Indian Institute of Technology Kanpur

National Programme on Technology Enhanced Learning (NPTEL)

Course Title
Applied Electromagnetics for Engineers

Module – 35
Electrostatics –III: Solving Laplace's equation in 2D

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Hello and welcome you all to the NPTEL mook on applied electromagnetics for engineers. In the previous module we looked at solving Laplace's equation in one dimension thereby we also derived the capacitance of a parallel plate capacitor and acquired serial cable. In this module we will first solve the Laplace's equation in two dimensions. And then solve Poisson's equation in one dimension.

Now my main motivation here is to introduce you to a technique called as variable separable method which we adopt later on for solving such two dimensional problems. Along that well, we will also show you how and what type of functions are necessary in order to now satisfy Laplace's equation once the dimension of Laplace's equation becomes two dimensions from the simple one dimensional case.

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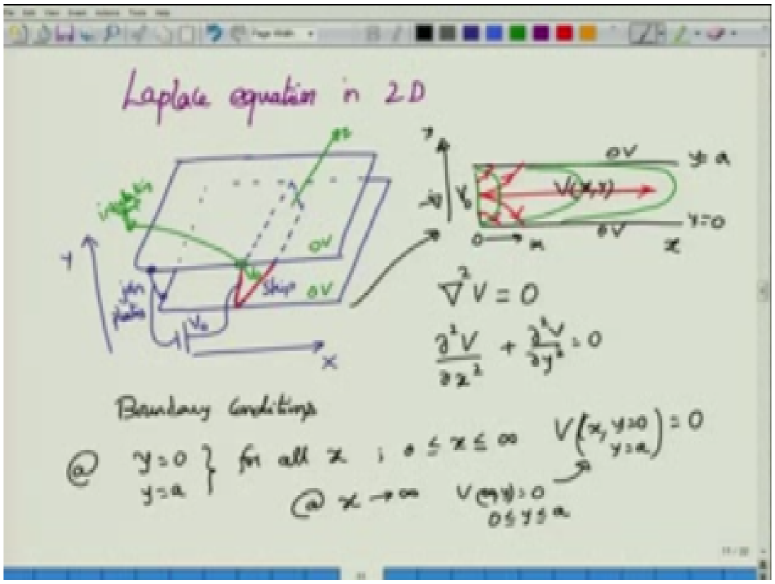

So let us begin by looking at the problem chosen to illustrate how to solve Laplace's equation in two dimensions. You have a pop and that in plate for now assume that the conductivity is very nice it is about infinity. And then there is a bottom conducting plate okay, and these two plates are actually joined together. Now in electrostatic phenomenon which is what we are dealing with here, you should remember that the electric fields that are going to be set up by this parallel plate structure will be time in variant.

That is no long be varying with respect to time and for that reason only we are able to actually define the potential or the voltage v between the two strips unambiguously right, which also means that the conducting plates themselves are the surfaces of constant potentials okay. So this phenomenon of equi potentiality holds only in the case of static condition and it will not hold in the case of a time varying situations such as the transmission line that we have studied in the first part of the course.

Nevertheless coming back to the problem at hand, you have the two plates, we will again assume that the plates actually have more point of bit of a width and length okay. So we are not bothered what happens at the edges of these plates, that is we will neglect completely the fact that there will be some edge effect in dealing with the problem over here. What we now do is, we join these two plates, so essentially both are at the same potential, the top and the bottom potential.

And then we imagine that we actually have inserted a strip at the middle of the two plates okay. The strip of course has to be inserted in such a way that there actually has to be a small gap at the top, so let me rewrite, you know at the, this one over here. So there has to be a small gap here which we will be filled with an insulating material, otherwise everything will be at the same potential okay.

So we need to give a small insulating gap as we call this one okay, and leaving this small insulating gap you we have a metal strip that is being connected to the center okay. And this axis essentially goes very long, so you can think of this as the z-axis and we consider that the z-axis is quite long. And therefore, what we want is what happens not along the z-axis, but what really happens along the y-axis.

So what is the condition that we are forcing with the strip? We have joined the two plates, the top and the bottom and then we hold that two plates with respect to the certain potential difference which we call here as V0 okay. Where the other point is connected, the source is connected to the metal strip in which is, so in other words the metal strip is actually at the higher potential V0 that the two plates, we can even imagine that the two plates are situated at the ground potential or zero volt, and the strip is actually at a higher potential of V0 volt.

And what we want is this situation, so we have a strip in this way okay. And then we have this direction along which this particular plats are going so this is my x direction so essentiality look what I am looking at is know in this particular cross section so this way is the y direction and there is certain gab hiver here as I told you and this fellow is kept at a potential V_0 the strip is kept at a point V_0 and what we want is a potential or the voltage V as a function of x and y okay.

The top and bottom plates are of course at 0 potentials, so it is essentially that you are looking at it from this particular point and then you know this view is what is relevant over here okay the z axis's is not relevant over here we are interested in what is happening to the potential in-between this region along the length x okay.

Now that we have step a problem we clearly understand that this problem is a 2 dimensional problem correct this is a two dimensional problem because if you just go by the physical understanding of what the potential means by placing a higher potential at one conducting plate and keeping other you know plates at 0 potential what I have created is potential difference between the two and whenever there is a potential difference okay we know that there will be an electric field between them.

And we also know that the electric field likes to go from the higher potential to a lower potential therefore you can expect we will later show that this is correct but you can expect that the field lines will actually go like this okay so the field lines will go this and the potential being 0 at the center right and it would be at potential V_0 as you go away from the x axis that is as you go away and away you can actually see that the potential would go something like the potential will 0 at the center and then will keep going away as you along the x direction.

So this red lines are essentially the field lines and the green lines are the constant potential contour that I have drawn clearly they have to perpendicular to each other from the operation of the gradient thing right so we know that this is essentially what physically would be happening so you have these constant potential contours and parsing them at normal will be the electric fields that you are going to find okay.

Now how do we obtain the same things using mathematical relationships for that we need to solve the second you know Laplace's equation in 2 dimension we already what is Laplace's equation applied in the region in between the two plates say at any point x and y I pick the corresponding Laplace's equation is $\nabla^2 v = 0$ so rewriting this as the partial derivative that is expanding this ∇^2 you have $\partial^2/ \partial x^2 + \partial^2 v / \partial y^2 = 0$, now we also know that it is not just enough for us to solve the equations we will eventually have to apply certain boundary conditions in order to obtain a unique solution.

So I also need to tabulate or I also need to give you what are the boundary conditions that I am actually going to apply to this problem clearly you have a top wall and a bottom wall so at the top wall and bottom wall potential v must be equal to 0 the top wall is at y equal to let us call this as y = A and this one as y = 0, so these are the two surface or you know two lines tell you the top which at y = A and the bottom which is at y = 0.

So at both walls that is at y = 0 and at y = A for all values of x going from so let us assume that x is 0 here and this is how the x axis is going around know so along the horizontal direction so for all values of x such that x is $0 \leq x \leq \infty$ the potential must be equal to V, so potential of V at x y can be 0 or y can be A in both cases the potential has to be = 0 so this is the you know stupid mathematical notation that I have used this is my own so do not confuse do not get confused what it say what it is saying this two conditions is saying V at any point x along the axis or along the direction of x on y = 0 bottom plate and y = a top plates and potential where must be = 0, where l is do we have the potential condition well we consider work should happen at a very large x.

So x turns to ∞ we do expect the potential v to go off to 0 right so potential V goes off to 0 as you go very far away into the condition and y of course is limited by 0 in y less then or = a okay then there is one more condition the final boundary condition as you might want to think of and that boundary condition happens to be on the left wall over here at which we have kept in the potential to be equal to.

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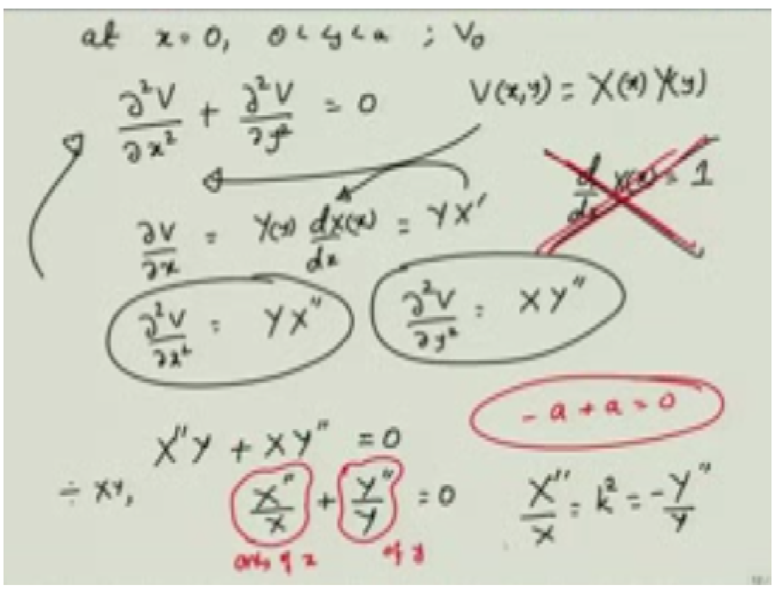

V0 that is at x = 0 and y between 0 to a right though we have a ∞ gap out there so y02a the potential V = V0 so this is the though final boundary condition that I actually have so subject this boundary conditions we need to solve Laplace’s equation what I am going to do is actually to skip a few mathematical details okay I actually expect that you will filling those details while yourself because there is nothing else like actually solving the problem along with the lecture okay you can pass and then you know filling all the details and you will you know see that all the equations are I am writing.

We can derived for yourself if you are able to do that one when you understand this particular problem how to solve it the reason why I want to do that one if because I want to also cover Poisson's equation in the same module okay, I will give you basic equations and then tell you what you need to do in order to arrive at the next step of the equation, so we go back to the second or the partial derivative equation and they recognize that in order to solve this one right, it is not just enough that I consider these some functions of x or some function of y.

Because V in this case happens to be a function of both x and y and this function is completely unknown to me okay I only know the boundary conditions so how do I actually try to solve it I cannot directly integrated it because if I integrate the left hand side then I do not know how to integrate on the right hand side because I do not know how this you know function would actually change as a function of x and y, therefore I cannot integrate the equations directly right so the direct integration.

Is does not work what we do is what is called as the variable separable method or separation of the variable method, in which we assume that the unknown potential V of xy okay satisfying this equation can be written purely as a product of two functions one of which is varying only with x and other is varying only with y, so you have to disguise here by the capital x and the small x small x is the variable X is actually the denote you know what we are denoting at a function I could easily equally have taken this f of x and g of y.

But this is a common notation that we moves in variables separable method this reinforces that the product solution that you are looking for this composed of a function only of x and the function only of y now substitute this one into this expression so if I now differentiate partial with respect to x this potential V of xy I see that because I am looking at a differentiate with respect to x I can treat y of y as a constant and differentiating x with respect to x will give you the derivative of x.

So I do not know that one because I do not know how it goes and please do not confuse that d / dx of X of x = 1 this is not correct okay so please ensure that this is not correct and you are not making this mistake okay because x is just an name of the function that we have chosen it is not equal to small x, there is a short hand notation for writing this is actually equal to Y x' where x' is one of the ways in which we write dx/dx or d/dx (x).

Similarly if I now differentiate the above expression once more with respect to x as we will obviously get the second partial derivative of V with respect to x which you can clearly see will be equal to y, x'' by the same logic $\partial^2 v/\partial y^2$ = xy'' so now you substitute for $\partial^2 v/\partial x^2$ here and $\partial^2 v/\partial y^2$ into this Laplace equation so obtain x''y + xy'' = 0 okay. Divide both sides by xy and while dividing this i9 am assuming that the product x and y never goes to 0, okay.

Otherwise tis division will be in a big tough of problem for us so when I divide this one by xy throughout what I obtain is x''/x + y''/y =0.Now the cracks of variables separable method is here this term x'/x is a function only of small x the function here is one y''/y is a function only of small y now how can I have one function of x and other function of y I mean they could be for example $\sin^2 x$ this could be some Bessel function of y.

It could be any crazy type of functions that I have but how can two functions at every point x and y sum upto 0 the only way it can happen is when this function x(x) is a constant and the function y''/y is a same constant except with a minus sign right because any constant – a + a will be erqaul to 0, right. So clearly this x''/ x must be a constant and y''/y must be the minus of the constant.

And in the you know to simplify matters we consider this x''/x to be the separation constant k^2 k is called as the separation constant and this should actually be equal to –y''/y okay, only then that the sum of these two will actually go to 0, so you now have two ordinary differential equation.

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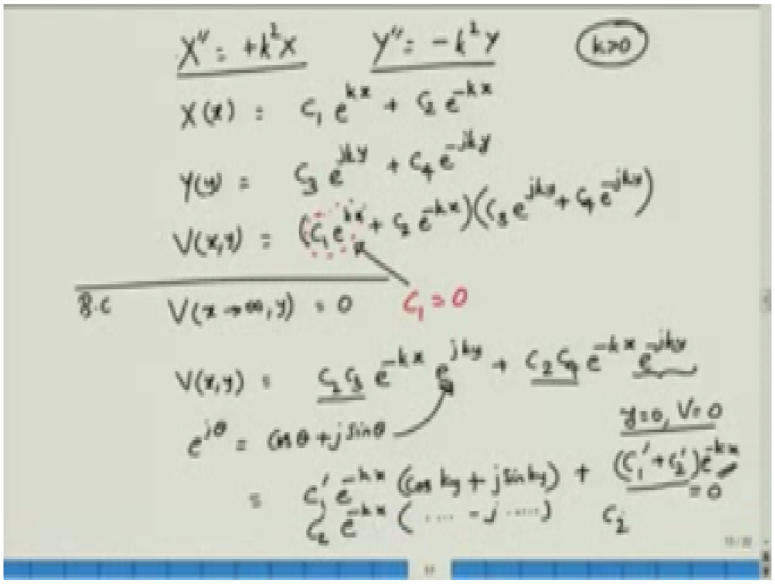

So you have x’’ = $+k^2$ x and then y’’ = $-k^2$ y the general solution for this are the exponential functions or they are the you know sign hyperbolic and sign functions I will use the exponential solutions over here okay so I can find the general solution of x(x) satisfying this ordinary differential equation as some constant c1 e^{kx} + c2 e^{-kx} okay assuming that k is a positive quantity so these are the solutions for x.

Otherwise you can also have the solutions in the form of sin hyperbolic and Cos hyperbolic functions but we will not consider this because we are not really familiar with sin hyperbolic an Cosine hyperbolic functions, similarly for this one because there is a negative sign out there the exponential term will become imaginary that is y(y) will be equal to some c3 e^{jky} + c4 e^{-jky} okay and remember the actual solution v(xy) is a function of these two or is a product of these two.

Which is (c1 e^{kx} + c2 e^{-kx}) c3 e^{jky} + c4 e^{-jky} I hope you have you know followed the thread throughout here and now what we need to do is to actually apply boundary conditions, okay. Let us apply a boundary condition which tells us that you can of course also expand this one right, so you have a eC_1e^{kx} and then you have a C_2e^{-kx} there is a e^{jky} so you can you know expand and then add the exponential terms and you can kind of simplify what you are looking at but if I left it as it also that is perfectly fine.

So first I will apply the boundary condition so this is the part where I am applying the boundary conditions okay, I apply the boundary condition that as x gets larger and larger for all values of y between 0 to A this potential must be equal to 0. Now examine which is the x dependence so this is the x dependence as x is made larger and larger when k is you know positive constant so unless even happens to be 0 this term will keep on increasing right, so this term will keep on increasing and eventually goes up to infinity.

Then you do not get any meaningful solutions for that, therefore because we want the potential and the electric fields to be in remaining finite we consider or we take the obvious conclusion that C_1 must be equal to 0, so the condition that potential must go off to infinity will allow me to say that this is actually equal to 0, okay. Now you can kind of simply what you are going to get okay, so you can now simply this expression so you get v(x,y)= sum C_2C_3 I am just opening the brackets and multiplying each of them.

So you have e^{-kx} $e^{jky}+C_2C_4e^{-kx}e^{-jky}$ now we also know from De Morgan's theorem that $e^{j\theta}$ is nothing but cos θ+j sin θ so I can apply that to this complex exponentials e^{jky} and e^{-jky} and then write the potential as sum C_1', where C_1' will be equal to C_2C_3, C_1' e^{-kx} and write this as cos ky+j sin ky okay, $+C_2'$ where C_2' will be C_2C_4 e^{-kx} and this is cos ky however here you will have a –j sin ky okay.

So this is the solution v(x,y) that we have, now we still on the with a couple of additional boundary conditions that we need to specify. Now what happens at y=0, so at y=0 we know that the potential for all values of x must be equal to 0 right, so now you substitute what happens at y=0 you will immediately recognize that it is the cosine term which actually becomes unity but the sin terms go off to 0, okay. So then you see that you know for what you are going to get the sin term will go to 0 okay.

So when you see that you know for what you are going to get the sign term will go to 0 and cosign term will become = one and what you get here is C_1' $+C_2'$ e^{-kx} = 0. Now under no condition we have the case that e^{-kz} is 0 that is we can you know evaluate this one x = 1 x=2 and then see that this has to be non 0 unless we take x = infinity but we have already consider that about condition earlier. So we keep x finite and recognize that the condition y= 0 v = 0 is telling as that $C_2' = -C_1'$.

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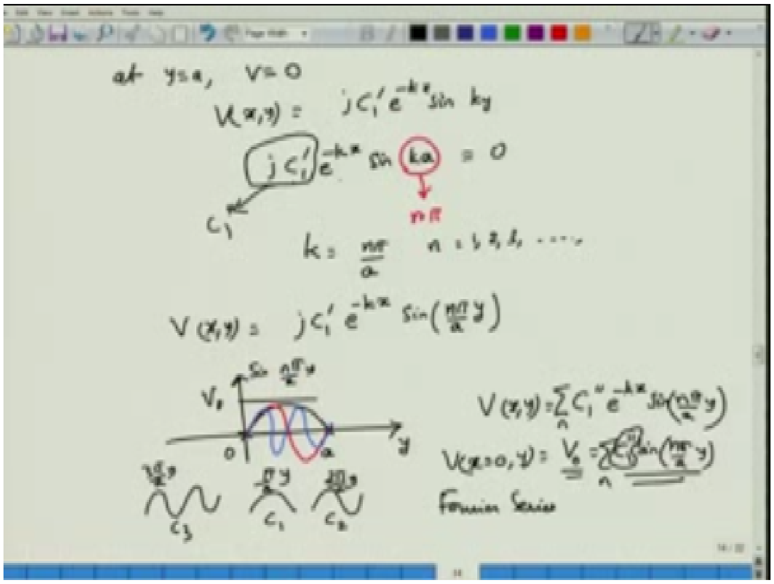

Similarly at y = a potential is again equal to 0 from the previous step you can show that v(x y) would have reduce itself to some j some C_1’ e^{-kx} and it would have been sign ky okay, now when you apply the condition at y = a the potential must be equal to 0 right, so what you get? You get j C_1’ e^{-kx} sign k x a = 0 now again here I do not want to make C_1 = C_1’ = 0 because if I make C_1’ = 0 then there is no solution for my potential.

So I would not make this 0 I cannot make j_0 I cannot make e^{-kx} 0 the only way that I am going get 0 here is when this ka actually becomes some integer multiple of π okay. For which the constant k which was earlier unknown to us must then become nπ/ k, and n will given integer, so n will be 1, 2, 3, and so on. What is the nature of this sign nπ/ a? So in general if I look at the potential that I now can write potential can be written as some j C_1’ there is still a constant do not worry about that one that gets fixed by the next boundary condition that we are going to apply.

This would be sign nπ/ a x y, so this nπ /a is nothing, but k if you sketch a few solutions so let us say this is along the y axis and this is a solution sign nπ/a x y that I am sketching. So you can see that at y = 0 and at y = a, the potential always as to go to 0, but when the n is = 1 you can go to 0 with only one half cycle and when n= 2 then you need to go in two half cycles to 0 or one complete cycle.

And when I consider n = 3 when you know you will have to go through three times or three half cycles, so these are all different solutions we call them as different modes of the problem that we are considering but these are all different solutions and for each value of n there will be a corresponding solution. So now we finally can write after absorbing j n C_1' x another constant called C_1'' so I can write potential v(x y) as C_1'' e-kx sign nπ/a x y final condition at have is at x = 0 y will be equal to some applied potential v_0 you can take this applied potential to whatever value so you let say it is equal to some V_0 we have taken.

So at x = 0 this is the condition, so V_0 = C_1'' sign nπ/a x y. Now this actually requires us to understand this one a little bit you know more carefully, on the right hand side I have a function of y on the left hand side I have a constant. Now there are of course not one term there are actually multiple terms that I actually have. So what I actually need to do and I cannot sum up all the sign functions right so I cannot sum up all the sign functions no matter what value of y to equal this one to V_0.

But the problem is no more difficult what this is telling you is that if I consider these as the sign sinusoidal signal right so these are the signals that I am considering when I adjust the coefficients of each of them so let us call this as some consistency event call this as some consistency C_2 this is nπ/a*y or rather π/a*y okay.

And this fellow will have 2π/a*y this will be 3π/a*y this will have the constant of C_3 what you are essentially trying to do is that if the solution is not just to that one particular n but the solution happens to be the sum of all n because every n will actually satisfied this condition so what you are trying to see is that instead of having a single constant.

And you know this particular function what you are trying to do is to take this infinity number of sign components and adjust the coefficients suspect there is sum over the range 0 to A will actually be equal to V_0 okay this process is very familiar to you in the name of what we call as four year series.

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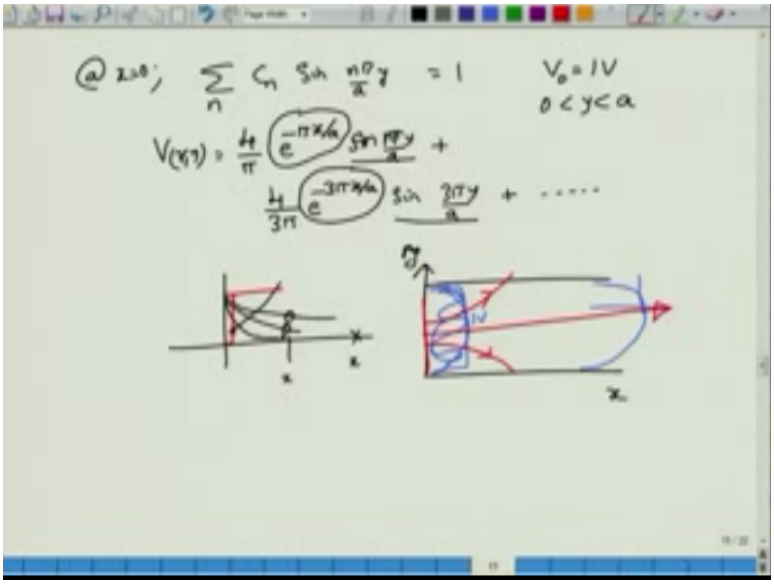

So what we are trying to do is that you take this infinite number of solutions okay so sign nπ/a*y at x =0 so write x=0 this is what I am trying to do and this one should be equal to some V0 okay we will consider this V_0 to be equal to 1 volt just to simplify so I have consider V0=1 volt so what I have this essentially here four year series ∑ and I will not going to details of how to find this n look at textbooks of signals and systems or look at the other courses.

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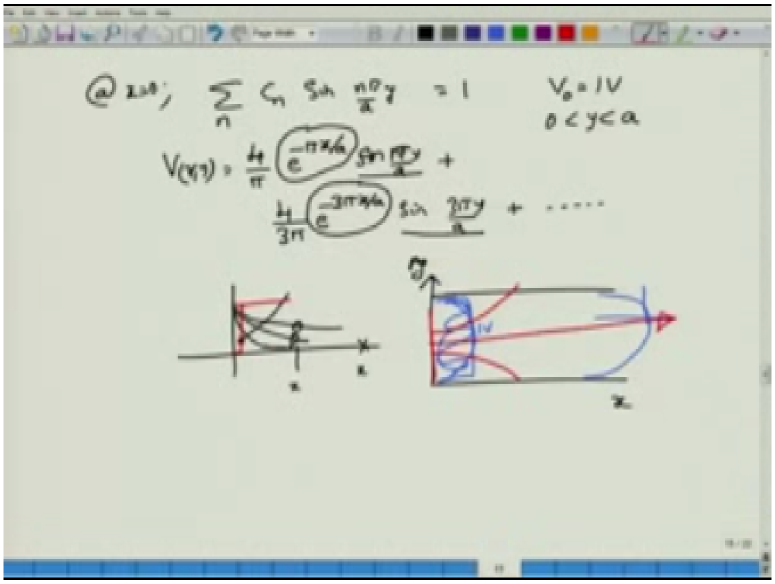

And then just understand how to evaluate this constant C_n so please remember that what you are trying to do here is you add all this individual solutions you know this π/ay is to 2π/ay 3π/ay and then 4π/ay and 4π/an and so on and sum of all this solutions with adjustable coefficients C_3 C_1 and C_2 must essentially sum up to the applier voltage V_0 here I am just considering the voltage to be 1 volt and when you do that you essentially obtain the four year series coefficients so y will of course be in the range of 0 to y less than a okay.

So this if you now calculate from the trigonometric series or the four year series the corresponding amplitude of C_L you get this as $4/\pi e^{-5x/a}$ sin π y/a+$4/3\pi e^{-3\pi x/a}$ so notice that even the K function that goes into that will also be different okay so sin 3πy/a+ something, something right so you will actually have $4/5\pi e^{-5\pi x/a}$ so you see that these are only odd harmonic that are going to represent and you can see that once you substitute this is the total voltage sorry C_n is not this one this is the total voltage of V_{xy} at any point that you are considering.

So you see that these are the solutions but they all are multiplied by the appropriate amplitude factors and as x goes to a very large value. So how does $e^{-\pi x/a}$ it will vary something like this. How does $e^{-3\pi x/a}$ vary, it will actually vary at a faster rate and then you start going you know for the values of higher order harmonics. Corresponding explosion function will actually start varying rather rapidly.

So which means if I am there at some intermediate value of x, know 1 end to may be at 2, may be at 3rd solution might be just sufficient enough to approximate this infinite series problem into a finite series problem. So with some approximation error okay, you can retain only a first few terms depending on what value of x I am there at okay. After any way at a large range of x does not matter, because all the solutions would have essentially gone down to 0 and that is precise what we want.

I mean we know that the potential at large value of x actually goes down to 0 but as you go near and near, all the terms need to add up with the considerable amplitude okay, such that they will total the applied potential of V_0. So this is what actually we can find, so if you now sketch these solutions, the two dimensional space x and y okay. So sketch this solution and then imagine this actually is your strip. At small values of x you will actually see a larger contribution from all of them.

But the contribution will be in such a way that, the potential will actually sum up to 1 okay. So this is the fundamental harmonic and then there will be another harmonic. So all these are summing up to the potential of 1 volt, but as you go to a very large value there is essential only single component sign by x/a and the amplitude of this one will also be very small okay. Here I am just showing it to the larger for the case of exaggeration but this will actually be small.

And the electric field will of course be directed in this way right, which would be perpendicular to the contour of the constant potential. So this completes apply the equation into two dimension I hope you really solve this step and understand what is going on and will some equation in the next module and then consider numerical methods of solving this equation, thank you very much.

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