**Indian Institute of Technology Kanpur**

**National Programme on Technology Enhanced Learning (NPTEL)**

**Course Title**
**Applied Electromagnetics for Engineers**

**Module – 31**
**Completing Maxwell's equations and Boundary conditions**

**by**
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Hello and welcome to the NPTEL mook on applied electromagnetics for engineers. In the previous module we looked at Maxwell's equations where we looked at all other equations except for Maxwell's contribution mainly the displacement current. In this module we will talk first briefly about the displacement current and then complete the Maxwell's equations, and then talk about the boundary conditions in this as well as in the next module.

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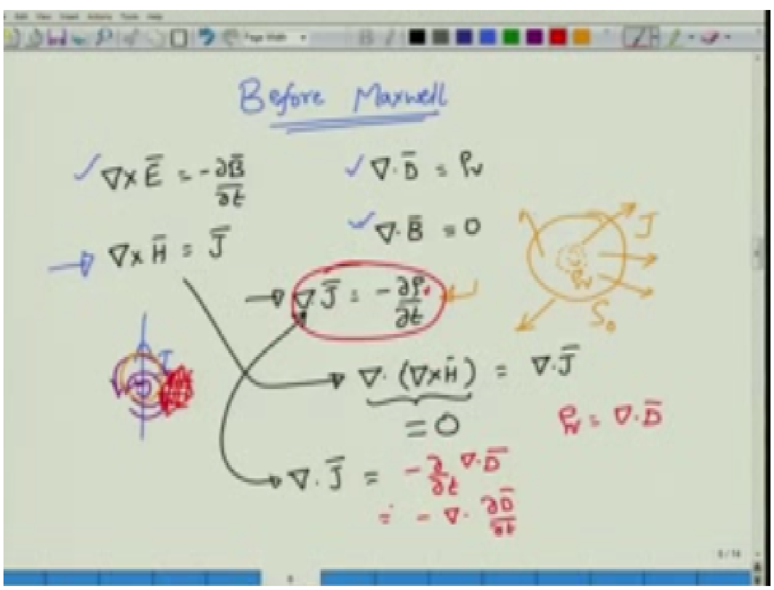


Now here is a set of equations which you know whose history we have studied in the previous module, we recognize that the first equation is Faraday's law, because it relates the flux linkage to a particular circuit to the emf that is generated in that particular conductive circuit. These two equations are Gausses law, one is our electric field which says that the source or the electric flux density vector D is the charge density in that particular region okay.

And it also shows Gausses law for magnetic case shows that there is no source for the D field, in other words there are no isolated positive and negative magnetic force and therefore ΔB is always equal to 0. This law which have saved for the final one to discuss is the Amperes law. We have already looked at Amperes law, you know in the context that if you have a current carrying wire, then there will be a magnetic field around it.

And if you then take any closed curve okay, and then integrate the magnetic field around that particular curve, then you will obtain quantity called as magnetomotive force and the source for this magnetomotive force will be the current that is propagating okay. So suppose this is my wire which is carrying a certain current I, and then the magnetic fields will be surrounding this particular current, you know they will be in the form of circles with decreasing amplitude as you go away from the current carrying wire.

And if you take any path okay, so for example if I take this particular path and then integrate the magnetic field, then I obtain what is called as a magnetomotive force and this magnetomotive force the source of this magnetomotive force must be the current that is coming out of this patch of area. So there is a patch of area here okay, the total surface, so whatever the current that you preserve through this one will constitute the source for the magnetomotive force.

In this particular problem this path which I have taken shown in the red curve does not enclose any current, because the current is directed along this blue line, and therefore the magnetomotive force for this case will be equal to 0. On the other hand if I take this particular curve, you know maybe a circle around this one, I know that if I integrate the magnetic field around this constant radius circle which is shown in this orange color, then it will enclose the current, because the current is coming right perpendicular to that, through that particular circle.

And therefore, the source of this magnetomotive force will be equal to the current that is being carried by the filament. So first so good, so this is what Ampere had in mind. But then there is a inconsistency between the equations that we have written here. So before Maxwell, you know when we put together all the equations along with that there is a current continuity or the charge conservation equation.

The charge conservation equation that we have discussed, in that any closed surface if you take, any closed surface S0 if you take, and then you find out what is the source for this J, that is if you find that there are this current density vectors coming out of this closed surface. Then the only way that can happen is when the charge density here inside is decreasing with respect to time okay.

So assuming that these charges are all positive charges, if the amount of positive charge decreases then it constitutes a conduction current change. So this is what people kind of understood before Maxwell but unfortunately if you look at this equations there is some inconsistency in the equation what is inconsistency let us take this amperes okay and the dot of both sides with ∇ okay operator so we actually calculate ∇. ∇ x H so you have to dot on both sides and clearly the left hand quantity you can show by actually writing out in various coordinates systems.

That this left hand side quantity will always be equal to 0 even when H field is dependent on time this quantity will always be equal to 0 this seem to indicate that if I take the divergence of J that should be = 0, but unfortunately this is not consistent with the charge conservation either we let loss of the charge conservation that is either we let go of the charge conservation and then I aspect the factor ∇. J = 0 but if we do that then you know you can create charges whenever you want and you can destroy charge when you want you kind of remove this conservation principal at all.

You can continue to create charges and you can create an infinite amount of charge clearly such a thing does not happen so conservation principle is more important because it has been observed that no matter you try we cannot create charges okay and have no current flowing it, so the charge conservation is a deeper principle of physics experimentally and therefore that cannot be violated it only means that the conclusion that we obtain form amperes law must be wrong how do we fix this well in order to fix this we simply use the equation that we have with the charge conservation.

What we do here is that we replace ρv/ ∇. D as in form the gausses law and the rewrite this equation, so what we obtain you know form this equation is that if I take ∇ . J the ∇. J must be equal to - ∂ / ∂ t ∇. D and interchanging the operations of time and space variables so you get - ∇

. ∂D/ δt now I can bring this quantity on the right hand side this divergence of the ∂ D / ∂ t on to the left hand side.

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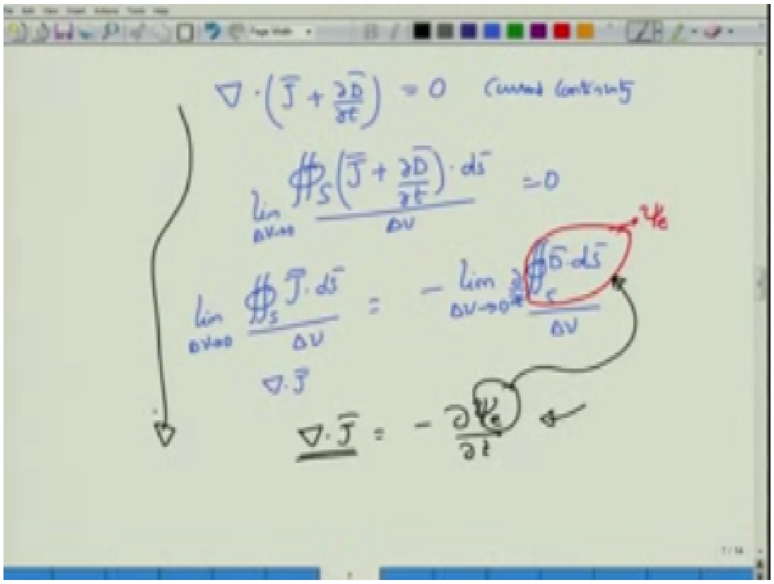

And the essentially obtained as simplified equation it tells you that ∇. J + ∂D/ ∂t is equal to 0 okay now this equation kind of you know make sense or this is called as the current continuity equation okay y it is current continuity equation because if you go back to the original expression for what is that divergence of a give vector field you would see that it has to do with the closed surface over which your evaluating that particular vector quantity in this case it would be J + ∂D ds and divide by a unit volume in the limit of volume going towards to 0.

So you can split this again into 2 integral so you will have J. ds/ ∇v in the limit of the volume ∇ v → 0 this of course is the restatement or rewriting this ∇. J itself right and this would be equal to – again in the limit of the volume going to 0 you have ∂D/ ∂t , so if I take if I interchange the operations of ∂/ ∂ t and the space operations so I first evaluate this D. ds over the closed surface that I am evaluating and then take the time derivative of this okay divided by ∇v now what is the on the closed surface if you evaluate the electric flux density what quantity do you obtain this quantity that you evaluate is simply the electric flux density.

So what you have is that ∇.j is actually = - ∂electric flux density / ∂t so this electric flux density change or the change of electric flux not the flux density electric flux itself give you the displacement current because electric flux is related to d it was called earlier as a displacement vector we now called it as electric flux density vector but earlier it was called as a displacement vector.

And therefore that time rate of change of the displacement is equal to the displacement current and this equation is called as the current continuity equation okay, so this is a simpler thing that you need to understand out here exactly not $\partial \psi e$ but $\partial$ I mean $\psi e$ is actually this quantity integral of d.ds but you get the idea right so here you have completely a conductive current or the current because of the motion of the charges in wires or in other conductive planes for example.

Whereas on the right hand side you have a quantity that is not really depended on the conductivity right, so this quantity can actually exist even in free space and insert such a current would actually exist between the two plates of a capacitor, as we will discuss very shortly from now there is only one small point that we need to note at this stage of our development of displacement current density is that whatever we did from the previous calculations showing that $\partial.\partial xH = \partial.j$ and then subsequently defining the displacement.

Current as $\partial d / \partial t$ d is not really happen this way because Maxwell did not know what vector analysis meant vector analysis first invented after Maxwell died, and therefore he did not followed this path at all rather than tracing what Maxwell did or what path he did because that is too complicated for this course we will assume that this law which does not contract with any other previous laws and his consistent with conversation charge conversation principle is that we take this displacement current as another.

Type of a current okay and we simply try to show where this displacement current might originate okay in order do that one.

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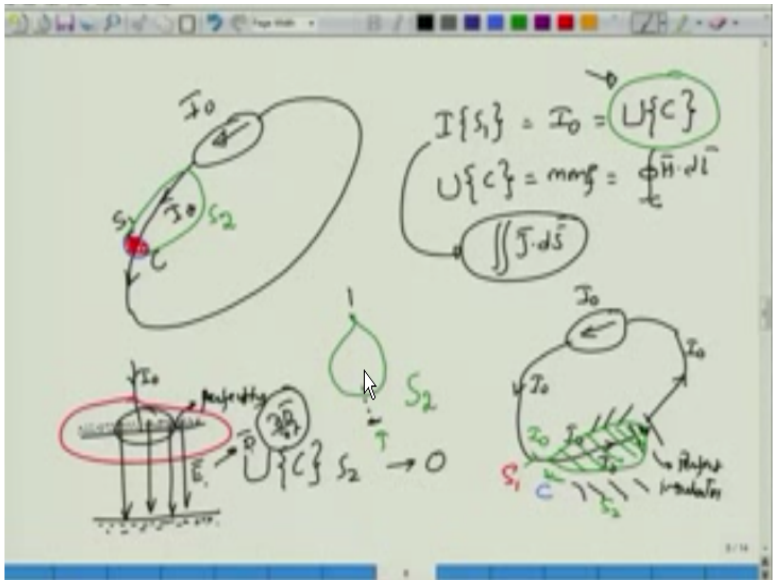


Let us go back to this simple experiment that you can think of let say we have a certain current source okay what is the objective of a current source there to supply a certain current let us call this current as some i0 and the current let us say is flowing through this thin filament of wire and then it comes back to the current source itself, so there is actually a continuous current flow which as it should be because the current cannot be in this way so current has to flow in a complete circle so this is the current that is flow it.

Now you imagine that I actually take a curve okay does not matter which where I take occur so since the current is directed along this one so I will take the curve in this particular way okay just to be consistent with what is called as a right hand rule, so I take this particular current and then now this is the contour which I will take okay. Now you know that if I take this .contour and evaluate the magnetic field around this contour, I will be generating the magneto motive force and that will be equal to whatever the current that would be contained by this patch of area. So by this patch of area because there is the current coming out right.

So there is the current coming out and since you are considering only the open surface, here let us call this as s1 because we are going to soon write a another surface here so the current through the surface s1 will be = i0 okay because that is the current that is being pass through that and this will act as a source of the magneto more is force over the contours see okay.

So just to refresh your memory the magneto mote is force MMF over this contour is simply integral of H over that particular closed curve that we have considered, okay. So far no problem and how did we calculate the current through the surface S1 which is the current density through DS and if you just you know kind of not worried too much about the problem that the current density kind of seems to be you know infinite.

Because the current is filamentary but just remember that the total current out of any surface that you are considering the open surface that is important the open surface whose contour is the same as C that current is exactly equal to Io the source of the MMF will be equal to the current as it should be, so far no problem right. Now there is also no problem if you where to consider a different kind of a surface.

So let us imagine that I actually have some sort of a balloon out there okay and this balloon is the surface that I am considering okay and I so now what is the surface I am considering so this was my contour okay so some direction C and the balloon that I consider will have the same contour okay and it will go through like this you can think of this as a sac or a bag or a sac this is the surface F2 let us call this, okay.

And this portion that I have shown here you know where my pen is going around this top portion is actually left empty, so you might imagine that this is a surface that I am considering so you can imagine that this is the contours C that we have considered okay if I now take a paper and then just you know take the paper on top of it, okay and then just look at what the marks that the paper does and imagine that this part is not actually existing, okay.

If I remove this path is not existing then you actually have a open surface whose contour is this one and if I now pierce that paper through this pen then there will be a current going into it, right? And because this current continues and comes out of this open surface you know you have to imagine a paper being wrapped around this and this current going in and then coming right out of the paper okay.

That would be the current that you have instead of doing that you can consider this as your open surface you can observe here that the contour is now left open and whatever this surface that I have draw so this edge combined with this closed path over here so this surface is the one that is now so this entire thing is now the surface okay. The paper has been removed and you just have an empty kind of a, you know emptiness inside here.

So there is a small whole inside and you can have a current going through and because there is a current coming out this current would come out okay again this current is a filament and it does not matter whether I consider this type of a curve or put a paper and then consider just the current going through that paper both curves essentially give me the same amount of current which will be equal to the MMF of that particular contour.

So this is the MMF of the contour, okay. Now so far that is alright so if I use surface S1 or I use surface S2 I essentially obtain the same current and same current means that the value of the MMF is un-ambiguous there is no doubt about what MMF I obtain that MMF will be the same it MMF will be in fact equal to I0 okay whether I evaluate it over surface S1 or evaluate over the open surface S2.

So far so good, okay. Now let us see what happens. Now what is do is I take the same scenario okay I have a current source and then I a pass through this and somehow ground here okay so somehow I ground here there is nothing coming out of this, okay. So there is clearly a discontinuity or if you are too happy with this one let us keep the current flowing completely but then output a capacitor okay.

So we assume for now we will talk more about capacitance in the coming modules that this capacitor is filled with a material which is perfect insulator okay that means there are no free currents or free charges anywhere to be found in this particular region. Now the current $I_0$ is coming through this current and obviously because the total current in the loop is $I_0$ from your normal circuit theory you simply expect that this current has to go you know and out of this plate and then essentially has to be pass through the current source and then come back.

Only catch is that now the MMF seen to become you know ambiguous watch what happens, let us initially consider the same you know loop okay, so since the current is coming out in this particular way I will orient my contour along this way so this is the contour so there is no problem with the MMF, the MMF is equal to $mI_0$ because $I_0$ is entering and $I_0$ is leaving so and if you integrate the magnetic around this blue curve we essentially obtain the magnetic of motive force.

Now as before we consider first one surface $S_1$ okay, which is given by this hatched area so going back to our example so we will come back that one the visual way of thinking of this one so over this open surface $S_1$ the current no passing through this open surface is equal to $I_0$ and therefore the magneto motive force you at you know U because of this contour C and the open surface $S_1$ also seems to be equal to $I_0$ so far there is no problem.

Now what we do is, we again consider this balloon kind of a surface and we consider the surface to be something like this okay, so this is my balloon surface that I have considered so please again understand that this hatched area has been removed so let me try and remove the hatched area for you okay, in the process I have actually also removed something else okay, let me retrace this, so I have a curve C here okay, and then I have this balloon which is passing through the plates in between okay, passing through the plates in between and this is oriented in this particular way.

So let us call this as $S_2$, so you can imagine that this sake which we have drawn here has been now put over this wire, please note that the line that I have shown here the black line is actually going through this it is not actually or it is not, there is no area to touch for this okay, the current would simply come here in between go through and it will only touch the bottom plate over here so at this point the current should touch. But unfortunately, we have just seen that the current there is no current coming out of this point right, so it is like this imagine this okay, so you have this current contour okay, so this is the sake that I am considering.

So now imagine that current is going in okay, so you can imagine that the current is going in but it kind of stops in between because I have put up a plate here okay, so I have to imagine that I have taken this contour and then I stretch this open surface into that and I am my current is going here and this getting stop because it encounters a nice perfectly conducting plate okay, so there is a space between what this plate encounters or rather the plate and the hole inside here.

So there is no current so this tip is not touching the bottom part or essentially not touching this and then coming out, so in that case what will be the current enclosed clearly the current that must be into that there is not current enclosed here, no current, no current, no current, no current, no current, no current the only current that should have enclosed is by this way but there is no current lining out there, there is no wire out there, and therefore there is no current coming out of this surface.

So it indicates is that the magneto motive force associated with this contour when you consider the surface $S_2$ must be equal to 0. Now clearly something is wrong because just by changing the surfaces whether it is a red surface which is the open surface or another surface $s_2$ which is again open but it gives you different answer you might seem to obtain two different answers so the k is when current seems to be discontinues is that the corresponding magneto motive force seems to be dependent on what type of a surface you consider in order to evaluate the magneto motive force.

Clearly something that is not to be having right, so you cannot have your answers dependent on you know whether on Monday you woke up and decided to take s1 and on Wednesday you woke up and decided to take $s_2$, so you cannot have such ambiguous values for the magneto motive force and therefore this is kind of indicating the failure of Ampere's law. To bring about more clearly what is happening we now let us close this surface so we actually instead of leaving it open, we close it we pinch the surface essentially obtaining something like this.

So we pinch and then the surface $s_2$ becomes closed okay, so here we go back and then make that one. So the surface $s_2$ is essentially closing so let me try and see if I can remove this one so let me that the current go back here and now I draw the closed surface right so this is the closed surface that I am considering and what is happening? To this closed surface $s_2$ there is some current i0 entering but there is nothing coming out of it or is there something coming out of it? Yes there is something coming out of it because this element lands on this plate this plate is a perfectly conducting plate okay what it means I that?

If current $i_0$ is coming through here and land on this perfectly conducting plate there will be charges induced on this plate and the amount of charge will keep on increasing, so the charge is nowhere to go but kind of pileup at the surface okay. The rate at which at the current comes in will the rate at which the charge is getting piled up because you can write down the charge conservation equation for a close surface that would look like this or the close surface sq2 and then you would see that 3this current i0 entering okay will give us to the piling up of the charges that the density of the charge increased because of the current coming in.

But these charges which have been you know piling up on this perfectly conducting plate has nowhere we go because, the only the conducting surface happens to be here so this is the second plate of the capacitor and here there is nothing out there. So there are no or nothing is happening but if you remember from your electro statics no all though the case here is not strictly electro statics if you remember the idea from electro statics is that if there are charges here there will be equal charge piled up on the other side of the capacity plate okay.

And between these two charges there will be e-field and e- field will lead to d- field and rate of change of d- field because the rate at which charges are getting piled is changing therefore e is changing with time d is changing with time and this ∂D/ ∂t will be changing with time.

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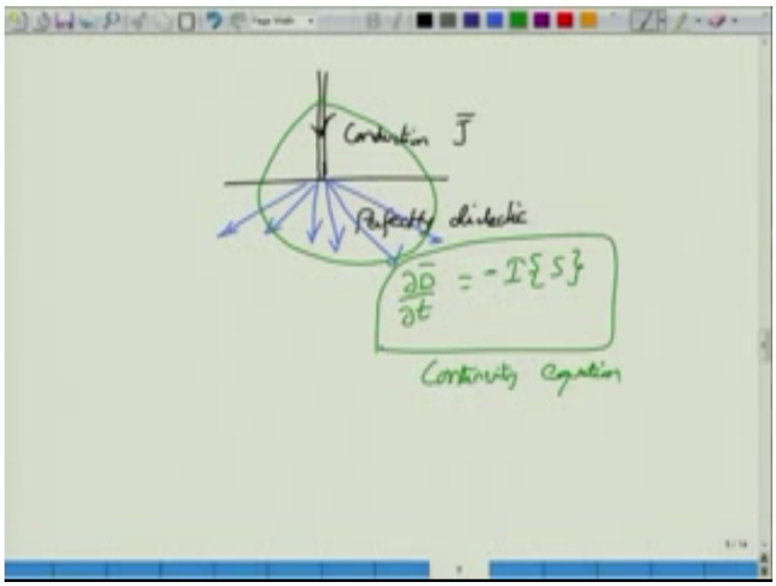


And it will take over when the conduction current hence so on this conducting plate until this filamentary current this is all conduction current okay which you can describe by the conduction current density J if you are not happy with that filament of the current you can imagine that there is achieve a small tube of the current okay and then this tube will have a certain conduction current density done.

Once the medium is perfectly insulating or perfectly dialectic medium dialectic is an another name for an insulator then there is a no hope of finding a J vector here we cannot find J vector simply because this medium is not a dialectic medium so that conduction current has to vanished what instead happens is that this region will generate the D field okay.

And D field would look like this okay because the charges are all coming but I am not assuming that there is another plate here okay I moved that plate very far away out to some other region so we have this D fields and then you along take a closed surface in this manner entering to the top of this surface will be the conduction current but leaving this surface will be the displacement current okay.

And this displacement current is given by $\partial D/\partial t$ and this must be equal to whatever the current that you are going to see over the particular closed surface okay there is a minus sign of there because one is entering the other one is leaving so in take leaving as a positive and entering there is a negative and what you obtain is the current continuity equation okay.

So this completes our introduction to Maxwell's equation so we can go back now with the fact that we have discovered or we have find out come up with the consistent way of writing down all the equations that agrees with all the experiments of course it is not that we came out of all this it was Maxwell's to introduce this displacement current and the importance of this displacement current is this in vacuum right.

So there is no way of having free charger for free currents so there is no conduction current there are no wires in the vacuum yet electromagnetic waves travel this electromagnetic waves carry a energy and the for the waves to actually happen you need a phenomenon in which electric field is changing magnetic field and magnetic field is changing in term the electric field and that link between the two is the displacement current.

Any time you have perfectly insulating medium you will have only displacement current any time you have a perfectly conducting medium you have only conduction current and if the medium happens to have both conduction and you know kind of dialectic that is it is not a perfectly conducting material nor it is perfectly insulating material.

Then you will always find a combination of conduction as well as the displacement current in such a way that the conduction current density entering through a close surface will be equal to the conduction current density entering through a close surface will be equal to the displacement current leaving that particular closed surface okay so we now gather all Maxwell equation in one place.

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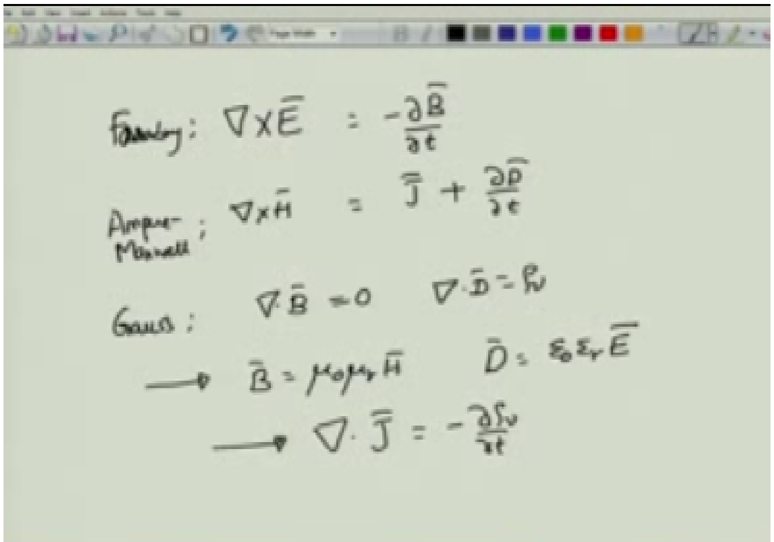


Write down the corresponding consecutive relationships and from then we will look at what you know how these fields interact when the medium property change okay. So we begin with faradays law, we have already seen that one so faradays tell us that curl of electric field is given by the magnetic field change with respect to time. Similarly curl of magnetic field H right, this is given by, this is what we called as ampere Maxwell law because Maxwell contributed the displacement current.

It could be given by the conduction current density change + the displacement current density ∂d/∂t so that when you take ∂. ∂H there is no problem in this. Then there are two Gauss laws, one for electric field and one for magnetic field, so what are those laws? Gauss laws are ΔB = ΔD= ρ and this B filed is related to H field by this quantity μ0 is the permeability of free space; μr is the permeability of medium. We will assume unless specify, we assume that μr is one in our course. We have D the electric flux density given by $\xi_0 \xi_R$ E for the simple case that we are considering.

R is called as the relative permeability it is the free space permeability. Add to this the current density vector Δ. J - ∂ρ/ ∂t and then we have the Maxwell equation, the constitution relationship or the constitutive relationship and then the conservation equation and we are good to go to study, how these field could be behaving in region one when material properties are different and in region two when material properties are different, and what really happens on this boundary okay. In order to that we need to study boundary condition which is the subject of next module thank you very much.

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