Indian Institute of Technology Kanpur

National Programme on Technology Enhanced Learning (NPTEL)

Course Title
Applied Electromagnetics for Engineers

Module – 45
Skin effect in round wires

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Hello and welcome to NPTEL mooke on applied electromagnetic for engineers in this module we consider the quasars static analysis of skin effect in a round wire of radius a they have already setup the preliminary problem statement in the previous module so please refer to that one, we have a wire of radius A.

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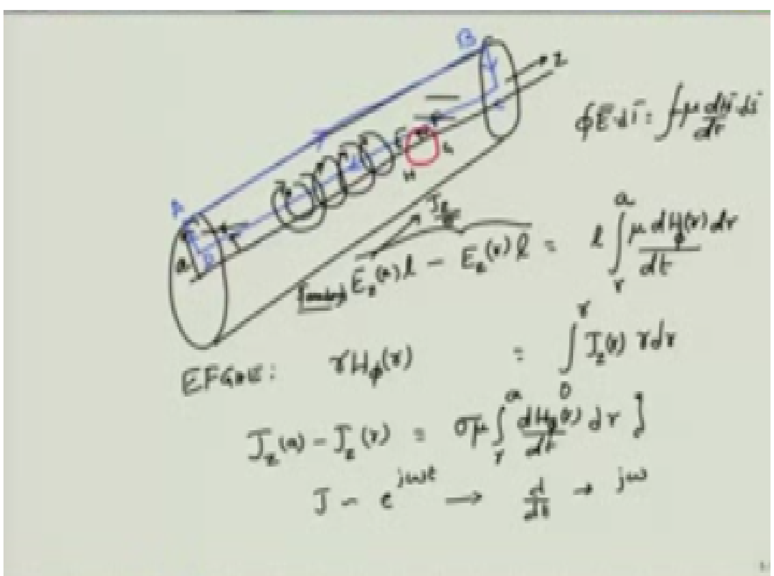

Which of course carries a current J okay, we have not discuss the field inside a wire we will leave that one for some other module to do so but our goal would be to see whether we are able to obtain the expression for the surface impedance z s that we obtain for the case of a plain wave and if you obtain that what about the values of r hint and l hint whether those values will go to the familiar values or they will in the limit code the familiar values when the frequency is made to 0. So that is the question that we are trying to answer.

And to do so, we actually construct to loops okay so I have a loop which is you know some length l long and this loop is cutting a constant value of 5 so this is the constant 5 loop let us label this loop as A B C D okay and then w ego along this particular loop in this manner that we have taken of course because of the current that is flowing through the wire there will be magnetic field and the magnetic field will form this circles at a constant radius power.

So if you consider the radius r so I am considering that this is you know 0 and this loop occupies from r to r = a okay, so the magnetic field of course will be coming out in this way it is in the five direction and it will have a constant value when it is at a constant radius r around the axis of the loop, that is essentially what you would expect from amperes law. We will of course neglect displacement current because this is the good conductor therefore ω ε is extremely small compare to σ.

So this is how the magnetic fields are coming out or coming around this one, so if I consider applying faradize law to this one that is evaluate integral of e. dl over the loop that must be equal to - μ dh/ dt over the surface that you obtain by looking at the area formed by this particular loop. Since we know that the electric field is along the z axes because the wire is assume to be along the z axes the current is flowing along the z axes, so when you apply this one over the loop what you obtain is ez of a where ez the subscript z tells you that the electric field, is the along the z axes and the value here is at a so ez of a time l – ez of r time l obviously this segments bc or da do not contribute to any emf because electric field is perpendicular to those cases.

This must be equal to what is the surface area of this one? This would be some value l because this is along the z axes but now you have an integration variable the second integration variable is along r direction and the field is in general not uniform therefore you have integral of μ dh of r or dh φ r because h is along the φ direction just to remind ourselves, and times dr dt that is the changing magnetic flux or the flux associated with that one is given by l which is the integration over the z axes that we have perform whereas this integration is being performed from r = r2 a okay.

So this is the open surface which encloses the magnetic flux time varying magnetic flux that should be equal to the emf induced around this particular loop okay. Now I also want to consider a different loop this time the different loop that I am going to consider will be center at the axes and it would go only to a distance of r okay, so this is the axes sot his loop that I consider will go only over a distance of r that is from 0 to r let us mark this loop as some e f g h okay.

So you have to imagine that this is the loop and this is how the magnetic so this is the axes of the wire and I have consider the loop e f g h such that you know circulating in the circum financial this one for the axes of the wire so this is the wire and then this is at the radius of r.

And of course the wire is assume to be reasonably thick having a radius a so that this small value of r is actually inside the wire itself okay. so over that one for that loop e f g h e if I apply ampere's law what do I obtain, the magnetic field there will be constant because it is being calculated at a constant radius r therefore h φ of r is the constant along that and loop has a circumference of 2πr which simply comes out because this would be integration over dφ correct. So this would be 2π r x h π r this however must be equal to the total current that is enclosed byt eh loop e f g h a.

What is the current enclosing by the loop? Well you have to go to r dr dφ correct you have to go to r dr dφ and go over φ 0 to 2 π because you are enclosing it completely and for r the integration variable is from 0 to r and then you have the current density will be along the z axes and it would be a function of r okay. So this is the expression so what we have here is one law which is the faradize law, which see you to calculate what is the emf chang4e or emf over the loop a b c d a and then amperes law without the modification of Maxwell, there is no displacement current because this is the good conductor okay.

I hope you understand these two equations, now these two equations do not really tell much about anything you know because we can first of all simplify this equation because the integration over φ will give you 2π and that 2π can cancel out with the 2π on the left hand side, so you have a slightly simplified expression which will be rh φ r given by integral of 0 to r on to this one okay.

So this is and of course you can rewrite jz as σ ez of r okay but we do not really want to do that because we want to eventually find out the current density, so I only retain jz here and instead convert from the faradize law whatever ez that I have in to jz / σ okay, so I rewrite faradize law so I retain the same law or the same form of jz in the amperes law but faradize law I will modifying.

So I obtain jz of a – jz of r to be equal to σ times integral r to a μ if you want you can pull μ outside the integration so you can have σ μ then dh φ r / dt time dr because only one small change that we are going to now make okay, we will assume that all quantities in terms of time expiator sinusoidal time variation, so they all go as $e^{j\omega\ tr}$ they essentially go as the cos ω t, which means that d / dt will be replaced by j ω is that okay.

So d/ dt can be replace by j ω therefore in this equation the equation on the bright hand side I can put j ω in place of d / dt and pull out this one right.

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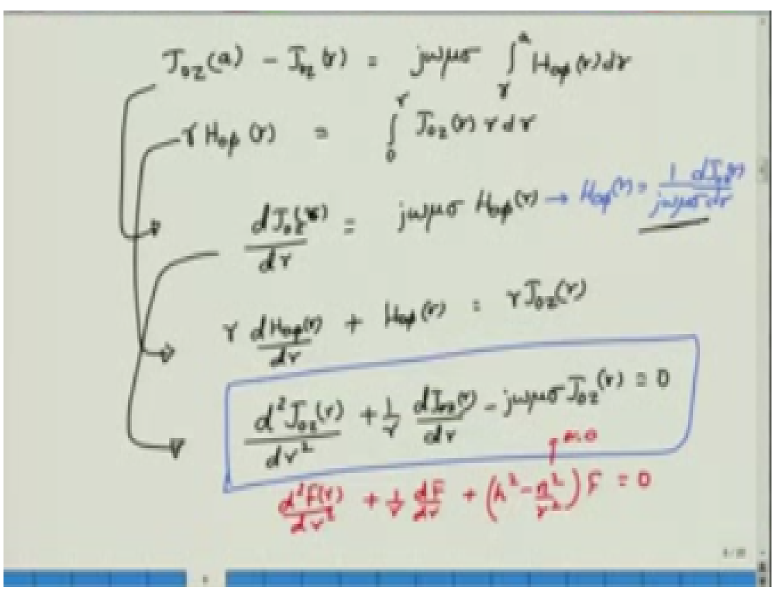

So if I do that you know have j o z of a this 0 simply indicates the amplitude of the cosine wave form or the complex exponential wave form okay so this is just for our reference. So joz of A – ja oz of r this is equal to the right hand side of that expression you know where you had dhφ / dt will become j ω μ σ integration will still remain r2 a h of φ of r dr and what will happen to the other expression with had the amperes law that becomes r h o φ of r = integral 0 to r j oz of r dr so there is a one more r here.

So let me rewrite this one so joz of r are dr here is a small caution or small kind of a clarification I know technically that you know if I have a integration variable r here I should not be using the same value of r inside I should be using some dummy variable maybe δ β λ right some other dummy variable I should be using but I hope that you are you know you do not get confuse between eh integration variable and the limits of integration because otherwise to use multiple dummy variables will complicate the notations slightly so I am assuming that you all are aware of the difference between eh integration limit and the integration variable itself okay.

With that small clarification let us go back okay, so I have two equations now what do I do with these two equations well unfortunately in these two equations I do not know h 4 φ of r I do not know j o z of r both quantities are unknown to me, all I know is j is along z h is along φ but I do not know how they are actually changing with respect to r okay with respect to time I know what with respect to r I do not know, so what I do is I differentiate both expressions with respect to r the left hand side of the first equation is easy to differentiate because this firs term will go away, so you have – d j ozx o dr / dr that would be equal to j ω μ σ because in the right hand side r appears at the denominator there will be minus sin further here time h of φ of r okay.

In fact this equation can be turned around to write h o φ of r as 1 / j ω μ σ d j oz of r / dr okay, so I hope that this is visible for you I have just inverted this one the minus sin on both sides will cancel with respect to each other so you have j ω μ σ, h o φ of r what about the second equation well the second equation if you differentiate you will have r dh o φ of r / dr because I am differentiating the magnetic field component plus h o φ of r I do not have to differentiate this r and already know what is h o φ of r h o φ r can be obtained from this expression..

So I can replace this one by 1/ j ω μ σ dj / dr this right hand side will of course you know will be equal to small r joz of r okay. The only quantity that now is remaining is this expression dh o φ r but we can you know overcome that one by differentiating this expression once more the expressions for the surface current if I differentiate it once more and substitute for h o φ and substitute for dh o φ of r by dr from this expression. So these two you have to substitute and carry out a little bit of an algebra I would not go in to the details of the algebra they are not very difficult you end up with the second order differential equation for the surface for the current density vector j o z as d^2 j oz / dr^2 will be equal to j ω μ σ j oz of r – 1/ r dj oz of r / dr okay.

Leave it as a small exercise you can of course pull these quantities to the left hand side so you do not need to keep them in the right hand side so pull them in to the left hand side and rearrange the equation okay, so that you get + 1/ r dj oz of r / dr – j ω μ σ j o z of r = 0 this is the very important equation okay of this problem that we have derived and I will show the importance by simply writing the block.

Now here is the problem, what is the solution of this equation? If have taken a course on differential equations you would recognize or at least now that I have told you, you would go back and verify that this is actually a Bessel function Bessel function equation of order 0. What is the Bessel function of general equation? Yu have suppose f of r as a Bessel you know as a function unknown function f of r then the Bessel equation Bessel differential equation of order n okay is given by d^2 f /l dr^2 + 1 / r df / dr + some h^2 h is is the constant – n^2 / r^2 times f = 0 of course in this equations f is a function of r.

Now you look at one by one term they all match f is the same as jo z 1/rdf / dr also matches, here you do not have any term which has some n^2 / r^2 therefore n must be equal to 0 in this case so when you map it to this one you get n = 0 but then you have h^2 as a complex number because h2 is the complex number because h2 is = - j ω μ σ okay.

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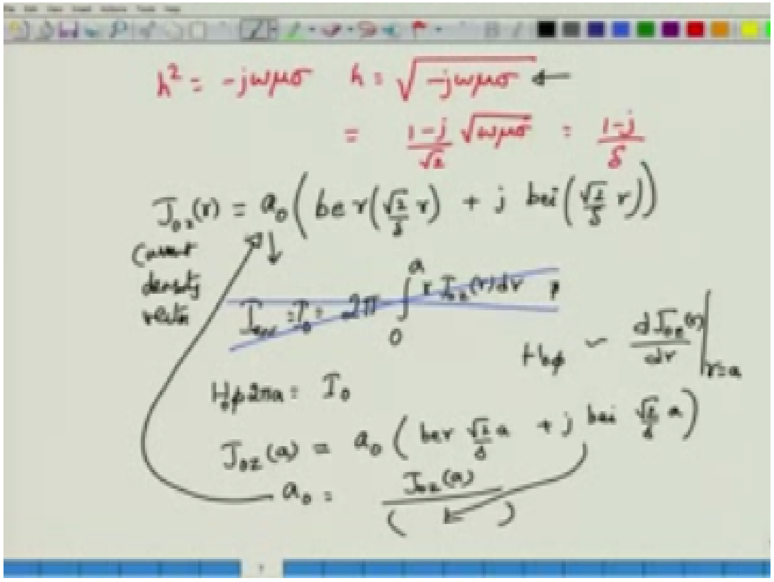

So I have h2 = - j ω μ σ and therefore h will be equal to √ -j ω μ σ which can be written as 1 – j / √2 / or x √ ω μ σ and you already know what is this √ ω μ σ this would be 1 – j / δ okay. well although this we know you have to actually know the solutions for the Bessel function the correct Bessel function solution for this equation which we have written this block or we have emphasize with this block is the Bessel function of order 0 argument being a complex number, the solutions for this equation the red differential equation which is Bessel functions is usually unfortunately for use also denoted by the j Bessel functions of the first kind and of order n and h times r h being the constant.

So unfortunately for you the notation for Bessel function is the same as the notation for the current density in order to you know avoid this problem of what we are talking of the current density or we are talking of the Bessel differential equation we will go back to we will write a different notation we will call this be as the Bessel function. So this Bessel function unfortunately because h is complex, so you can clearly see h is complex the solution must be broken up in to real Bessel function and imaginary Bessel function.

These functions you do not have to know them right now, you just take the solutions as what I am talking to you about it you just take it on faith, but you can use mate lap to plot real Bessel functions and imaginary Bessel functions you know for different values of h as a function of sorry for a different values of r as a function of complex h or a h value of there and when you do that you will be able to visualize how the Bessel functions would look, basically Bessel functions have different kinds so we have a first kind Bessel function which will be finite at r= 0 and you have Bessel function of the second kind which will go off to infinity at r = 0 because clearly I cannot have current density go in to infinity at r = 0 r = 0 is the center of the wire I have to chose a solutions the first you know kind of the Bessel function.

So with all that said the solution that I have for the current density let me rewrite this one this is the current density vector for this current density vector j of o z of r is = some constant a0 which we still need to find out and be you know Bessel real function √ 2 / δ x r + j because the constant h turned out to be complex. So you will have real and imaginary Bessel function so this is the imaginary Bessel function √2 / δ x r okay.

So this is the solution that we are looking for of course I still do not know what is a0 okay, there is one way in can find out ao right I can find out what is the total current enclosed we will call this as i0 enclosed by this round wire so that would be given by 2 π 0 to a r j oz of r dr tht must be equal to the current enclosed. Unfortunately this requires you to integrate Bessel function which is not at all simple okay, so we abandon this approach with seems to cause more competitions for us rather than that we already know the relationship between h φ or h o φ x 2 π a must be equal to the total current enclose which we will call as i0 okay. so i0 is our total current which then must be equal to h o φ evaluated on the edge of the loop or in the edge of the wire time 2 π a the circumference that must be equal to i0.

But what is the relationship between ho φ? This is proportional to dj o z of r / dr evaluate this one at r = a further more j o z of a that is the current density on the surface of the wire can be written as a0 the real Bessel function √ 2 / δ a + j imaginary Bessel function √ 2/ δ a okay, and with this expression I can replace a 0 as jo z of a / this term in the brackets, and consequently go back and substitute for a 0 x this expression obtaining the current density vector j o z as j o z of a.

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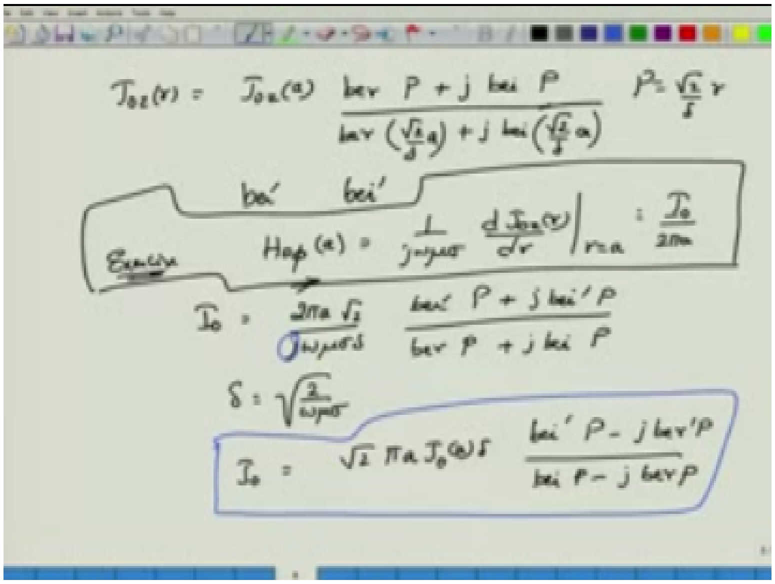

I still do not know what is the current density vector I have just know that this is the constant right, then I have b e r p + jb ei p / b e r √ 2 / δ a + jbei the imaginary fellow √ 2/ δ times a. where p I have defined as √2/ δ times r okay that is just my notation you and I could have defined this denominator as also some quantity q okay.

This is joz you have to differentiate, so when you differentiate you will obtain the differential of Bessel real function and the differential of imaginary Bessel functions okay, so you will have bi bei' or you know look at the relationship h o φ evaluated at a will be equal to one / j ω μ σ dj oz od r / dr at r = a okay. This must be equal to the total current i0 / 2 π a in fact I can write i0 okay after calculating the differential and rewriting all this I can write i0 as 2π a √ 2 / j ω μ σ δ ber' of p + jbe I' of p / berp+ jbei I will leave this as an exercise to you to note or to derive this one.

So you can use this equations that I have written on the top and then substitute that one the constant in the denominator will go away therefore I did not want to write that constant again, so the constant has gone away and what you obtain is this expression which relates the current i0 okay. You can further employ the relationship for δ which is given by √ 2 / ω μ σ in to these expressions and write io as √ 2 π a j 0 of a times δ okay these are some constants time the imaginary component bei' p – j ber' p okay / beip – jberp whys is there a –sin here because there is a j in the denominator in this expression for i0.

And therefore when you go to that j expression bei' will be positive beip in the denominator will be positive 1/j when take it to the numerator that becomes –j so this is the expression for the current that we have obtained okay. So we have this current expression i0, but what is the quantity that we are interested in.

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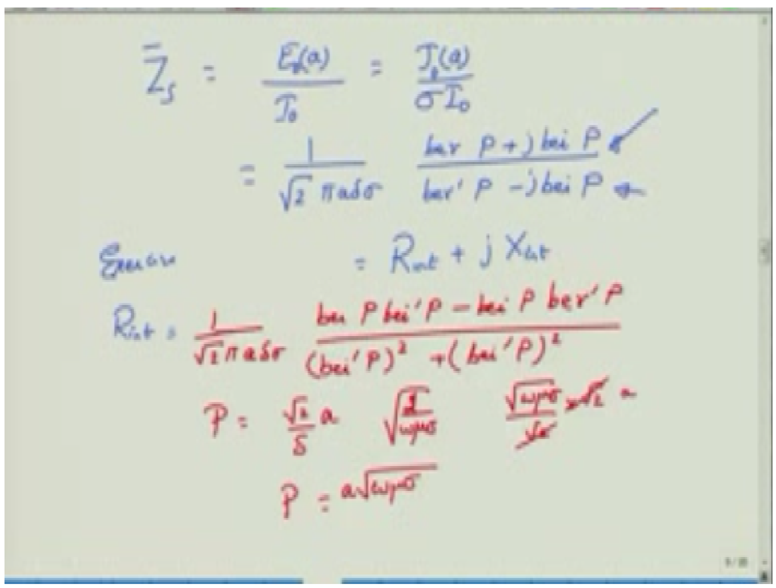

The quantity that we are interested is in the surface impedance right the complex surface impedance is what we are interested and this complex surface impedance is given b y the electric field E0 divided by the total current enclosed, but since the current is related to the current density this could be J0 (a) or j.a divided by σ, 0 correct because current density and electric field are related you carry out this expression or if you just do can you take the ration of these two what you get slightly lengthy expression $1/\sqrt{2}\ \pi a\Delta\ \sigma$, be r,p +jbi p divided by be prime of p - jbei of p okay.

This of course can be written in terms of R_{int} +X_{int} this is a complex number by the way so you will have to multiply the complex conjugate of the denominator and then adjust or rather separate the real and imaginary components, so you do all that I will leave that I can exercise okay.

So I will leave this as an exercise that you can show the resistance the internal resistance is given by $1/\sqrt{2}\pi a\delta\sigma$ times some here P bei' P-bei P ber'P this expression is quite lengthy and they cannot evaluate you know without some special numerical techniques for this but luckily mat lab and other programming languages do include the Bessel function, so you can go and substitute that for different values of t and then show that are plot or you have to give an internal resistance or hint.

In that you remember P P means $\sqrt{2}/\delta*r$ or in this particular case because we are evaluated this one at r = a everything is being evaluated r = a from this $\sqrt{p}/2$ $\sqrt{2}/\delta$ x a right, so δ you know is given by 1/ or other $\sqrt{2}/\ \Omega\ \mu\ \sigma$, so therefore $1/\delta$ will be given as $\sqrt{\Omega\ \mu\ \sigma}/\sqrt{2}$ there is a $\sqrt{2}$ in the numerator and a that $\sqrt{2}$ will be cancelled and P in this expression will be = $\sqrt{\Omega\ \mu\ \sigma}$ times a so you can actually plot this for different values of a / okay or you can try to find out what will happen when $\Omega = 0$.

When you take the limit of the major going to 0 you can find out what happens to the internal resistance as well as for the internal inductance okay.

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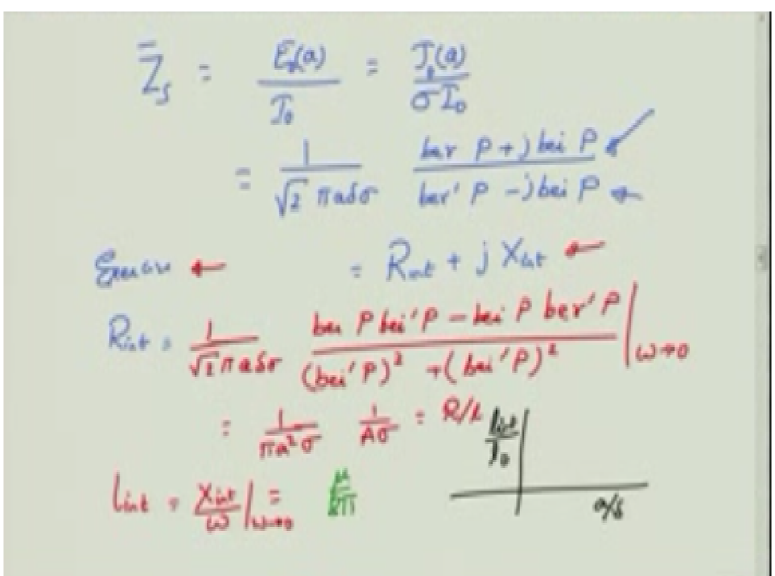

So if you evaluate this R_{int} under the condition that Ω tends to 0 you will actually see that this can be given by $1/\pi a^2\sigma$ and $1/\pi a^2\sigma$ is nothing but 1/area time's σ or this was precisely by resistance per unit length that you obtained in the previous module okay. Similarly L_{int} which is the internal inductance okay, obtained by looking the reactance and dividing that one by the frequency of operation under the condition that Ω goes to 0 by the way you have to find out this X_{int} from the expression here right. So there is an exercise which allows you to find this expressions or X_{int} and then divide that one by Ω and then take the ratio of Ω going to 0 and imply Bessel function through this you can show that this should be equal to $\mu/8\pi$.

I hope you are little bit surprised in this case or maybe you are not surprised because you have already seen this expression this is the expression for the inductance and this is the précised expression for the inductance that we obtained for the you know core of a co axial cable and that is making a comeback over here thus showing you that inductance concept that you normally defined at DC or at the frequency Ω = 0 is not quite the complete picture.

The Inductance is different as you go or it becomes different as the frequency increases, so the value that you calculate from that flux divided by the current strictly speaking applied only for DC case or the static case under the high frequency consideration the difference between these two will be different okay, and of course the differences actually depend on the material or the geometry of the material as well because.

If you plot the inductance okay of high frequency divided that inductance by the dc value of the inductance which I denote by l0 as a function of a / δ where a is the radius and δ of course is the skin depth, so you can show that at very low values of a / δ that is when the area is quite small the δ is quite large right, or really small and δ is quite large so the skin depth is essentially occupying the entire area a okay. You might have the skin depth of 1mm but if I consider the radius of the wire also has about ½ mm then δ is larger than A okay.

So in that case the inductions that you obtain will be something and as a starts to increase this fellows start to decrease okay, however if you go to the straight approximation which is called as a rally approximation okay, under the rally approximation you do not worry about all this you just take this one to be the δ x w kind of a thing we have not discuss we do not want to discuss that one right, now it is little complicated not complicated it will takes little bits of a time.

So this approximation is called as a rally approximation which simply assumes that the entire current flows in the skin depth δ okay, for that approximation will be different from the exact analysis that you have obtained and these two analysis eventually starts to become you know close to each other when a becomes much larger then δ okay. So we have shown you by co axis static analysis of skin depth how to calculate the complex surface impedance of a wire, so next when you operate or when you use a wire or a co axial cable please remember that has a certain impedance which keeps changing as the frequency changes. Thank you very much.

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