Basic Electric Circuits
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Module 11 - State Variable Analysis
Lecture 52 - State Equations

Namaskar, so in last session we were discussing about the Mason's rule. So, we will continue our discussion from that point onwards. We will also see the state equation in the today's session.

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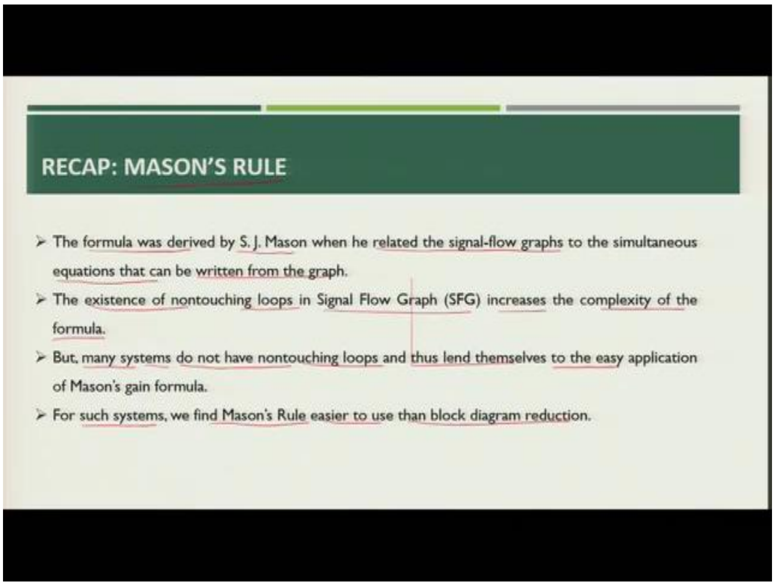

So, let us discuss first the Mason's rule which we were discussing in the last session. So, what we discussed about the Mason's rule that this formula was derived by S J Mason when he related the signal flow graph to the simultaneous equations that can be written from the graph. The existence of non-touching loops in signal flow graph increases the complexity of this formula which was given by S J Mason. But, what happens that the generally in most of the system which you see in real time do not have non-touching loops.

So, in this way the application of Mason's rule is easier for those kinds of systems. And for those systems we find that the Mason's rule is very easy to use than the block diagram reduction.

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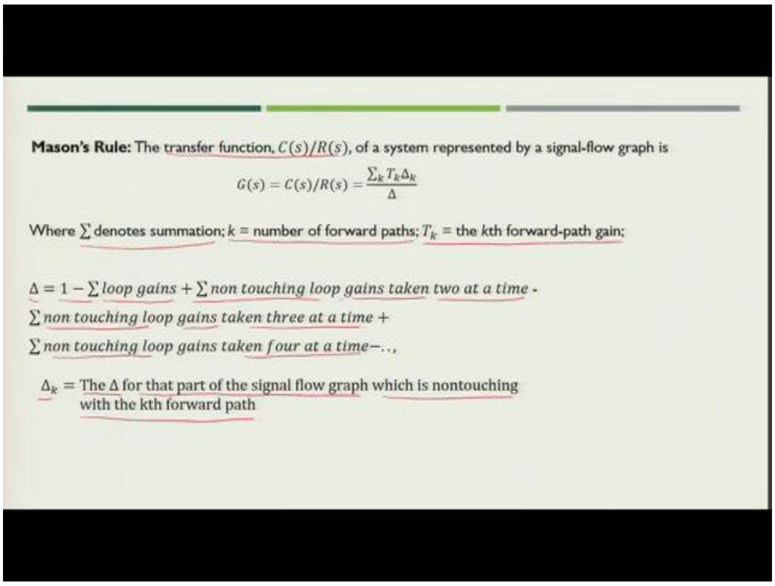

So, let us see what was the Mason's rule? If $C(s)$ is the output and $R(s)$ is the input Laplace transforms, the transfer function of the system can be represented as $C(s)/R(s)$ which can be evaluated with the help of Mason's rule as,

$$G(s) = C(s)/R(s) = \frac{\sum_k T_k \Delta_k}{\Delta}$$

Where $\sum$ denotes summation; k = number of forward paths; T_k = the kth forward-path gain;

$\Delta = 1 - \sum loop\ gains + \sum non\ touching\ loop\ gains\ taken\ two\ at\ a\ time$ -
$\sum non\ touching\ loop\ gains\ taken\ three\ at\ a\ time +$
$\sum non\ touching\ loop\ gains\ taken\ four\ at\ a\ time - ..,$

Δ_k
$=$ The Δ for that part of the signal flow graph which is nontouching with the kth forward path

So, this is the rule which was given by the Mason's. Let us try to understand how we will apply this particular rule with the help of one example.

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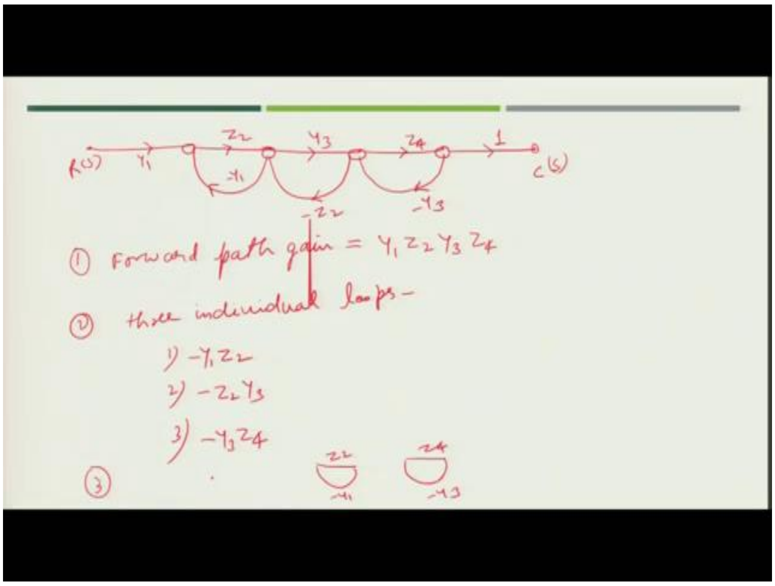

So, let us take one example say if the signal flow graph is like this where you have from here you have a input say Rs. And, then you have the output say Cs and the transfer functions are like Y1 for this set, then Z2 you have a loop which is say minus Y1 then Y3 you have another loop which is minus of Z2 and then this is minus of Y3, then Z4 and this is a unity gain the transfer function. So, this is the signal flow graph if you get the first thing which you have to find out is forward path gain.

So, if you see in this particular signal flow graph you will have only one forward path there is no any other forward path available. So, your forward path gain is equal to take Y1 then Z2 Y3 Z4 so this will be the forward path gain for the signal flow graph given. Now, next is we need to find out how many individual loops we have. So, here we will have three individual loop, what are those individual loops? One is this one, so first one will be minus of Y1 Z2, second would be minus of Z2 Y3 and the third one will be minus of Y3 Z4.

So, these will be the three individual loops which you will find. Now, next is you need to find out the pairs of non-touching loop. So, if you see in this particular signal flow graph and if you only draw the loops. So, there will be three loops which you will see, so this is Z2 minus Y1, Y3 minus Z2 and Z4 minus Y3. So, out of that if you remove the middle one you will get two sets of loops which are not touching to each other. So, in that case the value of non-touching loops will be equal to the value of the transfer functions that is let us write in the next slide.

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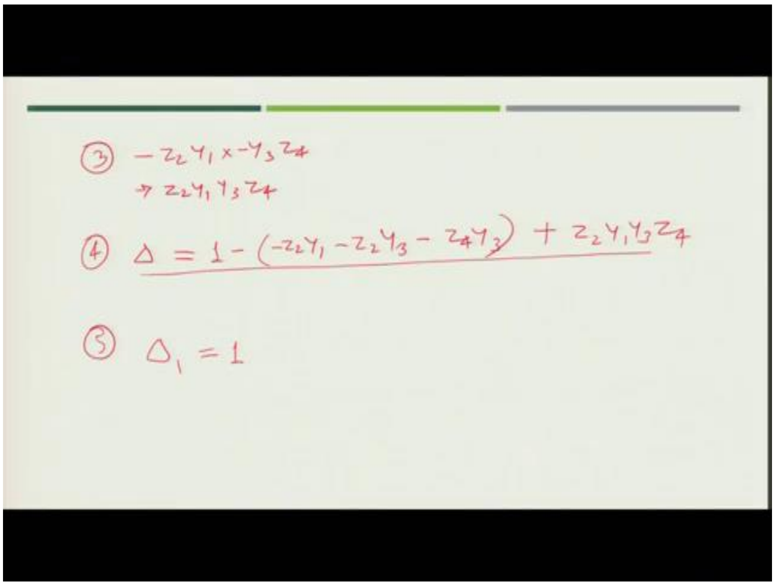

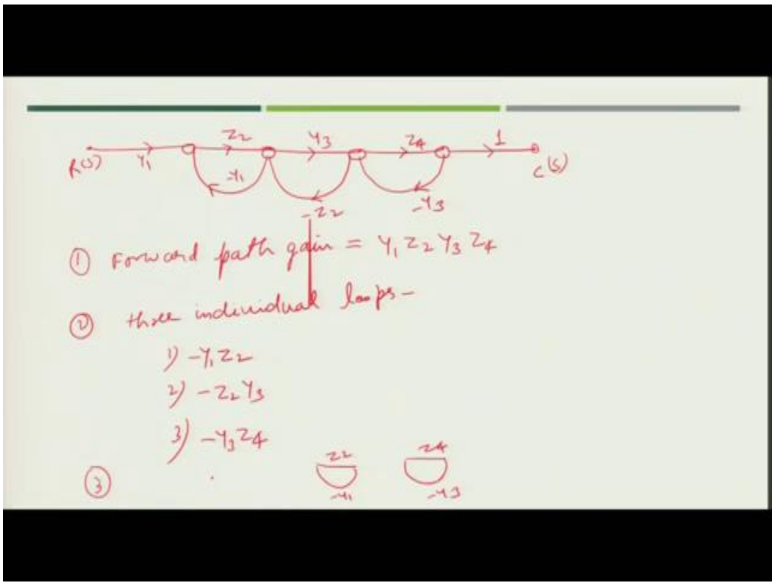

So, the total gain of non-touching loop would be minus Z2 Y1 into minus of Y3 Z4 so minus minus will become plus and you will get Z2 Y1 multiplied by Y3 Z4. So, this what you will get when you have two non-touching loops. Now, next we need to find out the other non-touching loops. So, if you see this signal flow graph you will not be, you will see that there is no any non-touching loop, which means there is no, not more than three non-touching loops you can see in this particular signal flow graph.

So, set of three or four non-touching loops are not available in this signal flow graph. So, finally the delta is which you can write is 1 minus summation of the individual loop gains. So, individual loop gains you have got like we discussed. So, what we can write? We can write minus of Z2 Y1 minus of Z2 Y3 minus of Z4 Y3 which you will get from this particular individual loops. And, then you will have set of two non-touching loops. So, this will become Z2 Y1 into Y3 Z4.

So, this is what you will get as part of your delta. Next is that we have to see how many loops which you can find which are not touching the forward path. So, if you see the particular forward path all three loops which you can see in the signal flow graph are touching the forward path. So, in that case what you can write? You can write delta 1 because we have only one forward path equal to 1. Because, we do not have any loop which is not touching the forward path. So, now what you can do?

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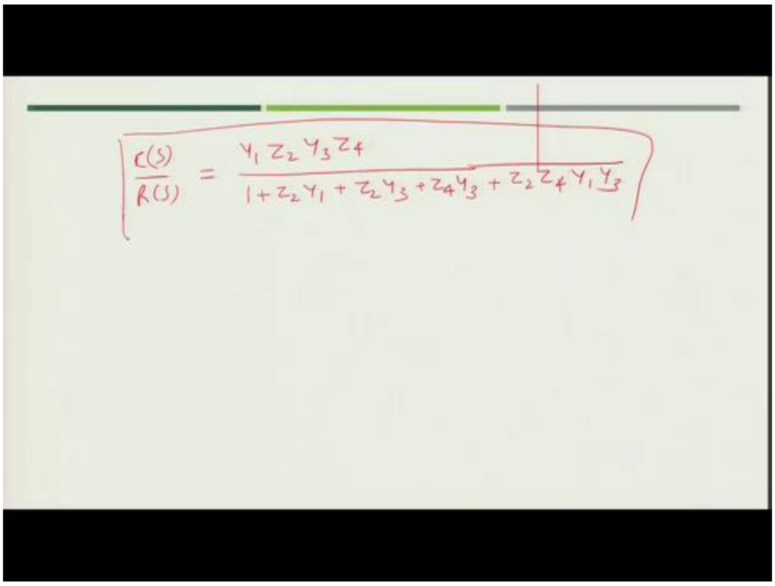

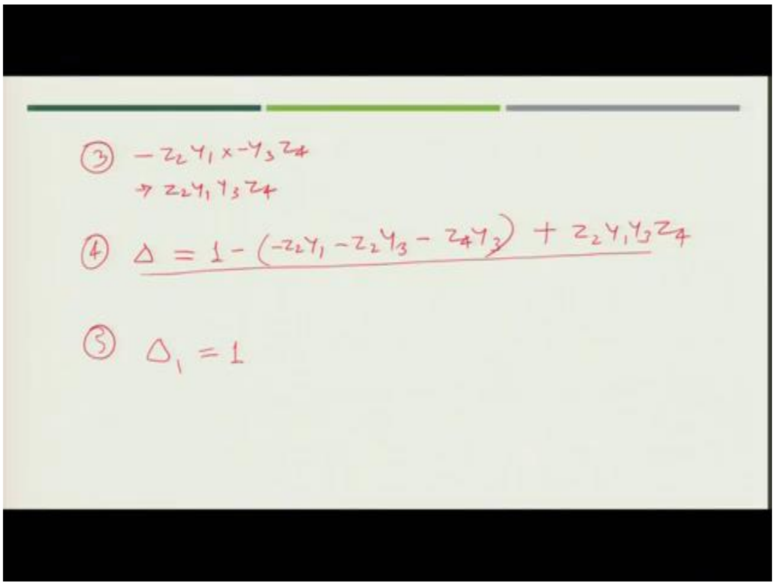

You now compile the value that is the transfer function of the complete system that is Cs upon Rs. What you can write? Now you can write Tk into delta k, so what is the forward path gain you have Y1 Z2 into Y3 Z4. Delta k is 1 so anyway that is you can write multiply equal to 1 or it will remain same. In the denominator you will write the value of delta. So, what is the value of delta? Delta will be 1 plus Z2 Y1 plus Z2 Y3 plus Z4 Y3 plus Z2 Z4 Y1 Y3.

So, this is the transfer function which you will get from the signal flow graph when you apply the Mason's rule. So, I hope you can now do the calculations related to finding out the transfer function for other signal flow graphs.

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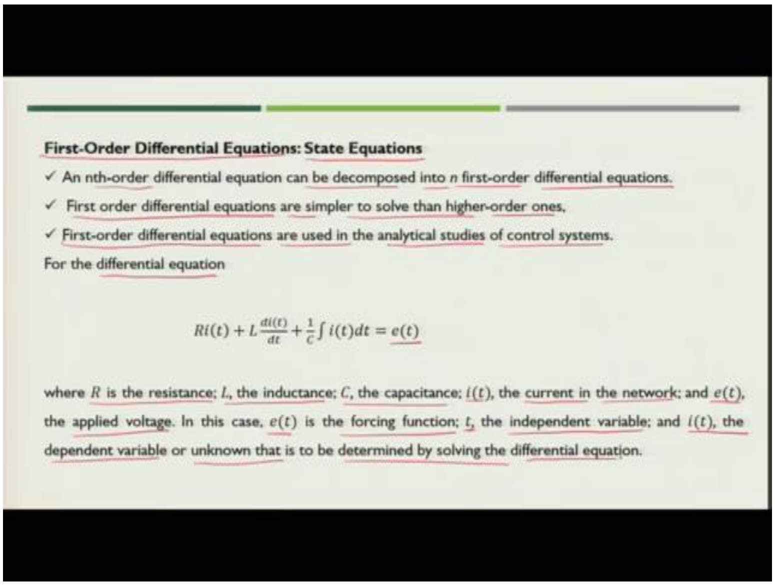

Now, let us talk about first order differential equation and we also call it as a state equation. Why? Let us understand that concept. So, an nth order differential equation can be decomposed into n first order differential equations. So, the first order differential equations are simpler to solve than the higher order ones. First order differential equations are used in analytical studies of the control system.

So, now let us take one sample differential equation say this is very common equation which you might have seen. When we discussed the series RLC circuit and having the external voltage source connected. So, we can $Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = e(t)$. R is the resistance, L is the inductance, C is capacitance, $i(t)$ is the current in the network, et is the applied voltage.

So, in this case $e(t)$ is the forcing function and what is t? t is the independent variable. And we will consider it that is current as a dependent variable or unknown which we need to determine by solving this differential equation.

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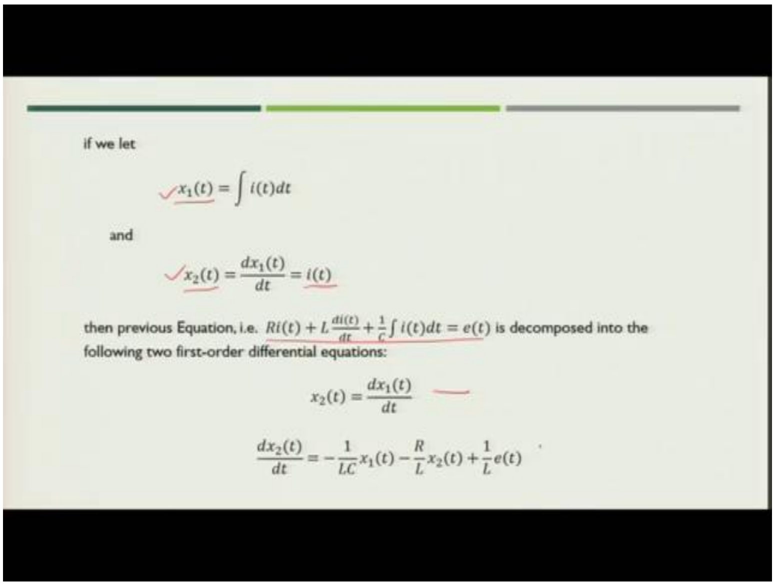

Now, let us assume

$$x_1(t) = \int i(t)dt$$

and

$$x_2(t) = \frac{dx_1(t)}{dt} = i(t)$$

then previous Equation, i.e. $Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = e(t)$ is decomposed into the following two first-order differential equations:

$$x_2(t) = \frac{dx_1(t)}{dt}$$

$$\frac{dx_2(t)}{dt} = -\frac{1}{LC}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{L}e(t)$$

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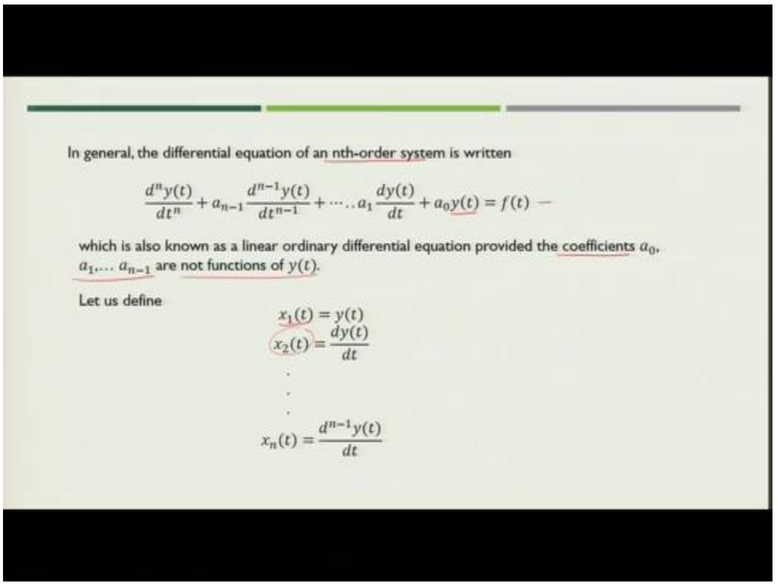

So, in general you can write this the concept which we just discussed. You can extend it for nth order system so what we can write,

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots .. a_1\frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

So, what we will do? We will assume that the a_0, a_1,… a_{n-1} are not functions of $y(t)$ means they are the constants coefficient. So, what we can do? We can replace,

$$x_1(t) = y(t)$$

$$x_2(t) = \frac{dy(t)}{dt}$$

.

$$x_n(t) = \frac{d^{n-1}y(t)}{dt}$$

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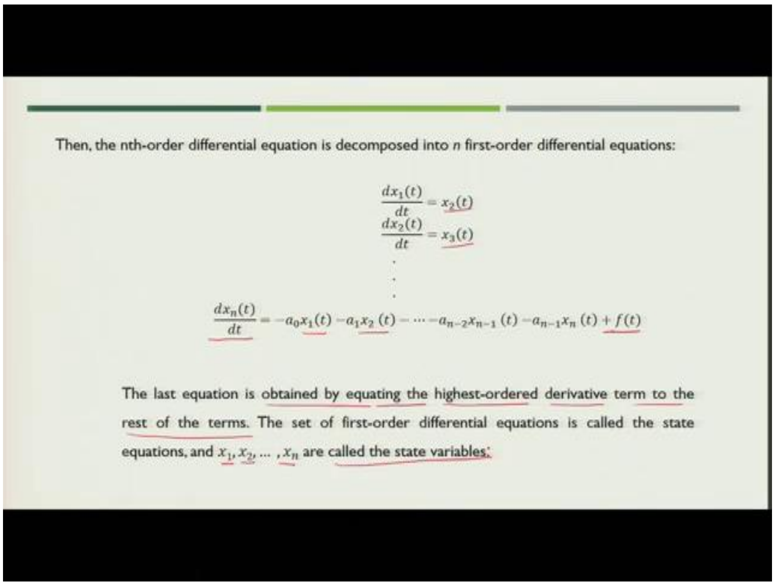

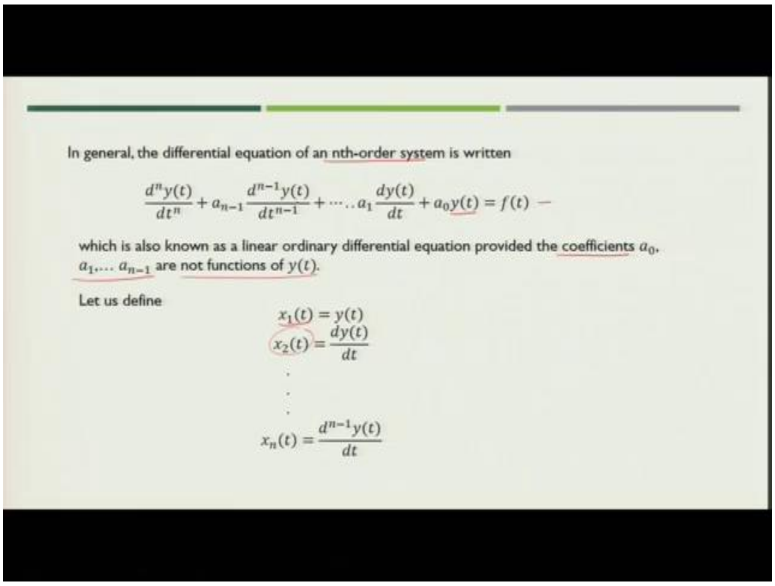

So, when you represent the individual component as a variable. You can simply write with the help of this equation that,

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = x_3(t)$$

.

$$\frac{dx_n(t)}{dt} = -a_0x_1(t) - a_1x_2(t) - \cdots - a_{n-2}x_{n-1}(t) - a_{n-1}x_n(t) + f(t)$$

We have put them into this particular equation and when we simplify we can write the $\frac{dx_n(t)}{dt}$ as the summation of other components. So, the last equation which we obtain is received by equating the highest order derivatives terms to the rest of the term. So, the set of the first order equation which we get in that the variables which you have define like x1, x2 to xn we call them as state variable.

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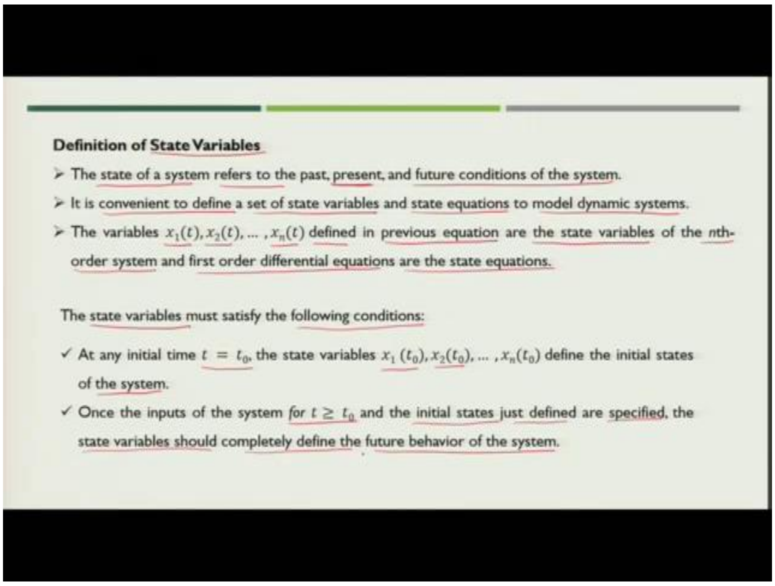

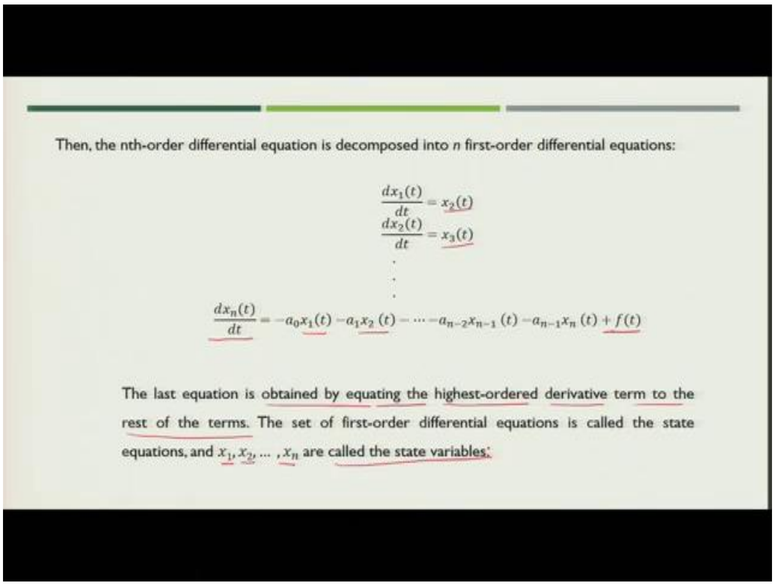

Why we call them state variable? Because, first let us understand what is the state, state of a system refers to the past and present and future conditions of the system. Now, it is convenient to define a set of state variable and set equations to model the dynamic system. Here if you see the variable which you have defined like x1, x2, xn are the basically the variable. Which define the previous equation are the state variables of the nth order system. And the first order differential equations are the state equation.

Now, what is the definition of the state variable? So state variable must satisfy the following conditions. So, at any initial time that $t = t0$, the state variables $x1\ (t0),\ x2(t0),\ \ldots\ ,\ xn(t0)$ define the initial states of the system. Once, the input of the system for time t greater than 0 and the initial states just defined are specified the state variables should completely define the future behavior of the system.

So, the variable which you have seen x2, x3 and so on are somehow defining the derivative of the previous variable so with this you can specify the system performance that is present or future condition of the system. So that is why these are called as a state variables.

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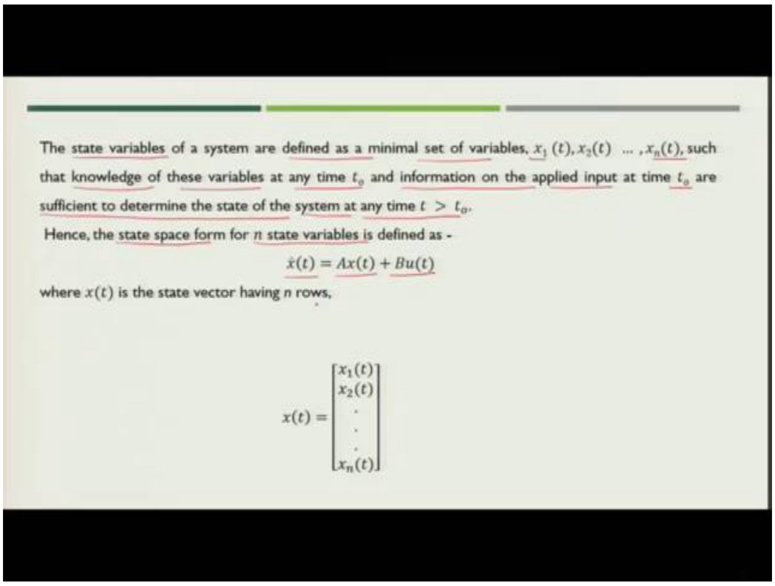

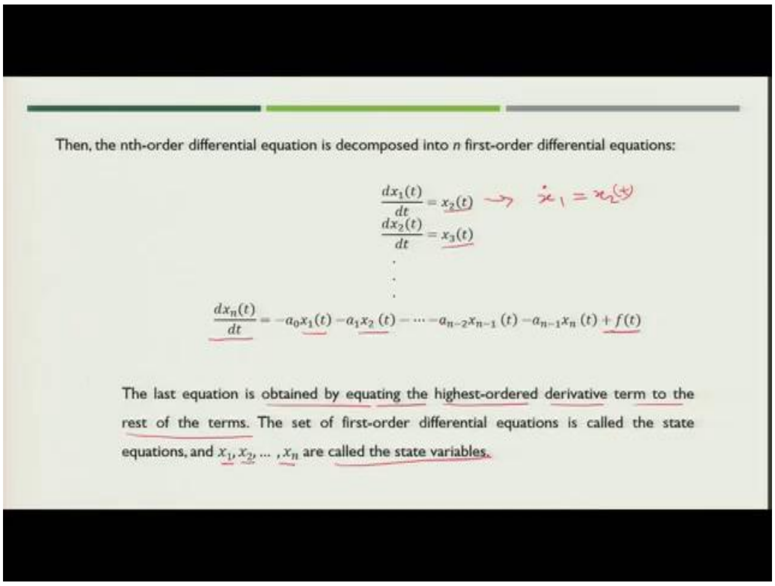

Now, state variable of a system are defined as a minimal set of variable that is x1 to xn in such a way that the knowledge of these variable at any time say t_0 and information on the applied input at that time t_0 are sufficient to determine the state of the system at any time $t > t_o$. So, if you have knowledge for related to this state variable at time t is equal to 0 you can specify the system state for any time t greater than 0 with the help of these state variables. So in state space form this n state variable you can define it as

$$\dot{x}(t) = Ax(t) + Bu(t)$$

When you compile all the equations which you just saw in this slide then you can represent them in matrix form as, $\dot{x}(t) = Ax(t) + Bu(t)$

where $x(t)$ is the state vector.

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}$$

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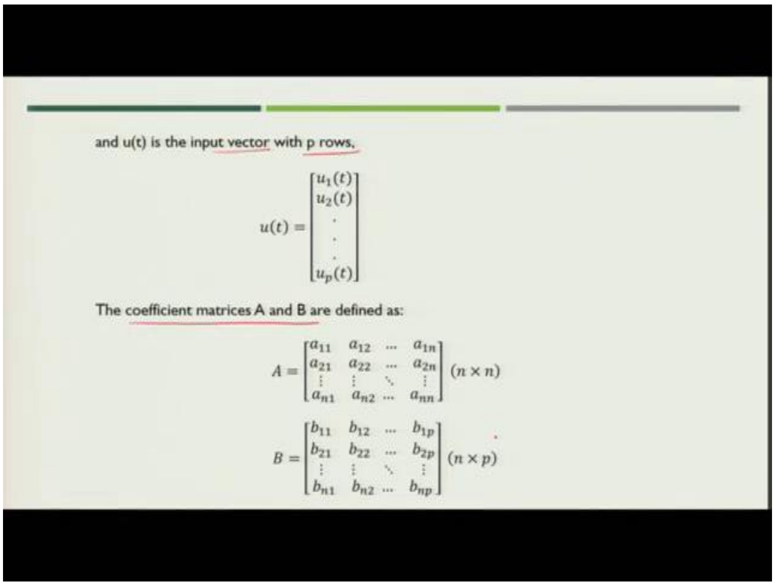

$u(t)$ is the input vector, say, if there are n input variables then the input vector will be,

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ . \\ . \\ . \\ u_p(t) \end{bmatrix}$$

Now, what are the coefficient? So coefficient matrix A and B you can define as,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} (n \times n)$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} (n \times p)$$

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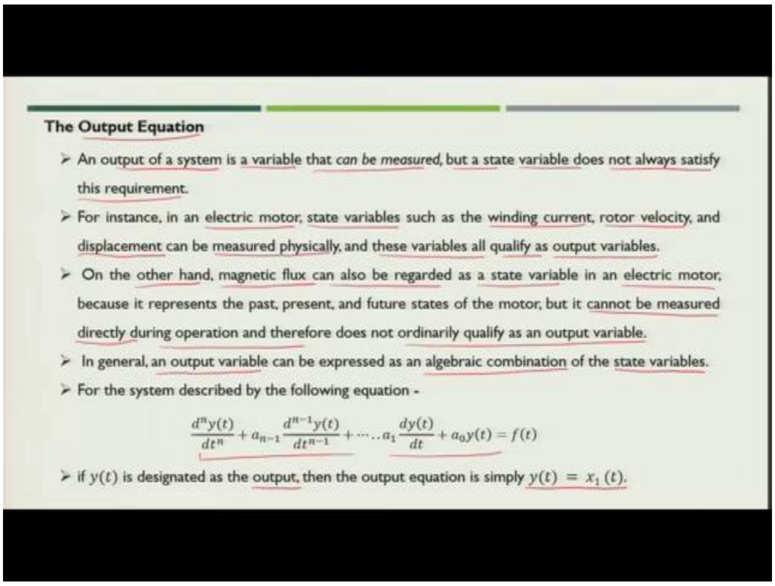

Now, let us talk about the output equation. An output of a system is a variable that can be measured. But, a state variable does not always satisfy this requirement. Let us understand what does it mean? Say for example in case of electric motor, state variables can be like winding current or rotor velocity and displacement. So, these are measurable quantity so that is way these variables can be qualified as an output variable.

While, on the other hand if magnetic flux is also considered as a state variable in case of this electric motor. You cannot take the magnetic flux as a output variable because you cannot measure the magnetic flux directly using any operation. So, that is why the magnetic flux does not qualify to be an output variable.

Now, in general, you will write output variable as algebraic combination of this state variable. So, suppose if the system we just now described as,

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots .. a_1\frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

If $y(t)$ is designated as the output, then the output equation is simply $y(t) \ = \ x1\ (t)$.

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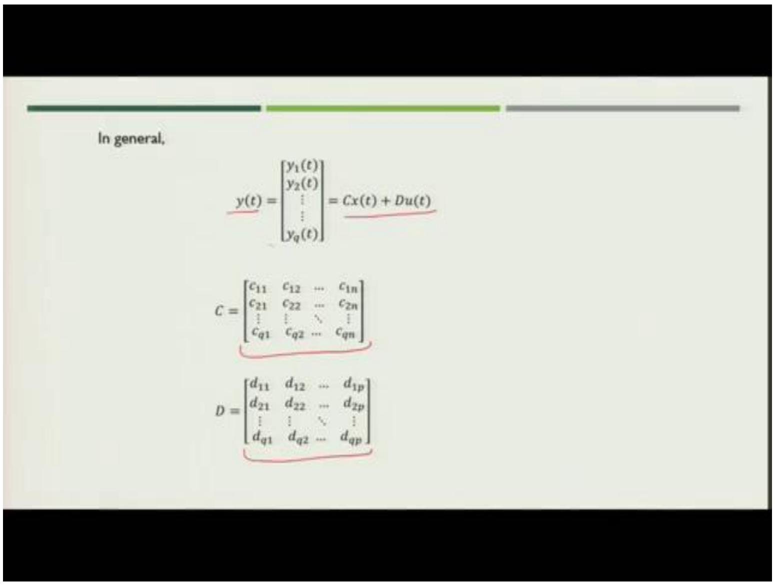

So, you can write,

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ \vdots \\ y_q(t) \end{bmatrix} = Cx(t) + Du(t)$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{q1} & c_{q2} & \cdots & c_{qn} \end{bmatrix}$$

$$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1p} \\ d_{21} & d_{22} & \cdots & d_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ d_{q1} & d_{q2} & \cdots & d_{qp} \end{bmatrix}$$

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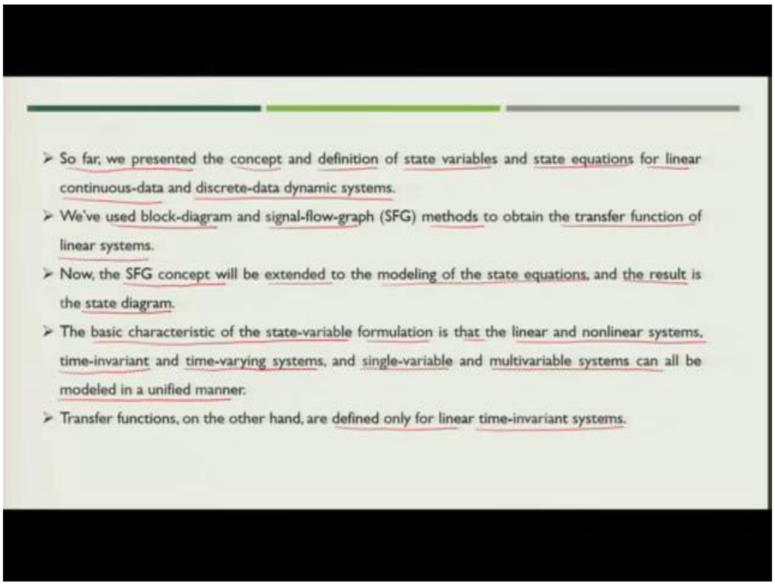

So, what we have done so far? We have presented the concept and the definition of state variables and state equations for linear continuous data and discrete dynamic systems. We have also used block diagram and signal flow graph methods to obtain the transfer function of linear systems. Now, in the next session what we will do, we will use the signal flow graph concept to extend in the modelling of the state equation and the results which we will use in the state diagrams.

The basic characteristics of a state variable formulation is that the linear and nonlinear systems, time invariant and time varying system. And single variable and multivariable systems can be modelled in a unified manner. While transfer function on the other hand are defined only for linear time invariant system. So, with this we can close our today's discussion, we will continue our discussion while we start modelling the multi variable system where you will see multiple input and multiple output and how you will analyze those set of models, thank you.