**Fiber - Optic Communication Systems and Techniques**
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**Lecture – 19**
**Properties of modes of step-index optical fiber**

Hello, and welcome to NPTEL MOOC on Fiber-Optic Communication Systems and Techniques course. We will continue in this module, what we had started in the previous module that of analysing the steps of a single mode fiber or rather steps of a step index fiber, that is fiber whose refractive index profile was constant inside the core having value n 1 and outside the core that is in the cladding which based on it all the way to infinity on both sides would essentially have the refractive index n 2 which is less than n 1.

And, following our systematic procedure of understanding or deriving the modes of the waveguide or the fiber; we had stopped at a certain point, where we had derive the wave equation for the z component of the electric field.

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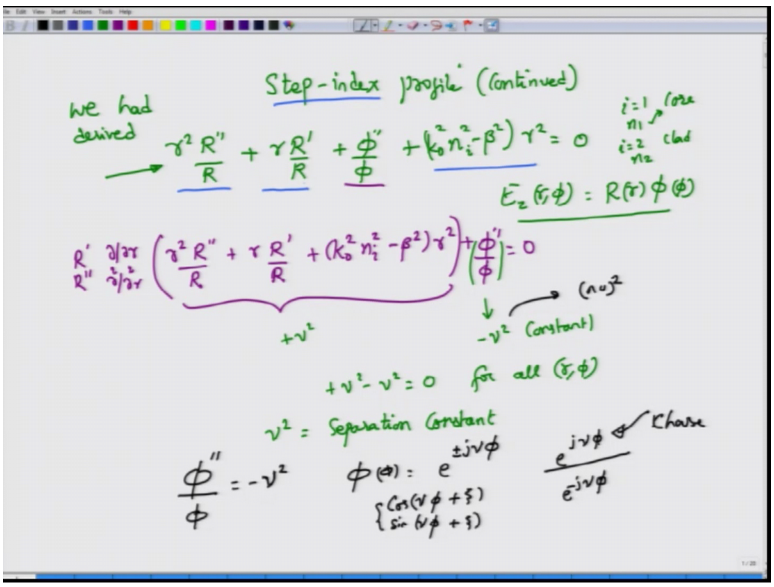


Of course, you could derive a similar equation for the magnetic field as well and that resulting partial differential equation because it would couple the r and phi terms in the circular cylindrical coordinate system that we have chosen, the solution was not so straight forward. So, to solve that a differential equation the partial differential equation we started employing the method called as method of variable separation or sometimes called as variable separable method.

The essence of variable separable method is to actually write down the unknown solution as the product of functions of individual coordinate. For example, since we know that E z depends on r and phi, it is z dependence is already very well known to us. So, we wrote E z of r phi as R of r phi of phi, where R of r is exclusively a function of the small r itself which is the radius and phi being the azimuthal angle the capital phi of phi is just the function only of this phi. And, we substituted this one into the wave equation for E z and at after simplifying that we actually derived this particular equation at which point we had stop because we did not have sufficient time to discuss the solutions.

Now, that is what were going to do in this module in this module we will first understand what this differential equation is what are the solutions of this differential equation and once we have the solutions of this differential equation which is essentially giving us E z because H z also satisfy the equation which is very similar in this form except maybe different amplitude constants. So, the form of the solution would remain the same. So, the solution that we are going to discuss now will be applicable both for E z and H z and you know that once E z and H z have been obtained, the solutions of E z and H z have been obtained then all the other field components can be readily given by the various combinations of E z and its derivatives H z and its derivatives right.

So, go back to this equation that you are seeing now and you have to understand that this equation will have to two such equations; one for core and one for cladding because the refractive index in the core is different from the refractive index in the cladding. Inside the core n will be equal to n 1, outside n will be equal to n 2. If I were to say i equals 1 corresponds to core then n 1 will be the core refractive index, i equal to 2 correspond to the cladding and therefore, n will be equal to n 2 and that is why I have written this k 0 square n i square minus beta square. Beta is of course, the unknown propagation constant which you are trying to find out.

So, now you see this equation, right. I will take another colour pen here, and then underline those terms which are functions only of r. So, clearly the terms which I have underlined here with a blue colour are functions only of the small coordinate r. Of course, I am assuming the beat is a constant it does not depend on any other coordinate and of course, because we have assumed step index profile the value of n is independent r and phi it is just equal to n 1 in the core and n 2 in the clad.

Now, this term which I am underlining with a different colour is a function only of phi, ok. In arriving at this equation of course, I have kind of you know divided everything by R and adjusted the equation such that I looked at I mean I get this kind of a functional dependence. Now, let me do something I am going to rewrite this equation by grouping all the terms which are functions of small r in one side and the functions of phi in the other side and then what we will get.

So, I have r square R double prime by R, please note that R prime and R double prime denote differentiation with respect to r only. So, this is the differential that you are actually looking at. So, plus r R prime by R plus k 0 square n i square minus beta square times r square plus phi double prime by phi equal 0. Now, I have two functions night. So, there are three terms which may be collectively combined into one big brace. So, whatever that term that is here in this braces is a function only of r. So, I can change the value of r I can go from core to cladding I can do whatever that I want to do with the small coordinate r. The function or the terms that have shown in this bracket would represent only functions of that small coordinate r.

Similarly, what I would now indicate with another brace is just a function only of phi you have to understand that r and phi are independent variables. They can change to whatever value of the they can change over whatever the limits that they have. Of course, phi is limited to typically from 0 to 2 phi which covers the one complete revolution around the z axis, whereas, r can go from 0 to infinity at r equal to a u hit the boundary between core and the cladding right.

But, you have a term which is completely function of r you have a term which is completely function of phi and for all values of small r and small phi that sum of these two functions must be equal to 0. The only way the, this can happen is when this is separately equal to some integer and this is separately equal to the opposite of the integer, correct. So, only when this phi double prime phi is equal to some constant I mean and I have denoted that constant by nu square minus nu square or nu square plus nu square in case you are looking at the dependence in terms of only function of r.

So, what we are saying is that the term which is function only of r is equal to a constant and the term which is function only of phi is another constant, but this then the constant with a negative sign as that of the other one. The reason because plus nu square minus nu square the sum of these two will be equal to 0 for all r and phi and this nu square is called as the separation constant, ok. That is the reason why you call this method as variable separable method.

Because, we separate the variables by writing the unknown solution as the product of the functions of individual variables and then when you look at the resulting differential equation you will actually have terms which are functions of only one coordinate, the function for another coordinate the sum of these two will always be equal to 0 of course, that is the original equation condition. And, because that has to be 0, these two terms individually must be equal to constants, and further this sin of these constants must be opposite to each other. So, that is the reason why I have chosen the separation constant to have a minus nu square and plus nu square for the function which I have written as a function of r alone.

Now, there is no reason as to choose this as minus nu square and this as plus nu square you free to interchange them, but it turns out that if you choose in manner that I have indicated then the solutions can be simplified, ok. Remember, the goal is not to keep dragged in the mathematics the goal is to move forward understand how these functions of r and functions of phi would look like because your ultimate interest is to find the modes as well as the propagation constant, ok.

So, all the possible modes of the waveguide plus the propagation constant is what your after. Anyway, now that I have these two equations individually I will separate them out and write something like this I have phi double prime by phi equals minus nu square writing it sometimes my nu does not really look like nu it may look like v please excuse that please remember that this is nu. So, this is nu square.

So, this equation is essentially a second order differential equations so, nice homogenous equation whose solutions as you can verify are given by e to the power plus or minus j nu phi, right. Now, either I can choose e power j nu phi or I can choose e power minus j nu phi, whatever the solution that I choose is perfectly fine, I mean I can let say choose e to the power plus j nu phi for example. So, this is the solution that I will choose.

And, this point that you can actually have solutions which are e to the power plus j nu phi or e to the power minus j nu phi will be exploited later when we discuss something called as linearly polarized modes. There we will see that it is possible to you know use this e to the power j nu phi and e to the power minus j nu phi in a specific manner to combine and actually make the longitudinal components completely disappear. But, that is a story that we will have to wait for another module, ok.

For now I had the option of choosing e power plus or minus j nu phi I chose e power j nu phi as my solution. I could as I told you equally well have chosen e power minus j nu phi. Of course, you also realize that the solution could be cosine nu phi and sine nu phi right or in general with some additional phase zeta these are the generalized solutions, but I will not want to go to the cosine and sine form of the solutions because there are little bit of problems with that one. So, dealing with exponential is much easier. So, I have chosen e to the power j nu phi as the solution.

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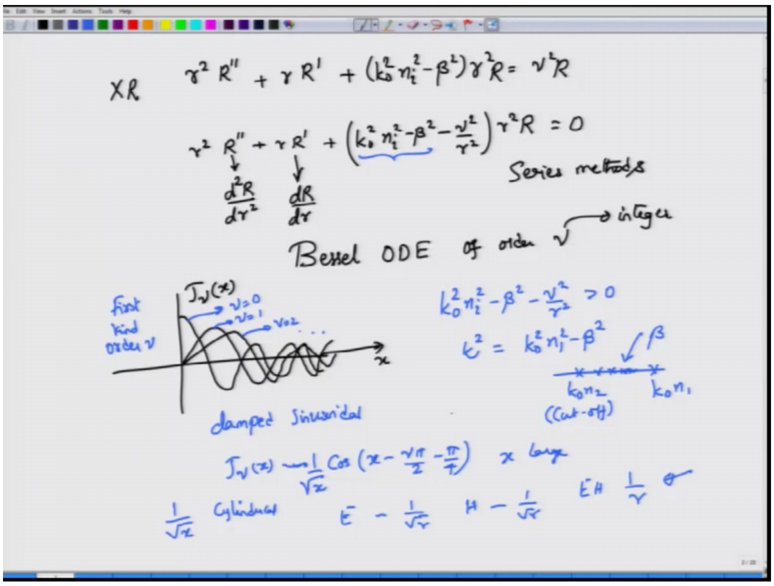


So, that was about phi. What about r? Well, you had this equation, correct plus r R prime by R plus k 0 square n i square minus beta square times r square is now equal to nu square, correct. So, now, let me put this nu square to the left hand side and obtain a slightly different equation, ok. So, I have r square R double prime I also do something I will multiply the entire equation by R, so that this R which is in the denominator can be removed and moved to the numerator here. So, I can write this as R and write this as R, ok. So, please ensure that you understand the manipulation that I am doing.

So, now, I have r square R double prime plus r R prime plus k 0 square n i square minus beta square minus nu square by r square times r square capital R equals 0 and R double prime is basically d square R by d r square and R prime is d capital R by d r. So, this is the equation that I now have which we need to now solve, right.

So, the equation that we now have has to be solved. It is a second order differential equation alright, but this is a very special equation which has to be solved by what is called a series methods, something that you must have studied in your engineering mathematics courses. So, I will not go into the details of this differential equation, but suffice to tell you that this differential equation is an example of Bessel ODE; ODE stands for Ordinary Differential Equation. This is a Bessel ODE of order nu and because we want solutions which are actually propagating inside the waveguide on inside the fiber we take this nu to be an integer.

Now, there are different types of Bessel functions the ones that are actually useful for us will be called as Bessel functions of the first kind and Bessel functions of the second kind of this particular thing. So, these are the different Bessel functions you must have seen them. So, if this is some j nu of x, this is x, ok; x is the independent variable. The function here will be nu equal to 0, the function here is nu equal to one the function here is nu equal to 2 and so on. So, these actually remind you of damped sinusoidal waves indeed for large values of x you can approximate this j nu of x as cosine x minus nu phi by 2 minus phi by 4 divided by 1 by root x, ok. So, this can be written for very large values which means far away from the origin.

So, the reason where we call is the this damped sinusoidal is because the exhibit the oscillatory solutions and they have these oscillatory solutions are important because that would correspond to standing waves inside the fiber which is precisely what we start of hoping that we actually get to this type of a solution, because you want standing waves inside the fiber and you want the evanescent or the radiating modes outside the fiber core.

So, this situation is very similar to the kind the slab waveguide situation. In the slab waveguide also you had the standing waves because you know this waves were actually getting reflected and therefore, creating an interference pattern inside the waveguide, but the interference pattern created in the fiber is slightly more complex than the simple slab waveguide that we have discussed and that standing waves are actually in the form of, they are not in the form of cosine or sine waves there in the form of Bessel functions which look like damped sinusoids, right.

So, x goes to a larger and larger value the oscillations look more like a cosine wave with some phase shifts give or take. But, the amplitude drops because there is a 1 by root x term out there. This 1 by root x is a very peculiar characteristic or it is not a peculiar, it is a characteristic of assuming cylindrical waveguides because when electric field goes as 1 by square root r and the magnetic field goes as 1 by square root r together the product E H which would be your pointing vector I am not writing the vector I am just giving you the basic idea will go as 1 by r, right. So, this is something that we have the characteristic of a cylindrical wave guide system, ok.

So, this is one type of Bessel function and this Bessel function will be the solution provided k 0 square n i square minus beta square is greater than 0, ok. We want this particular term when k 0 square I need to also include argument with nu square, right. So, minus nu square by r square is positive. So, we will have this kind of a solution which is called as Bessel first or I mean first kind Bessel functions of order nu is the solution provided this k 0 square n i square minus beta square minus nu square by r is positive quantity.

Now, let us introduce two symbols the symbol that you know earlier which we called as kappa f which was the transverse wave number this time I will call kappa square as k 0 square n 1 square minus beta square. Why? Because beta is bounded by two values; it can utmost reach k 0 n 2 at which point the corresponding mode actually reaches the cut off, and it can go at most up to k 0 n 1 in which case the transverse wave number becomes 0 and beta value will be anywhere in between, ok.

So, these allowed values of beta are discrete as we have seen earlier and the transverse wave number can go to a minimum to a maximum minimum of 0 and. So, minimum of 0 at when beta equals k 0 n 1 and a maximum of k 0 square n 1 square minus n 2 under root when beta reaches k 0 n 2 and gets cut off, ok. This we have seen earlier in the slab waveguide exactly same procedure I am using the same notation as well in fiber optic literature it is more common to call this kappa square as u square. But, I have avoided that because I want you to constantly remember the significance of this kappa.

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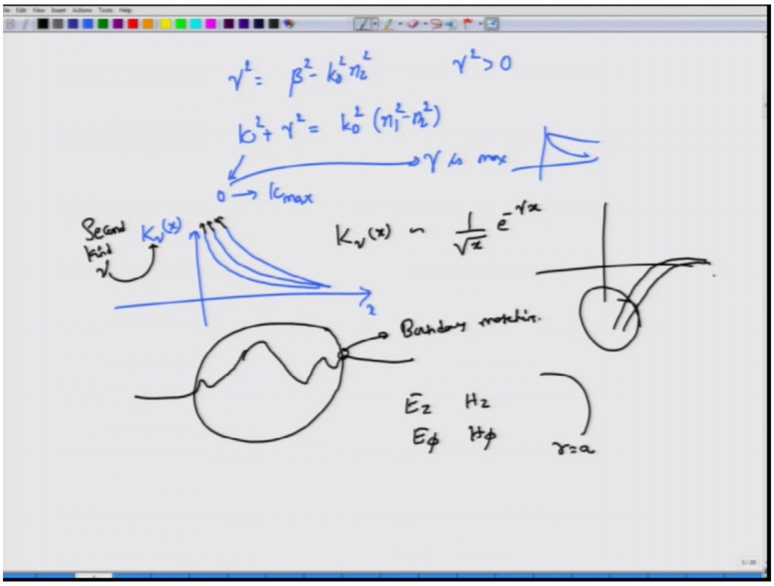


And, of course, since kappa f has made it entrance the next one that to define would be gamma square which is beta square minus k 0 n 2 square with the stipulation that gamma square be a positive quantity, which means that beta is greater than or at most equal to k 0 square n 2 square. So, this gamma is the decay constant as you have seen you know and it would be something that would describe the evanescent field moving away from the core, ok. So, the so you have the core the fields are guided by here, but the field component that is in the cladding would essentially go and radiate away, and there is another relationship that you must know which is kappa square plus gamma square is given by k 0 square n 1 minus n 2 square, ok.

This is very important because kappa can go from 0 to kappa max, ok. When kappa is equal to 0 gamma is at its maximum value the sum of this two will always be constant assuming n 1 and n 2 are constant. So, when kappa is 0 gamma is maximum, ok. So, at this point gamma is maximum and the field essentially decays rapidly outside the cladding, when gamma is not maximum it is lower than the field decay will be much slower, but when kappa is you know at it is value of 0 the field is decaying outside rapidly, but as kappa rises the field outside will be decaying much less rapidly. So, this is something that you have to keep in mind.

Now, this was one Bessel function that we talked about. The other Bessel function that will be the solutions outside the cladding which you want anyway is because you are looking at you know evanescent waves are called as the Bessel functions of the second kind. These are the second kind and order nu, again the order is nu and I have list k to denote the second kind some authors use y, some authors use n.

So, these are all different notations. So, please do not worry about those, what is important to note is the shape of this orders. right. They are all exponentials. But, they are all exponential which are actually going towards infinity and at x equal to 0 and then drop off exponentially. Anyway, this is fine for us because if you imagine that the solution outside or inside the core is kind of an oscillatory solution. At this point you need to have an exponential decaying solution which is provided by fitting one of these orders, right.

So, if this is the my optical fiber core, then the solution would look like this and outside it has to decay out, right and these points at which we are going to merge the two solutions one from the core which is oscillatory and the other one in the cladding which is decaying are called as boundary matching, and we will have to do lot of boundary matching here. So, what sort of boundary matching that we need to do? We need to do boundary matching of E z component, H z component, E phi component and H phi component. So, you have match this four components because all these four components are tangential component at the boundary r equal to a.

So, these solutions k nu of x also have a asymptotic form, that is for large values of x they go something like 1 by square root of x therefore, at x equal to 0 clearly this is going off to infinity and then the decay with some constant gamma times x. This gamma need not be the same gamma, but it could be the same gamma in case you are applying these equations to fiber. So, you have these two type of Bessel functions; just for record there are other categories of Bessel function, some of them go like this. These are clearly not allowed solutions because they are not going out alright. So, we have only Bessel function of first kind of order nu second kind of order nu as the solutions for E z.

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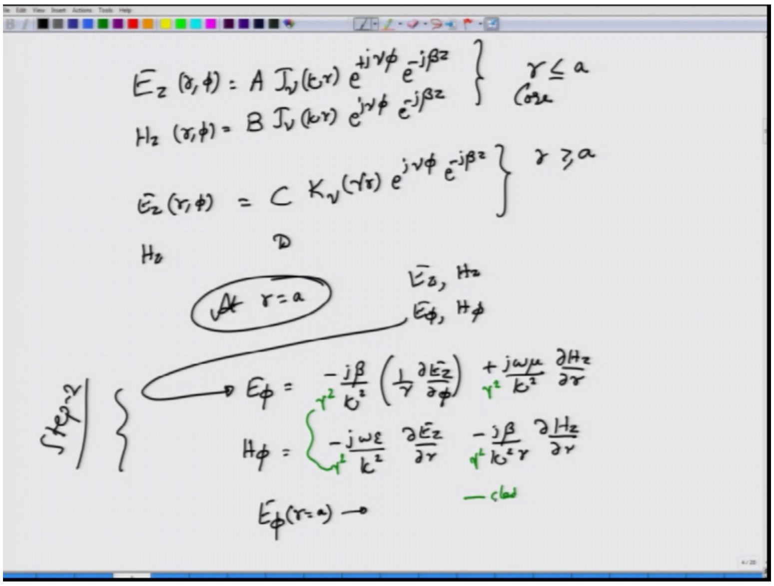

So, now I can write down the solution for E z in the core as a function of r and phi as j nu some constant amplitude A; so, A j nu kappa r e to the power minus or rather plus j nu phi, ok. Of course, if you want you can also put in e power minus j beta z into this one it is up to you and then the solution for H z because E z and H z satisfy the wave equation of the same nature the solution will be with the constant b here same j nu of kappa r e power j nu phi e power minus j beta z and then you have solutions.

So, these are the solutions for r less than or equal to a, meaning this are the solutions for core region of the fiber. Whereas the solution for the cladding region will be again some constant C and instead of j nu which is oscillatory K nu which is exponential. So, K nu of gamma r gamma r e power j nu phi e power minus j beta z. For H z the solution will be the same equation accept that the constant instead of C it will be equal to D. So, this is valid for r greater than or equal to a at r equal to a you need to match E z I know H z, E phi and H phi to get this expression E phi and H phi you should refer to step 2. For example, E phi is given by minus j beta by kappa square and then I have 1 by r del E z by del phi then I have minus j or rather plus j omega mu divided by kappa square del H z by del r, ok.

So, clearly if you know both E z and H z you can use the equation for E phi similarly you can obtain an equation for H phi these are all coming from step 2. So, I am hoping that you have done that exercise. So, that you would you are not seeing this equation for the first time you must have been seeing these equations earlier as well because of the exercise that you solved. So, minus j omega epsilon by kappa square del E z by del r and then you have minus j beta by kappa square r del H z by del r. Notice that you are going to differentiate with respect to phi and differentiate with respect to r when you differentiate it with respect to phi the assume solution with respect of phi you will pull out plus j nu, and when you differentiate with respect to r your differentiating Bessel function of the first kind, of order nu and Bessel function of second kind of order nu and these derivatives are not so simple. In fact, there are set of relations called as recurrence relations which are valid for Bessel functions which allow you to replaced its derivatives in terms of Bessel functions or superposition of Bessel functions of different orders.

So, I would actually urge you to have a brief look at what this Bessel functions are go back to your engineering mathematics textbook and then look at the recurrence relationships, look at how Bessel functions are actually solved or Bessel differential equations solved to really understand what we have done. Alternatively you can go to one of your software packages in MATLAB and print out a few values of this Bessel functions or few plots of this Bessel function for different orders and generate pictures from your side and then look at what happens to these pictures as you change some of the orders or some of the properties or the variable range.

So, I hope that you done all that or you will do all that and once you have done that and also derived or made appropriate derivations of del E z del phi del H z by del r you will be able to find expression for E phi and H phi and these expressions E phi at r equal to a must be continuous in the core and cladding. Of course, you may I ask what about the equations are these equations valid for both core and cladding? Not really, this is for core and for cladding you replace kappa square by gamma square, ok. So, for cladding you replace wherever you get kappa by gamma square ok.

So, please note that this gamma square or not part of this expressions I have, that is the reason why I am writing them in or different colour, ok. So, this gamma square I have to be written in place of kappa square. So, when you write the fields in cladding so, this is the field for cladding and the black expressions or expressions written in the black colour are the expressions for fields in the core.

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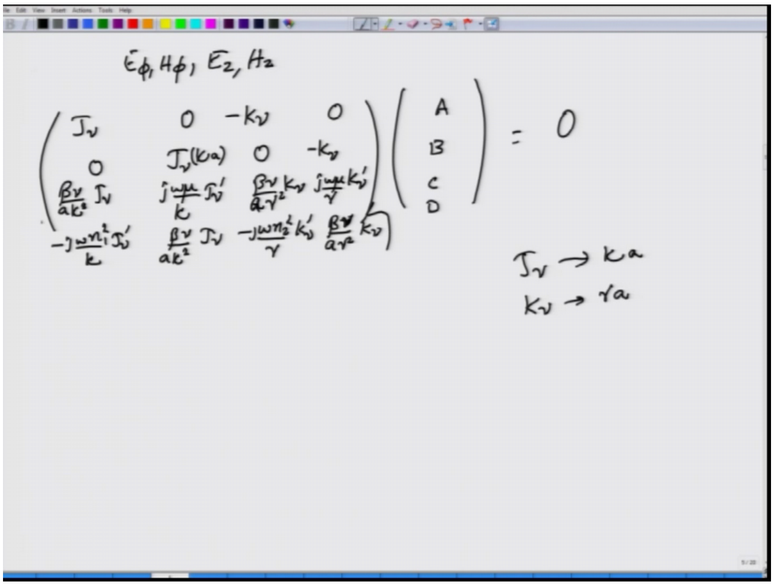

Now, this is an exercise that I am going to leave for you that is deriving the value of E phi or rather expressions for E phi H phi and then using the expression E z H z write down the boundary condition. You will actually get four equations. There will be four unknowns in your equation A, B, C and D. And, you can put all the equations in terms of a matrix, ok. I will write down the matrix it is important, but I would not derive the matrix which I will leave it as an exercise for you. You are going to use this matrix in the next you know module to actually discuss the solutions.

So, you can see that this mattresses are of course, with an argument kappa a because that is the point where you are matching the fields what r equal to a is very o match the fields and these are the expression that you are that you will get. So, these are not easy expressions in the sense that they are tedious to write down and then solve, but once in awhile if you do it it would actually be better.

So, you have third row given by this one. So, note the appearance of j nu prime let me also simplify this equations by removing the argument of kappa a from everywhere, ok. It is understood that the argument is either kappa a or gamma a if it is associated with j nu or j nu functions that is j nu functions would always be associated with an argument of kappa a, k nu would be associated with an argument of gamma a.

So, with that is the final equation or this one you are going to get j mu by k nu prime. The last row is this is beta nu. This equation, please verify that you actually written them down correctly. In the next model we are going to discuss the solution of these equations and then discuss the properties of modes of the step index optical fibers.

Thank you very much.