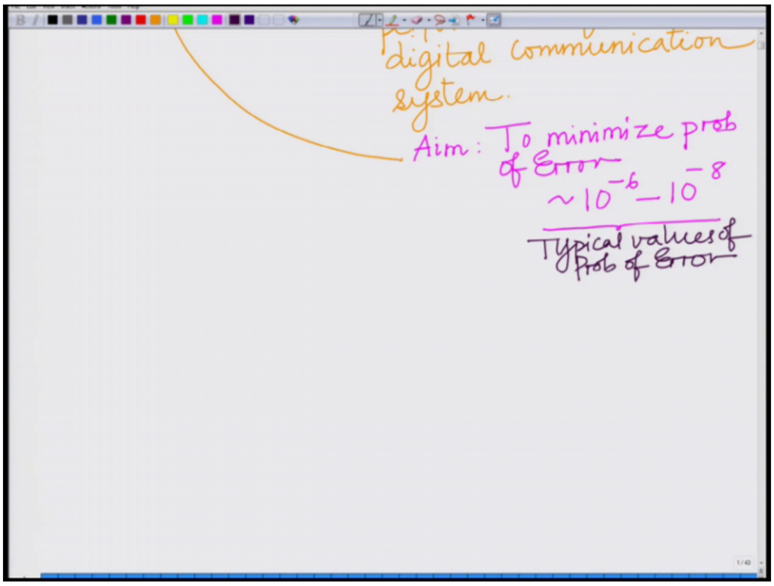

And of course, the lower the probability so we try to minimize we aim to minimize the probability, the aim is to minimize probability of error; typical values of probability of error approximately 10 to the power of minus 6 to 10 to the power of minus 8 these are the typical probability we also know that we are going to look at this is the known as the bit error rates or symbol error rates etcetera, these are the typical values of probability of error.

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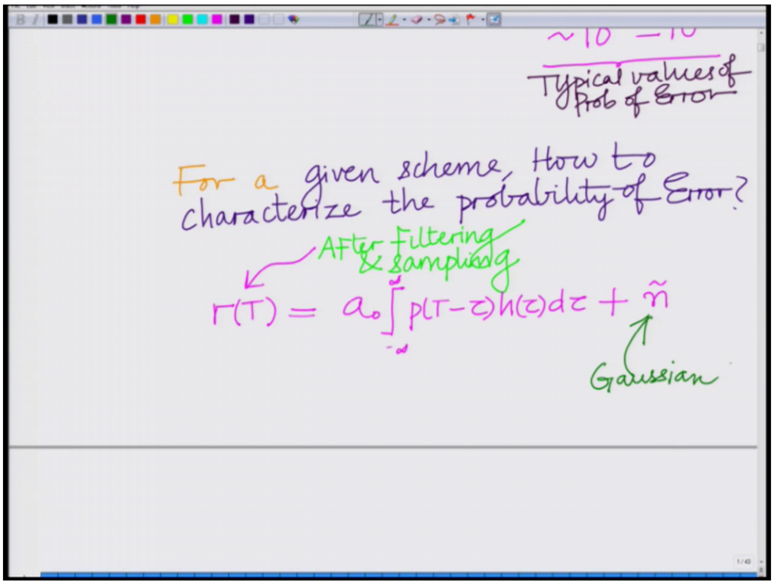

Now, how do we characterize now how to characterize the probability of error that is the next question how do we measure or how to for a given scheme that is the idea for a given scheme how to characterize the probability of error for a given scheme, how to characterize the probability of error. Now basically we have seen that our received sample r of T is given as a naught, after sampling. Remember after filtering and sampling you have seen this in the previous modules after remember after filtering with h of t and sampling at t equal to after filtering and sampling this is a naught correct integral minus infinity to infinity, P T minus tau, h tau d tau plus n tilde this is the noise remember we also verified that the noise is Gaussian.

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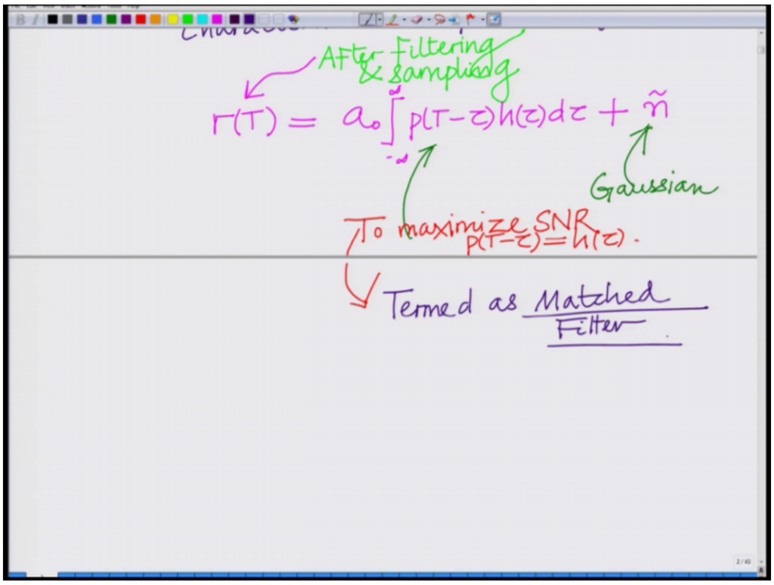

We also verified that this noise is Gaussian in nature and further we have seen that to maximize the signal to noise power ratio we have to use the optimal filter which is the match filter. To maximize SNR we have to choose h of t equals h of tau equals P T minus tau, this is termed as the matched filter and this is one of the important most important principles in digital communication.

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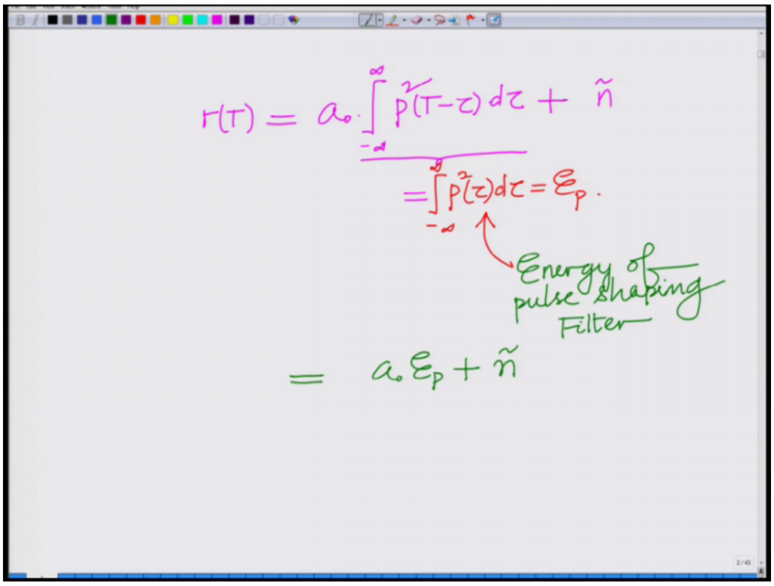

This is termed as the this is a very important principle this is termed as the matched filter and therefore, employing the matched filter we have that is h of t minus tau equals p of tau we have r of T equals minus infinity to infinity, that is substituting h of tau we get h of tau equals P of T minus tau, we get P square T minus tau d tau plus n tilde.

Now, remember we have also seen this P square d minus T minus tau d tau is nothing but this is also equal to this is nothing but this is also equal to integral minus infinity to infinity P square tau d tau which is basically like a denote it as E P the energy of the pulse. So, this quantity here integral minus infinity to infinity is nothing but the energy of the pulse shaping filter, this is the energy of the pulse shaping filter. So, I can say the output is very simply is simply a naught E P plus n tilde where a naught remember is a transmitted symbol, E P is the energy of the pulse shaping filter that is integral minus infinity to infinity, P square tau d tau plus n tilde where n tilde is a Gaussian noise.

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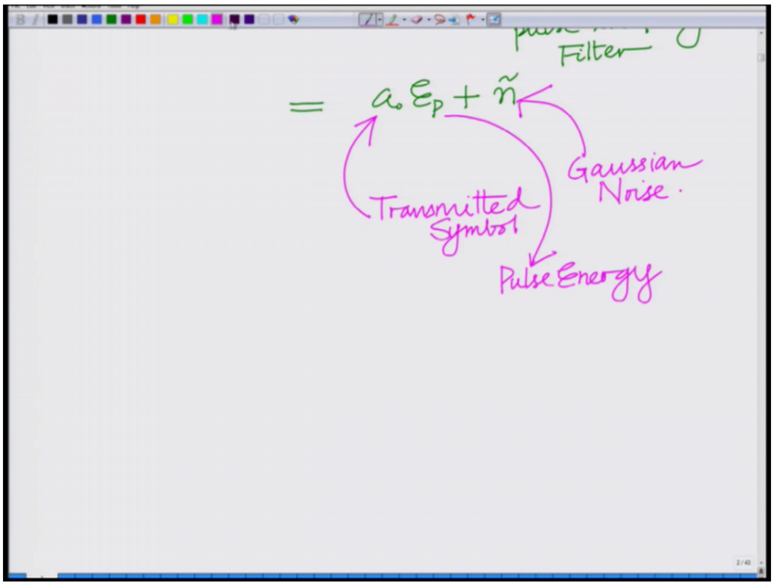

Now, let us look at these aspects further. So, first we have this is the transmitted symbol, E P is energy of the pulse shaping filter we have already seen this is the pulse let me simply write this as the pulse energy in tilde is Gaussian noise.

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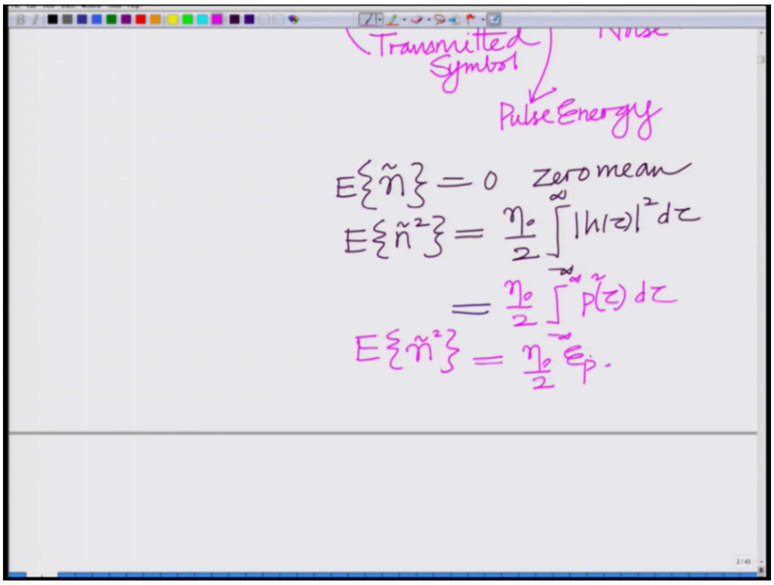

And remember we have also seen n tilde we also proved in the previous modules that the mean of n tilde equals 0, that it is 0 mean and further the variance of n tilde that is expected value of n tilde square for additive white Gaussian noise is eta naught by 2 integral minus infinity mod h tau square d tau, but we have also seen that the energy of the pulse shaping filter is nothing but the energy of the receive filter is nothing but the energy of the pulse shaping filter for a matched filter. So, this is eta by naught by 2 integral minus infinity to infinity, P square tau d tau which is eta naught by 2 times E P that is the expected value of n square.

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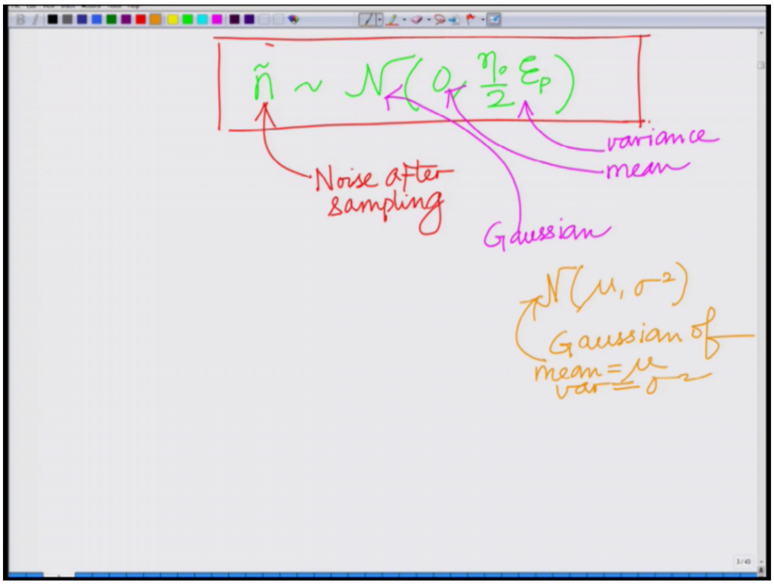

And therefore, we can write n tilde is Gaussian, basically this can be summarized as n tilde this n calligraphic n denotes normal distribution or Gaussian. So, n tilde is Gaussian with mean 0 and variance eta naught by 2 E P.

So, this is basically what we have derived for the Gaussian noise. So, n tilde this is the noise after sampling correct noise after sampling or noise at output of sampler, this denotes Gaussian this denotes that the mean equals 0, this is the mean that is the first entry and this is basically the power of the noise or the variance that is we are using the notation for a Gaussian which is standard notation, Gaussian of mean mu and variance sigma square is denoted by n mu comma sigma square, that is Gaussian of mean equal to mu variance equal to sigma square. So, here our noise n tilde remember the noise n tilde is the argument when test follows that is n t that is n t the noise additive white Gaussian noise is passed through a linear filter that is h t alright.

Hence the output noise process n tilde t is also Gaussian and when we sample it the sample of a Gaussian random process, Gaussian random noise process is a Gaussian random variable. Therefore, n tilde is a Gaussian random variable we have derived the mean of that, the mean of that is 0 the variance of that is a eta by naught by 2 integral minus infinity to infinity, h square tau d tau that is the energy of the filter, but when you use a matched filter the matched filter is same as the pulse therefore, the energy of the matched filter is same with the energy of the pulse therefore, the variance of the Gaussian noise at the output of the sampler is eta naught by 2 times E P where E P is the pulse.

So, that is the complete argument and I think it is important that you understand that. So, please go through the entire if it is not clear I urge you to go through the previous modules where we calculated the nature of the gauss noise at the output of the sampler, and also the mean and the variance once again.

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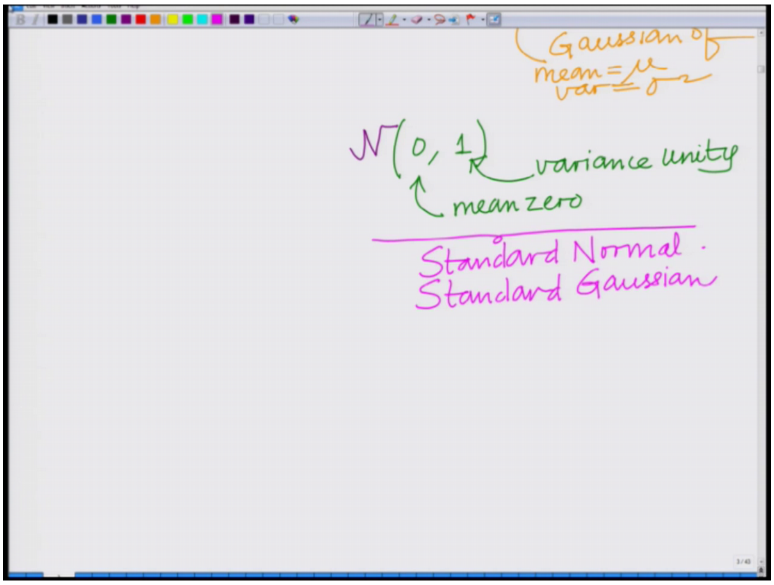

Now more importantly the other thing that one can also notice note here that if I have a Gaussian random variable that is as an aside, you can note that if I have a Gaussian random variable with mean 0 and variance one that is Gaussian random variable mean 0 variance mean equal to 0 variance unity. Such a Gaussian random variable this is known as a standard Gaussian or standard normal variable standard normal random variable, this is known as the standard normal or the standard Gaussian; this is known as the standard normal or standard Gaussian random variable or standard Gaussian random variable.

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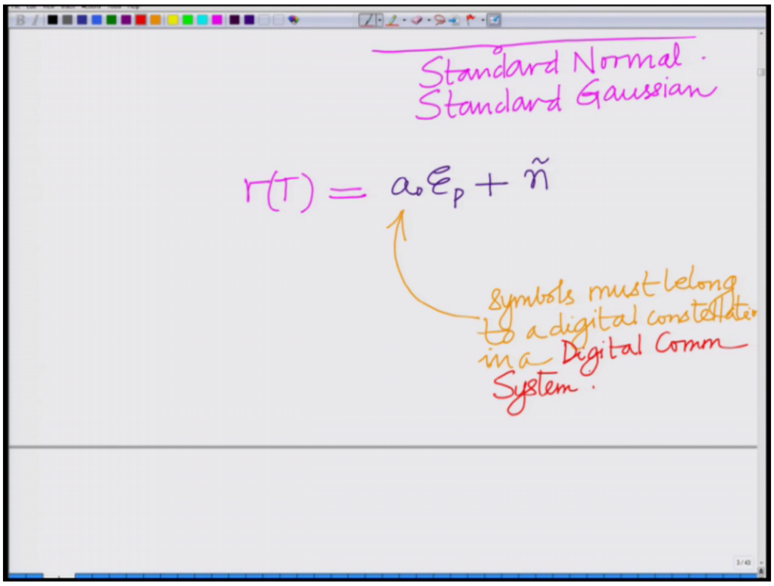

Now therefore, basically we have a very elegant model if you look at this we have the received symbol r T that is not the received symbol, but the sample r T which is equal to a naught E P plus n tilde, let us now of course, we said that the symbols must belong to digital constellation. Now the very first modules we have realized that this symbols in a digital communication system symbols must belong to a digital constellation in a digital communication.

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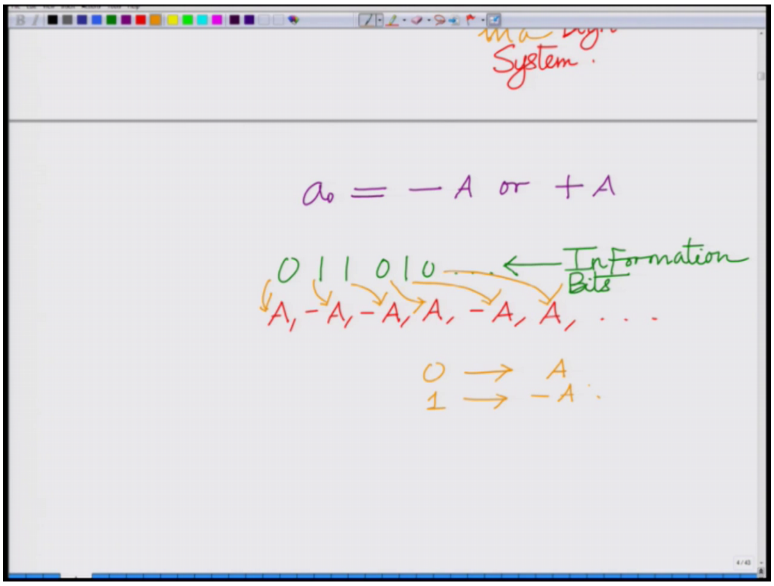

So, let us choose a very simple system in which basically we have very simple constellation in which a naught equals either minus A or plus A. Remember we can have remember how this arises we have the information symbols if you go back, we have these are your information stream is the bits basically the raw bits information bits. Now I can map 0 to the voltage level A and 1 to the voltage level minus A for instance or so I will get a minus A, A minus A 0 is mapped to a one is mapped to minus A, A minus A 0

So, these are basically the symbols. So, I will just let me write it clearly. So, A minus A minus A, A minus A, A and so on here we have 0 mapped to a, one mapped to minus A, one mapped minus A, 0 mapped to A, one mapped to minus A. So, we are using the mapping 0 is mapping to A and information bit 1 is mapping to minus A and now, therefore, if you observe this at the output you will have r naught r T equals plus or minus A times E P plus n tilde.

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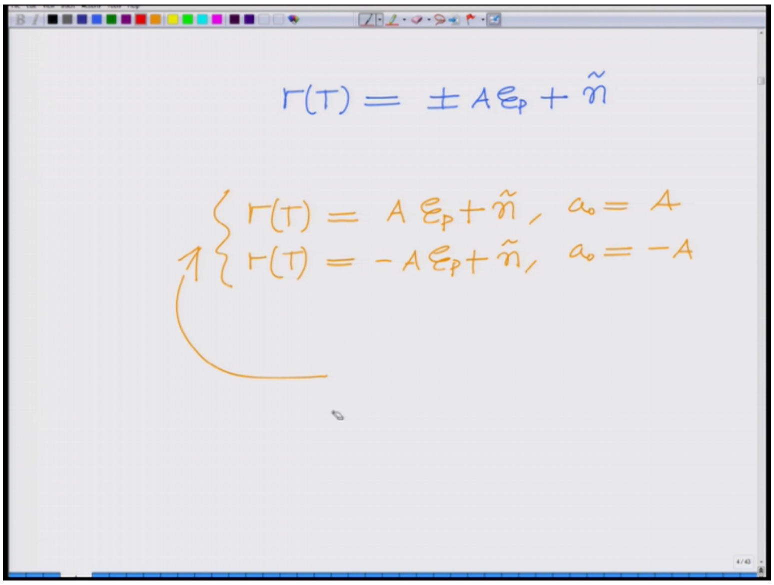

So, what you will receive is the signal level plus A. So, since you are transmitting either plus A or minus A. So, what you will receive after matched filtering and sampling these are the plus A times E P, where E P is energy of the pulse or minus A times E P in the presence of this noise n tilde.

So, let us consider these 2 scenarios let us write them clearly. So, we have r T equals A E P plus n tilde, if a naught is equal to A that is corresponding to the information bit 0 or r T equals minus A E P plus n tilde if a naught equals minus A. So, this is basically a complete characterization of the output.

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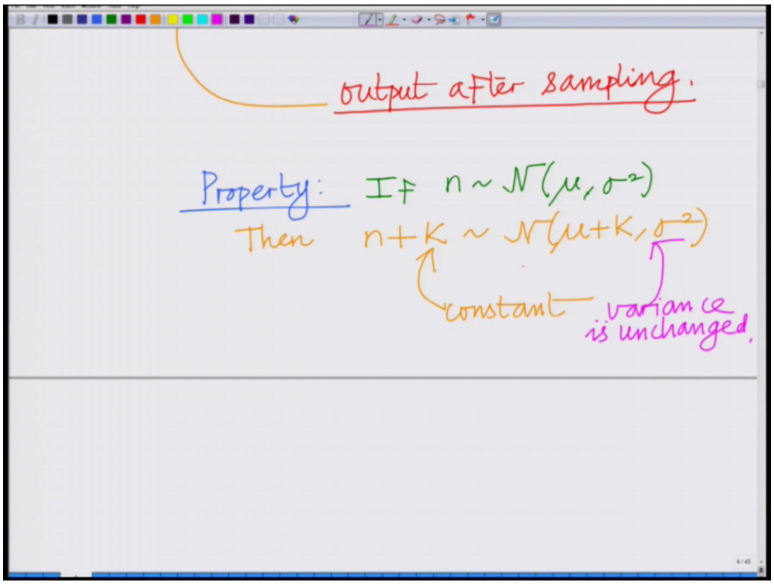

So, this is output after sampling and now you will realize here something interesting that is if you transmit a naught equal to A what you get is A times E P plus n tilde, which means n tilde. Now look at this A times E P is a constant you are adding it to the Gaussian noise n tilde which are 0 mean. So, the mean of this Gaussian noise n tilde is going to shift to A times E P.

Similarly, when you transmit a naught equal to minus A output is minus A times E P plus n tilde, which means the in this case the mean of the noise n tilde will shift to minus A times E p. So, if you look at the output right r of t which is either A times E P plus A n tilde or minus A times E P plus n tilde corresponding to a or minus A the mean will be shifted to basically a times E P or minus A times E P because n tilde itself is 0 mean Gaussian noise, and the variance will remain unchanged because when you simply add a constant quantity to the Gaussian noise to the Gaussian random variable, the mean shifts the variance which is the spread of the Gaussian random variable remains unchanged. So, the variance of the Gaussian random variable will remain fixed that is your eta naught by 2 times E P and that is an important point.

So, let us note that. So, if we take a Gaussian random will. So, let us look at this property I think because it is a fairly important property which we might invoke later also, if you have n which is distributed as a Gaussian random variable according to mean, mu, variance sigma square, then n plus K where K is a constant is distributed according to the

Gaussian random variable the mean is shifted by K variance is unchanged to variances is a this is an important property, variance is unchanged because variance is remember it is related to the spread of the Gaussian random variable variance is a.

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So, basically what we are saying is if you have an initial Gaussian random variable which has this probability density function, which is centered at the mean which has mean mu if you add. So, this is let us say your n probability density. In fact, this is not n, but this is the probability density function of n.

Now, the probability density function of n plus k will be something that is shifted. So, this is basically your mu plus K, but when we say the variance the variance is related to the spread its pop it is proportional to the spread. So, this spread remains unchanged or basically the shape of the Gaussian remains unchanged, it is basically simply translated. So, the spread here is sigma square, spread here will also be sigma square this is unchanged.

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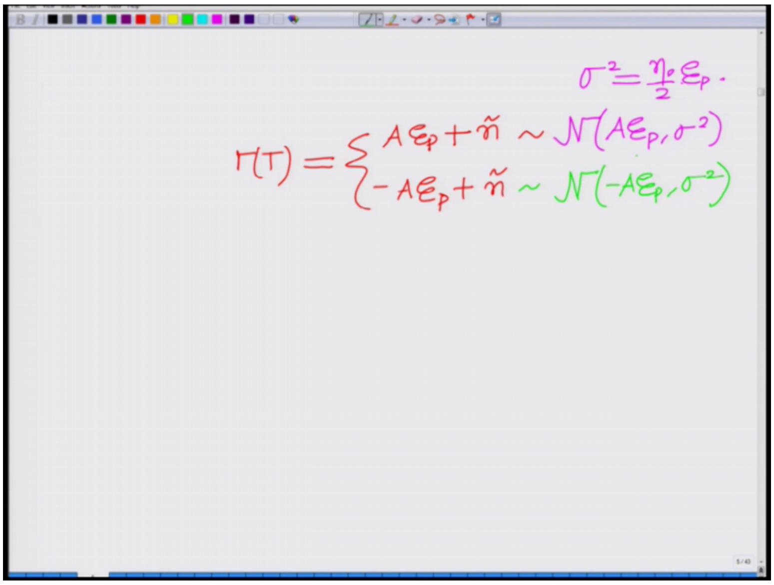

Now, what is the relevance of this 2 a digital communication system problem now what we are seeing is something very interesting go back and look at r T equals either plus A E P plus n tilde or minus A E P plus n tilde, which means this is n tilde shifted by E p. So, this will be simply Gaussian with mean shifted because n tilde is 0 mean. So, it is it is mean will be 0 plus A t e p which is A E P sigma square of course, sigma square we also know what is sigma square, sigma square is eta naught by 2 times E P the variance is unchanged this is also Gaussian with A E P minus A E P; minus A E P plus n tilde is Gaussian with mean shifted to minus A E P and variance sigma square.

So, basically corresponding to both these, so the way to distinguish the outputs is basically corresponding to the transmission of plus A, the output is a Gaussian random variable. Remember because you are always observing the output in the presence of noise n tilde which is random.

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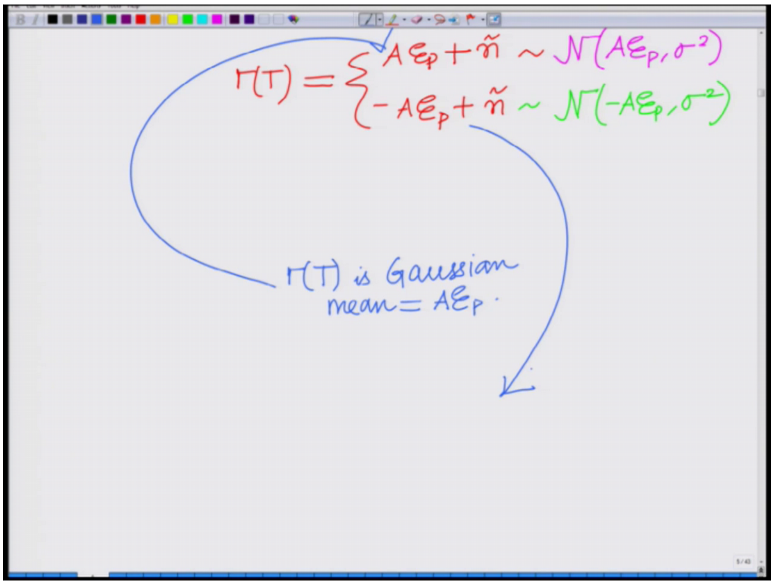

So, naturally the output is also a random variable correct; however, when the symbol plus A is transmitted the output is a Gaussian random variable with mean A times E P when the symbol minus A is transmitted the output is a Gaussian random variable with mean, minus A times E P and in both cases the variance is the same variance is identical which is sigma square equal to eta naught by 2 times E p. So, let us note that.

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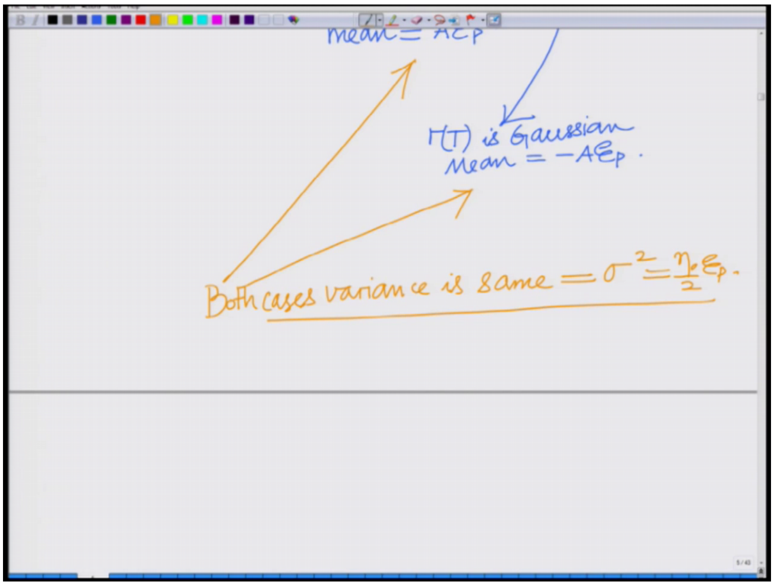

So, in this case r T is Gaussian mean equals minus A times E P and in both cases the variance is the same, we will call it sigma square as a eta naught by 2 for the matched filter this is a eta naught by 2 times E P as we have shown above alright. So, that is what we are going to. So, what we are going to do.

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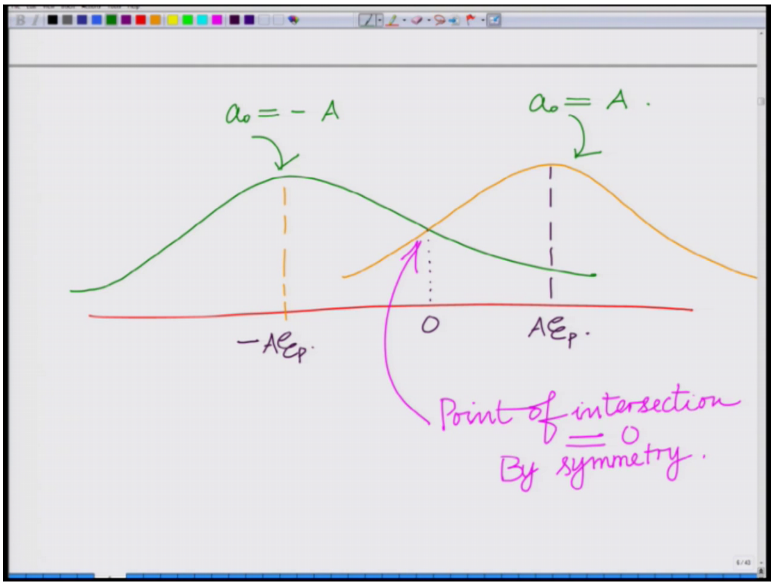

So, let me just draw it I think it makes sense to draw it. So, we have 2 Gaussians and the other Gaussian looks something like this, both of them have the same shape. So, this is mean equal to. So, this is basically at minus A or let me just this is minus A times E P and this is A times E P, variance is the same and by symmetric this for symmetry this point of intersection is 0 this is purely you can seen by symmetry.

So, the point of intersection is 0 by symmetry and of course, now you can see this corresponds to this Gaussian distribution corresponds to a naught equals minus A, this Gaussian density corresponds to a naught equals plus A. So, naturally you can see that when you are transmitting a naught equal to A the Gaussian is shifted towards the right that is shifted towards a times E P, when you are transmitting a naught equals minus A the Gaussian is shifted towards the left that is the mean A times or minus A times E P ok.

So, we have 2 Gaussians with the same variance that is eta naught over 2 times E p, but one is shifted to minus A times E P, the other is shifted to A times E p. So, this is basically the 2 the output the probability density function of the output basically comprises of 2 shifted Gaussians.

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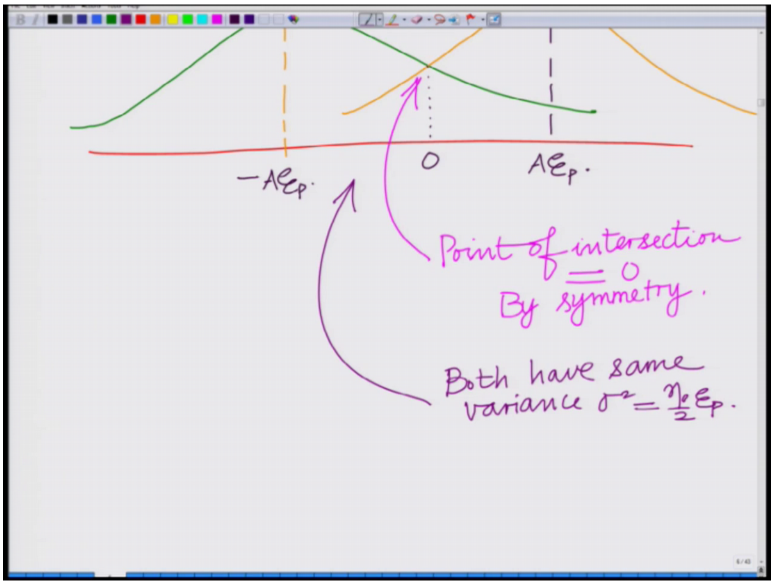

That is something which is very interesting and both have the same variance and again I am making the same point I am repeating this point both have the same variance, sigma square equals eta naught by 2 times E P and this is an important property. So, please go over this to understand this better and we will complete the derivation of this from this probability density function the different probability density functions of r T corresponding to plus A and minus A, we are going to compute the decision rule at the output at the receiver and also what is the corresponding probability of error. So, let us stop here.

Thank you.