Indian Institute of Technology Kanpur

National Programme on Technology Enhanced Learning (NPTEL)

Course Title
Applied Electromagnetics for Engineers

Module – 33
Electrostatics –I: Laplace and Poisson's equations

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Hello and welcome to NPTEL mook on applied electromagnetics for engineers.

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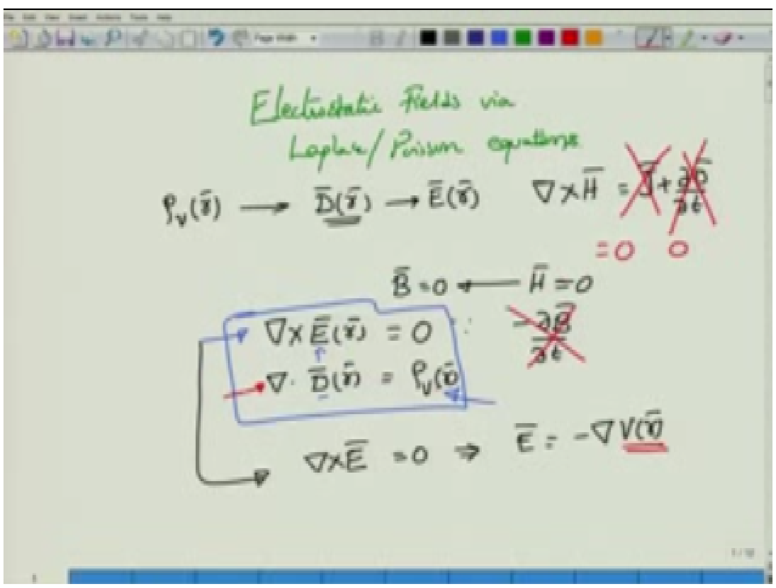

In this module we will begin the study of electrostatic fields via Laplace and Poisson's equation right. What is this electrostatic field, we are already acquainted with Maxwell's equations where we discussed how the electric field E, the magnetic field H, the magnetic flux density vector B, and the electric flux density D, they all are related to the charge distribution, in general the charge distribution can be volume charge distribution ρB, and current distribution J, which in general can be a surface current distribution.

So how does all these equations or how do all these quantities relate to each other is given by Maxwell's equations and we have studied these Maxwell's equations, we have looked at those equations, and we have also seen what happens when you actually have two different media, in that case we have studied boundary conditions which relate the electric fields and magnetic fields in one region, the electric fields and magnetic fields in the other region.

In general solving these Maxwell's equations which are most general case okay, is actually quite complicated okay. There are few scenarios where these Maxwell's equations can be simplified okay, and these scenarios are quite practical. In fact most of your introduction to Maxwell's equations could not have been through the general time varying Maxwell equations, but rather through the subject called as electrostatics.

In electrostatics Maxwell's equations can be simplified simply because we assume them to charge distribution, that is the way the charges have been spread, the way the charges have been constructed, these charge distribution is constructed in the space, this charge distribution does not vary with time. So if you, you know get to the simplest of the scenario, if you consider a simple point charge and you fix the point charge with some particular point, then this point charge the amount of the point charge will always remain constant with respect to time.

However, there will be a electric field generated because of that particular charge and that electric field also happens to be independent of time. In other words time has no effect when you consider electrostatic fields okay. So that is the meaning of static, static is with respect to time. So these fields can also be considered as time in variant electromagnetic fields or rather time in variant electric field, because we will be mostly interested only with the static electric field.

And we describe the static electric field or we try to find this static electric field by way of solving set of equations known as Laplace and Poisson's equation okay. As I said static conditions arrives when the time dependence is taken out. In other words, when the charge distribution happens to be constant with respect to time and of course where it is only with respect to space, then you know that the D field that this charge generates will be independent of time.

So no more D or r, t, but you simply obtain D(r) which in turn generates the electric field, this is the model that we have considered how the charges are related to electric fields inside a medium.

If it is vacuum then ρ_v the charge density will directly tell you what is the electric field intensity E(r) okay. So you have this electric field also independent of time which means that the corresponding quantities that we had looked at for example, when you go to Faraday's law or modified Amperes law or modified Ampere Maxwell law, this have this condition that the curl of the magnetic field must be equal to some current plus some $\delta D/\delta T$ which was the displacement current.

Now because our D happens to be only function of the space coordinate and not the time coordinate this quantity will actually be equal to 0, so this quantity will be 0 further if I assume that we are dealing with a scenario where there are no currents because currents mean charges have to be changing with respective time that is what the relationship between the current density J and the charge density ρ_v that we have seen in the charge conservation equation or the continuity equation.

So we for now consider the scenario where our displacement current is 0 because the charge distribution is static with respective time and hence the D filed is also static with respect to time correspondingly $\partial D/\partial t$ term is 0 and we also consider the scenarios where we have no current what so ever which means J is 0 so you now have very trivial field for the magnetic field H and the magnetic field F must be equal to 0, there are no magnetic fields because there are no currents okay.

Now if H is 0 because B and H are related by the permeability B also happens to be 0 now because B is 0 the conditions for faradays law when you write the faradays law in the differential form you see that $\nabla \times E$ which describes the curl of the electric field and since the electric field is now only a function space the proper way to write this one would be E(r) so this $\nabla \times E(r)$ must be equal to 0 since this term is the flux density or the flux linkage to that particular point where we were considering which is given $-\partial B/\partial t$ this term is actually equal to 0.

Why is this equal to 0 because we have considered the case where there are no currents means no H no H means no B and no B means this right hand term is equal to 0, so you see that all of the elector statics can actually be described by a set of two equations the two curl equations which have nothing to do what so ever with the magnetic fields in this case, so what we say is that the electric fields and the magnetic fields became D coupled with each other.

That is these two are D coupled with respect to each other the electric field that is when the charge is to fixed I am sorry here this should have been a charged distribution ρv so the charge distribution density is fixed with respect to space will generate a time in variant or static D field static D field will be related with the static electric field okay so these two equations are sufficient to describe the entire field of electro statics.

However these two equations still a little bit of a problem for us to solve because you need to know the charge distribution if you know charge distribution you find out what is D once you know what is D, the you find out what is the electric field E but unfortunately many particle scenarios the charge distribution is an unknown quantity in fact you want to find out the charge distribution as a result of solving these equations, so you cannot specify electric field because that is what your trying to find out.

You cannot specify D field because that is again what your trying to find out nor you can specify the charge density ρv because that is something that is usually very difficult to measure and hence that is also unknown if everything is unknown then you that seems to be a case that there is no this is a no hope situation because everything is unknown and therefore I cannot solve for any felid quantity.

However this vector field relationship that we have $\nabla \times E = 0$ service us to introduce another auxiliary electro static quantity which we call as the potential or in general words we call this as a voltage okay so this voltage and potential are essentially one and the same there is a small difference between the two which we do not have to go through here now this potential which will be a function of space will actually be scalar field and it can be found out even if you do know the charge distribution in fact in most cases you specify the potentials that need to exist at different points in your circuit based on that information you can find out the potential everywhere from potential you can find out the electric field from electric field, you find out the d field from d file you find out the charge distribution so this is the route a little circumstance kind of route that we are going to take it is not a straight forward path.

Seems to be but this is one of the most practical pass that you will take because, it is easy in laboratory to specify potentials of voltages at different points okay rather than to specify the charge distribution and the corresponding electric and d field vectors, so that is the route that we go so what is this potential that I am talking about well I know that curl of electric field is = 0 so this actually know from some vector algebra you can rush up that vector later on, or refer to appropriate text book.

Tells you that if there is any vector field quantity whose curl is = 0 this is an irrotational kind of field, then it is implies that it is possible for us this electric field or any vector quantity whose curl is = 0 in terms of a certain scalar field okay, this scalar field is denote by V of r and V is something that would remain us of voltage and voltage is a quantity that is related to potential in fact you measure potential in terms of fields so you have a nice scalar field here, okay which of course if still depended.

And change in respect to points in the space but note that this is a scalar field, and if I somehow and able to solve for thi8s field V of r everywhere in the region that I am interested in when using this gradient I can obtained the electric field and from the relationship between E and d I can obtained d and using Gausses law I can then obtained what is a charge distribution or we will also circum of that problem by actually you know going to what is called as Poisson's equation which relates v to ∂d okay around being the volume charge density.

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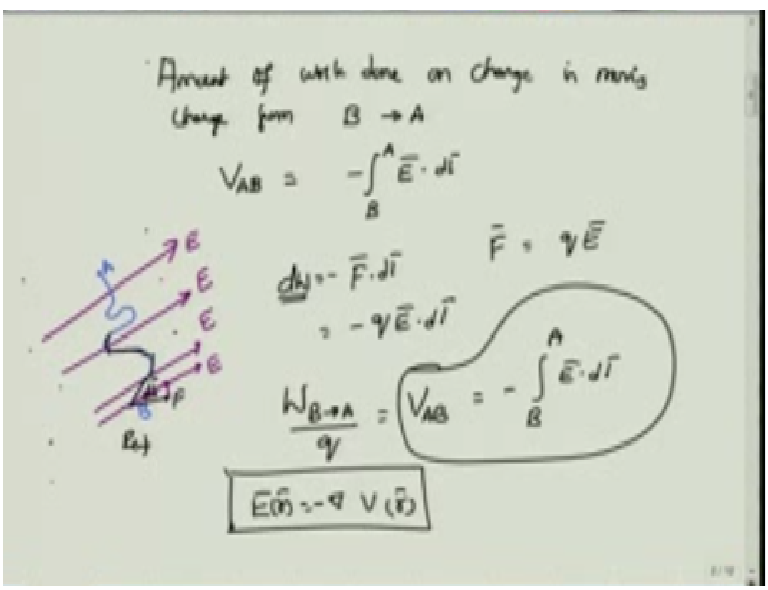

To gain to understanding as to what this potential E is the basic idea of potential is how much you know work you do, so this is the amount of work that you do okay in order to move a charge amount of work done on the charge okay in moving this particular charge from a point B to point A we then say that there exist some potential difference in AB between these two points and this potential is given by I mean integral of the electric field, so this just the way you go above to you know the gradient equation and then you invert.

Or integrate this gradient equation in order to obtained this line integral so we say that point A is at potential V_{AB} with respect to point B when you know by defining this particular thing as the amount of work done on the charge in moving the charge from point B to A, to give you slightly you know better idea if what is happening so imagine that everywhere you have an electric field E so these are all your electric field E okay so these are all your electric field E directed in this particular weight and you want to move a charge.

From some point be here along this particular path and reach the point A here okay what is the work done in moving the small distance dl along the path here, to work done must be equal to whatever the charge or whatever the force that is represent at this particular point on the charge okay, so let say this is the way the force is present and force dot dl that is the small amount of distance that you have moved along this particular path will give you the amount of work that is done.

However when you have only electric static field here the force that is a particular charge in experiencing is given by the amount of the charge that you have q times the electric field at that particular point so for example I am considering this at a point B and I move a small distance dl then the work done will be given by F .dl where F is the electric field at that particular point times the charge.

So at B if I say that this is the electric field then I should actually orient my F along E because F is now equal to Q time’s E, okay. So this is how the force will be presents or the electric field that is present at this particular point and then multiply that one by Q in order to obtain the force at that point P and then you now take the dot product of this force along this dl or with respect to dl you know vector.

And then you obtain the amount of work that is done, however this work is being done by the change but the work done on the charge will be denoted by a negative sign and this would be given by –f. dl okay and substituting for F you get Q E. dl this is the amount of work done in moving a small distance dl if you now think of this different paths okay, so you can think of this small paths over here.

And then everywhere you have the electric field out there take that dot product you know sum up all this quantities you get that total amount of work done in moving a charge of magnitude Q from B to A and this charge or the work done on this particular charge per charge so work done in moving a charge from B to A per charge is what we call as the potential V_{AB} and this is given by - ∫ B to A E. dl, okay.

And it is this V which actually satisfies so if you know what is the potential difference at any every point so you can select one point as a reference point and then you calculate potential here potential here everywhere you calculate the potential in essence knowing what is this scalar quantity V_R done inverting this relationship differentiating this and then adjusting it for the nature you obtain the relation E(r) = - Δ R(r), okay.

So this is the relation that actually is you know that we have discussed in the previous slide and this is what is the starting point for obtaining Laplace equation, what is Laplace equation?

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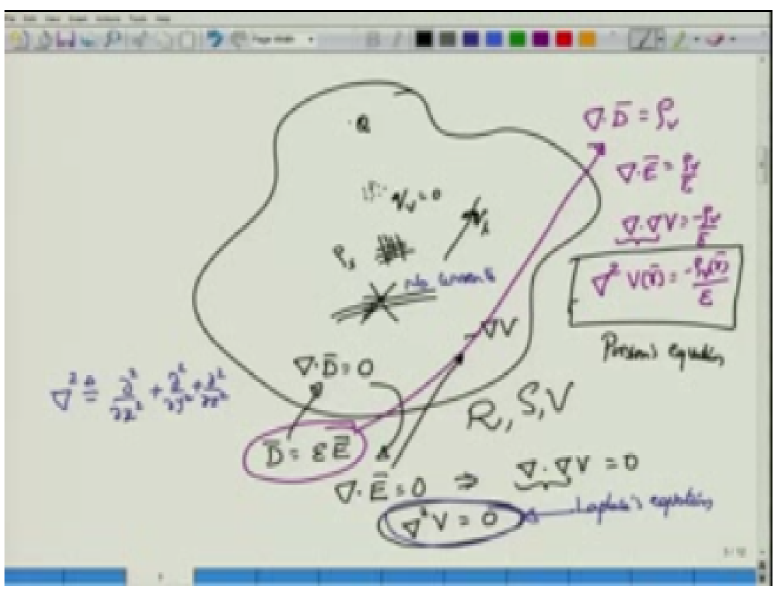

Consider some region okay so this is a region that I am considering it has some surface S and some volume V and it is surrounded by a certain contour out there, okay. Inside this if I assume that there are no charges so there are no charges here it could be volume charges it could be line charges or surface charges it could be line charge okay so this is your line charge this could be the surface charge. Or in general it could be a point charge as well.

So some Q and also neglect that there are any currents that is there are no currents inside this region then Gausses law if you apply this over this entire closed surface will tell you that $\nabla.\bar{D}$ must be equal to 0 there are no charges inside right. However, $\bar{D}$ is something that is related to electric field in a simple matter in the way of $\varepsilon\bar{E}$, where ε is what is called as the dielectric permittivity of the material. So what we are saying and you know taking ε which is a constant out of this dot product is that $\nabla.\bar{E}$ is also equal to 0, okay.

Because you substitute this $\bar{D}$ into $\nabla.\bar{D}=0$ equation Gauss's law and then essentially what you obtain is this $\nabla.\bar{E}=0$, but I know what is $\bar{E}$, $\bar{E}$ is nothing but $-\nabla V$ so I substitute that one here so this equation tells me that $\nabla.\nabla V=0$ and we give a special name for this $\nabla.\nabla$ and denote this one as ∇^2 okay, and this ∇^2 is what we call as a laplacian operator okay, the laplacian operator is actually given by the dot product of ∇ with itself okay, and this is the notation that we use for a laplacian which is given by ∇^2 okay in the Cartesian coordinate system.

These expressions are very easy to remember in the Cartesian coordinate system at other coordinate system unfortunately I have to look at a book or you have to take a look at the reference hand book or something because these expressions are not so simple in other coordinate systems. We will look at mostly rectangular coordinate system and in this rectangular coordinate system this ∇^2 is defined as you know partial derivative of you know any quantity that on which this ∇^2 operates.

So $\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$ okay, so this equation which has circled okay, is a very important equation and this equation is called as Laplace's equation, okay. Suppose now I assume instead of having no charges inside I continue to assume that there are no currents but there are charges okay, in that case what will happen to the equations so you now no longer can write $\nabla.\bar{D}=0$ but you have to write down what is in general the amount of volume charge density in this region.

So you have $\nabla.\bar{D}=\rho v$ do not worry $\bar{D}$ is still related to electric field E by the permittivity ε so I can substitute this $\bar{D}=\varepsilon\bar{E}$ into this equation and write this as $\nabla.\bar{E}=\rho v/\varepsilon$ and then I say electric field $\bar{E}$ is related to the potential or the voltage v out there, so I go back and substitute for electric field so I get $\nabla.\nabla V=-\rho v/\varepsilon$ and we have already discussed what is this $\nabla.\nabla$ which is nothing but laplacian so you have $\nabla^2(\bar{v})$ at particular point r we given by the volume charged density at that point ρv divided by the permittivity ε, so let me put a box around this because this equation is what is called as Poisson's equation, okay.

So Laplace's equation and Poisson's equation both tell you how the voltages are related in a particular region if $\nabla^2(\bar{v})=0$ that is there are no charges enclosed by this closed surface, then the condition that holds is that the $\nabla^2(\bar{v})=0$ and that is called as a Laplace's equation and when there are a few charges enclosed within that this charges could be charge density in various forms and that charge density divide by ε with minus sign will then be equal to $\nabla^2(\bar{v})$, so that is called as the Poisson's equation.

Amongst these two equations it slightly easier for us to solve for Laplace's equation that is what we are going to do now, and then we talk of Poisson's equation later on.

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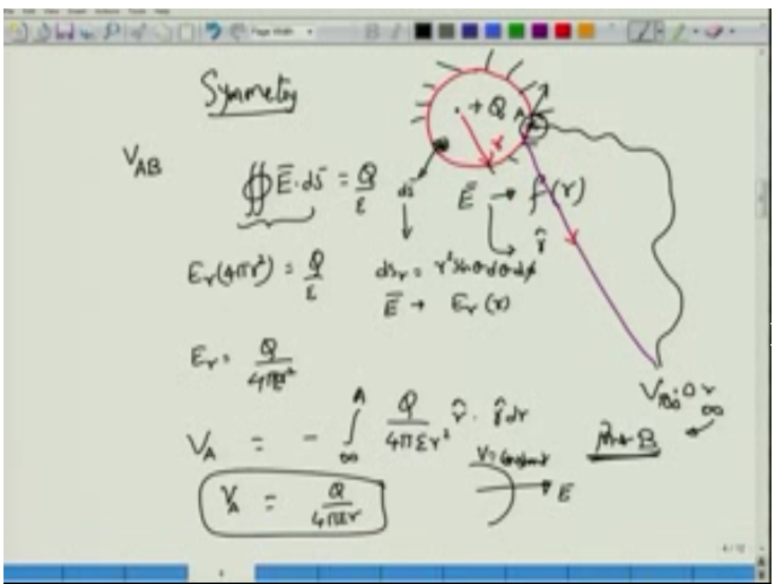

So for the next minutes and for the next few modules or at least one module we will be talking about how to solve Laplace's equation let us actually go back once more to this voltage difference VAB and then actually try to find out you know in if the charge distribution is given then can I find out what is the voltage now that is a straight forward case right.

So if charge distribution in the space is known to you then I can use that information to find out what is the potential, so we will actually do that one first and then come back to Laplace's equation. So let me not just jump directly to Laplace's equation first you get a idea of what V_{Ab} is when I consider a single point charge also assume that the point charge is that you know having a magnitude of + q and this the charge that is present okay.

The simplest way to solve this problem is to actually imagine that you know a spear which is closed okay this did not close itself so there is a spear which you can assume it be closed and let us say this spear has a radius of about R okay and w e are considering this spear in the three dimensional spherical coordinate system okay and now apply Gaussian's law because this problem has too much of a symmetry in the since that there is a point charge now you imagine that there is a point charge here okay and then you are going around the spear as you keep going along a particular circle on the spear okay you keep going does the charge distribution look different to you the charge distribution does not look different to you.

So it looks the same charge right, so I am standing maybe here I get the same charge you know I see the same charge I move here on the circle of constant radius R I see the same charge everywhere. I might even decide to look from the θ direction, so I might start looking from the top and then go either like this or like this whenever I you know turn my neck the charge distribution does not really change because it is just a point charge therefore you can immediately conclude that the electric field that is generated by this charge must only be in the radial or rather it should be only dependent on the radial distance.

The only way you can see different charge is when you keep moving away from it, if your faraway at infinity you do not see in a charge at all, and if you keep coming closer to end closer you actually see in the charge. So the electric field is actually function only of R also the electric field can have only the radial direction okay because if it has to have a five direction so let us say there is some electric field because of the point charge here and since you know that electric field have to originate from a charge and then terminate on a charge if I need to have an electric field quantity along this five direction they has to be a some charge here and there has to be some other charge here over here okay which unfortunately there is no charge out there.

Even if you assume that there is not second charge over here you simply cannot extend this lien because there is absolutely no charge at this particular point. So you actually see that electric field has only the radial component and this arguments holds a same for θ as well, so you have only the radial component and this radial component is only function of R. So what is the corresponding Gausses law.

So Gausses law is essentially the closed surface integral okay where you consider this particular this one as a surface area Ds and likely this surface area is actually the constant radius surface area which is given by r2sign θ d θ dφ and the electric field which is unknown to you but it has only the radial component is given by Er ® and at a particular value of r this electric field is constant it will be directed radically outward but it would be constant and DSr will only dependent on φ and φ because small r is constant.

So at a constant radius small r then this particular Gausses law will tell you that integral of D. DS is Q but then I am writing DS ε you know ε to be constant so this fellow will BQ/ ε and then integrating the electric field which is a constant at a particular radial distance from the charge okay will give me $E_r*4\pi r^2$ which is the surface area of the sphere at a constant radius r and this is equal to Q/ ε and you have E_r is equal to $Q/4\pi r^2$ now imagine that I want to find out what is the potential at that particular point okay.

What I do is I imagine myself first that there is infinity point here whose potential you know V of infinity I take it to the 0 so this infinity is your point B for μ in this calculation so I imagine that potential of that infinity will be equal to 0 and then bring this charge from infinity all the way to the point A over here so which is on the sphere that I am considering so when I have to take this part let me actually take this part which is radial you know I am coming in the radial direction.

So that way it is easier and I have to contained with the electric field so if this is a positive charge to a electric field will be in this way and if I a moving this charge I am moving this charge against the field okay no matter what I do the corresponding potential V_A can be written some were expression with infinity as a point B and we have already seen that the potential at point B if taken to be equal to 0.

So VA is equal to –integral of infinity to A and bringing this one from infinity to point A the electric field is $Q/4\pi r^2$ long the radial direction the part that I have taken is also the radial direction so I can take this dot product and then simplify this relation or whether integrate this equation and obtain that the potential on any point at a radial distance r from the positive charge +Q is given by Q/4πr.

So this was the problem which essentially told us how to calculate and how to utilize symmetry can be extended to many, many cases and I am sure you have seen lot of those electro static field you know calculation some of this field calculations are quite hard but then in principle you can you know makings of Gausses' law you can make use of symmetry properties.

And then you can know with some amount of mathematical maturity you can calculate electro static fields in many, many different charge configurations okay and once you calculate what is the electric field you can then find out what would be the potential at a different points only thing that you have to note down is that the potential field will always be perpendicular to the electric field.

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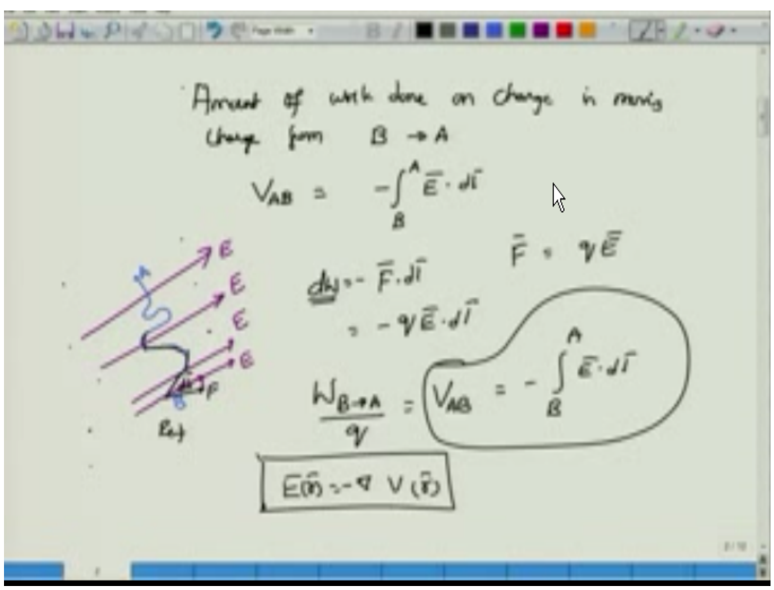

So if this is the constant V equal to constant contour then the electric field will always to perpendicular to that one okay and that relationship happens simply because electric field is related to the potential V using the gradient operation and gradient tells you how the scalar field is changing you know that potato example that we gave in one of the previous modules can be thought back over here and then, understand that whenever I have potential gradient okay, which is the electrical field that gradient will be perpendicular to the direction of constant or to be perpendicular to the contour of constant potential V okay.

So this was the example of you know solving the straight forward problem but we are really interested in solving this straight forward problem. We are interested in actually trying to solve a problem in which the potential is specified okay. And I consider very simplest case of this.

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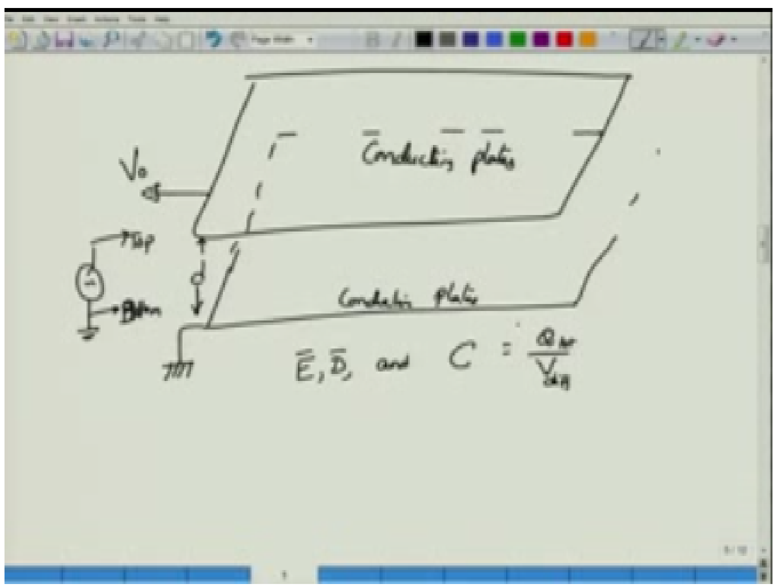

So I consider 2 conducting plates, whose areas are very large okay, so I neglect or consider the area to be very large and I consider the distance between these two plates. So these are both perfectly conducting plates okay, so these are all nice metal plates with infinite amount of sigma that you can think of okay. And then I have the area A, you know this area A, which is as I said is quite large.

So below this plate at the distance of say about d I place another plate this is also a conducting plate okay, made out the metal of the same as a top plate and it also has the sigma going towards ∞. Now what I do is? I ground this bottom conductor and I apply a potential of V0, so I can do that by simply connecting the voltage generator out there okay. Connect this one to the top conductor and this one to the bottom conducting plate. So I actually specified that the potential everywhere on the bottom conducting plate is 0.

The potential everywhere on the top is actually is the constant value of v0. What I want is to find out what is the electric field E? What is the D field E and an important quantity called as the capacitors. Capacitors if you remember is the amount of charge in the single plate, so this is the total charge in the given plate, divided by the potential difference between the two plates okay or between any two conductors. This is the quantity that we want to solve for. We will do that in next module, when we will solve the equation. This is the case where the lipases equation helps us to solve and then find out the corresponding values of E, D and C, and thank you very much.

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