Basic Electric Circuits
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Module- 11
State Variable Analysis
Lecture - 56
Characteristic Equation, Eigenvalues and Eigenvectors

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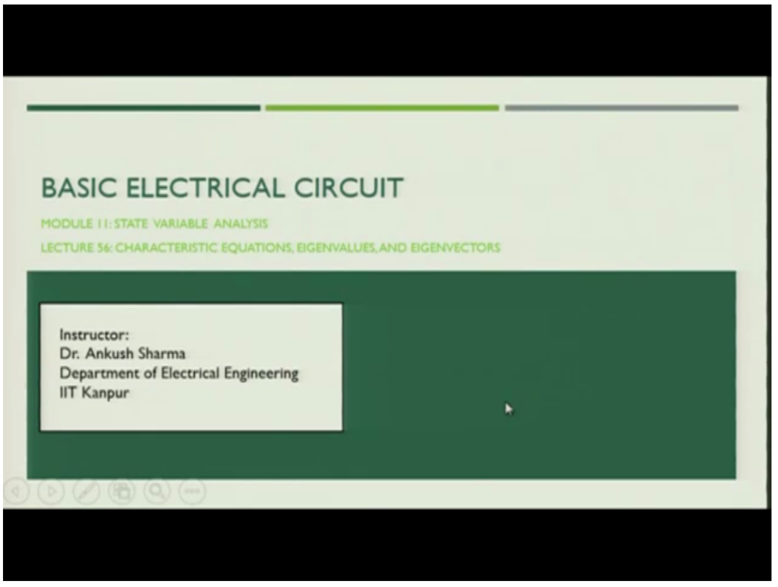

Namaskrar, in the last class we discussed about the State variable analysis in the electrical circuits. Today we will continue our discussion further and we will try to understand few properties of the State equation. And we will see how we can use those properties in our circuit analysis.

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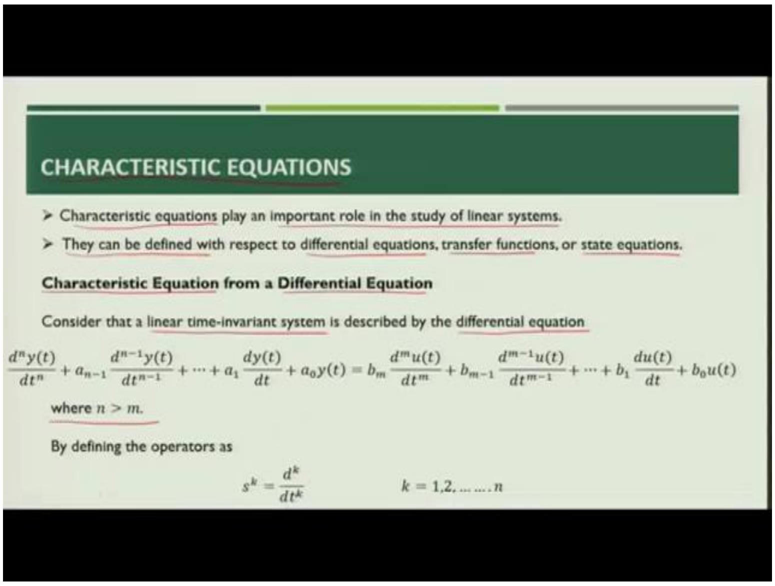

So, let us start the discussion of today’s lecture. First let us try to understand what characteristic equations is. So, the characteristic equations play an important role in the study of linear systems. So, they can be defined with respect to differential equations or maybe the transfer function or state equations.

So, first we will understand how we get the characteristic equation from a differential equation. So, let us consider one linear time invariant system which is described by the following differential equation. The differential equation is given as

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m\frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t)$$

Now here, we assume that $n > m$. Now, let us define one operator that is your Laplace operator,

$$s^k = \frac{d^k}{dt^k}, k = 1,2, \ldots \ldots . n$$

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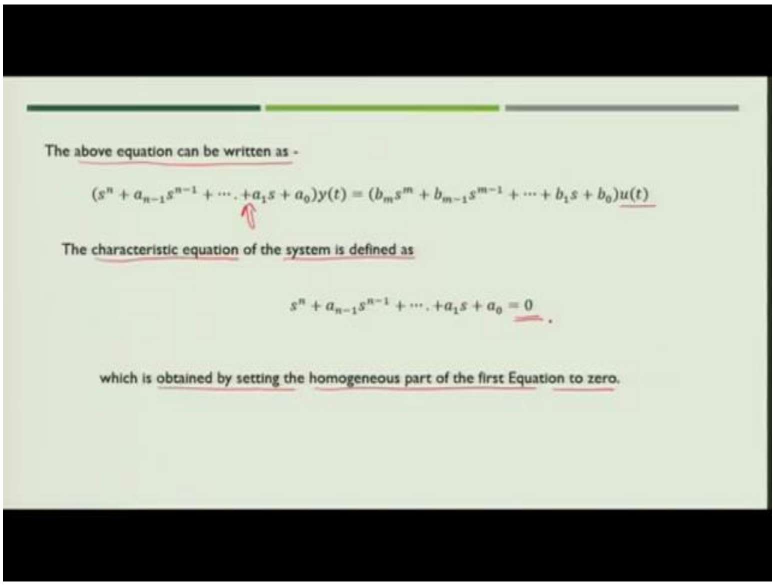

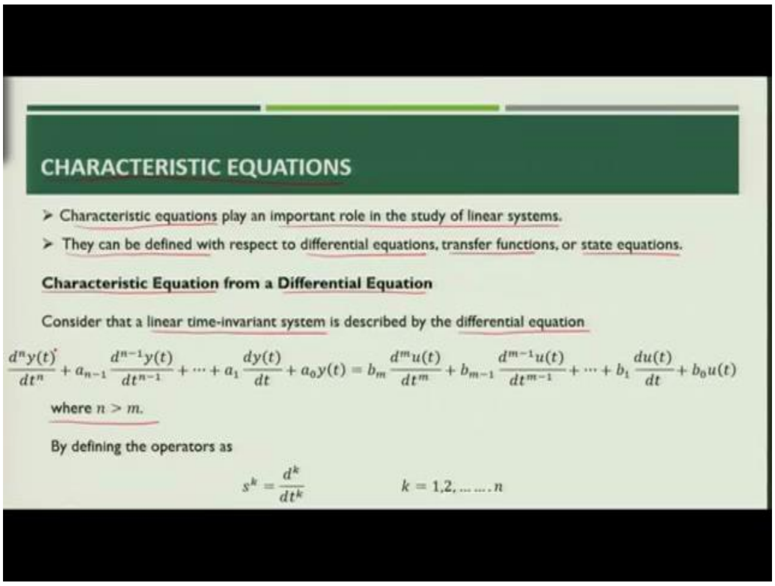

Now, if you put that into the above equation, so the above the differential equation and what we have saw, recently we can write in terms of Laplace transforms. So, what we can write? Instead of writing the differential equation it can be expressed as,

$$(s^n + a_{n-1}s^{n-1} + \cdots . + a_1 s + a_0)y(t)$$
$$= (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)u(t)$$

So, we will have, we will take $y(t)$ as common out in the parenthesis and within the parenthesis we have $s^n + a_{n-1}s^{n-1} + \cdots . + a_1 s + a_0$. On the right side similarly, we have $(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)u(t)$ because $u(t)$ is also common in the right side terms. Now, what is the characteristic equation in this? Characteristic equation of the system is defined as the left side portion. So, we write the characteristic equation as $s^n + a_{n-1}s^{n-1} + \cdots . + a_1 s + a_0 = 0$. We obtain this characteristic equation by setting the homogeneous part of the equation. Homogeneous means you set the input to 0, so we get the homogeneous part of the equation to 0. So, with this you get the characteristic equation of the system which is nothing but the coefficients of the differential equation. When you put the input as 0, so you got the characteristic equation of the system.

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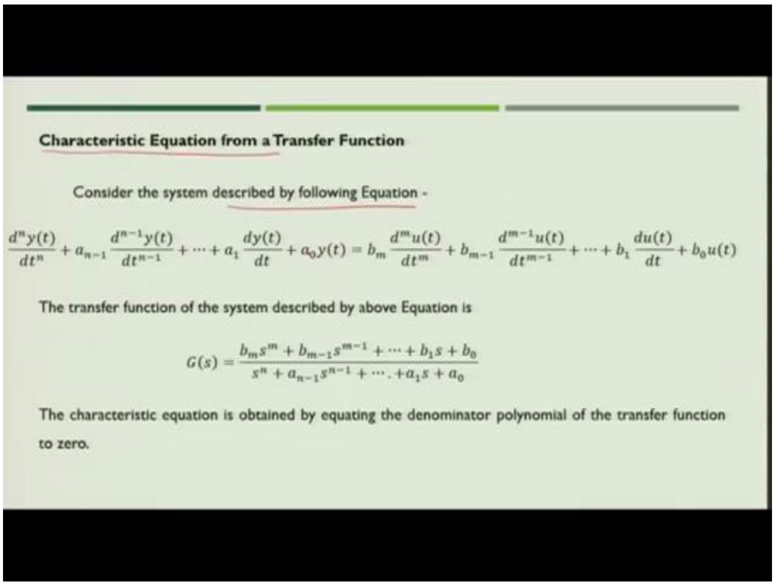

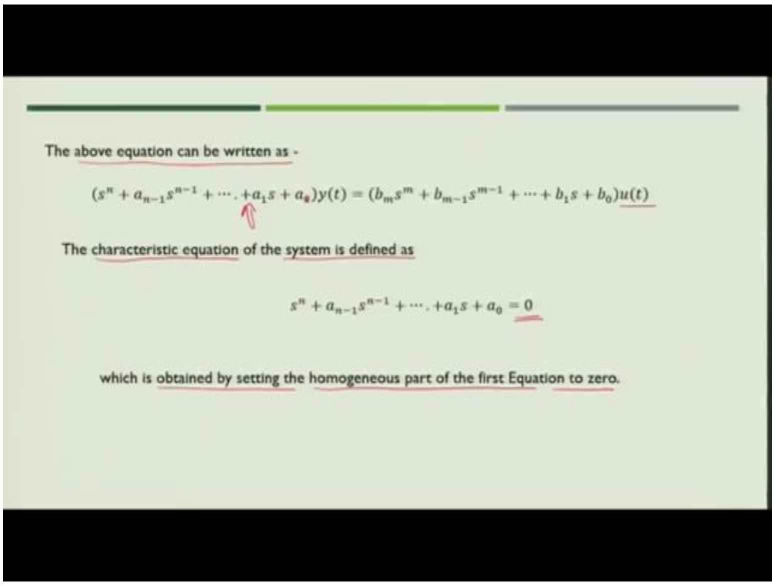

Now, if you are given the transfer function, how to find the characteristic equation? Now, let us consider the system given by the differential equation same differential equation we are using which we saw in the previous case. Now, the Laplace transform of this we saw equal to this. Now, what is $y(t)/u(t)$? This is nothing but the transfer function of the system, so

$$\frac{y(t)}{u(t)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots . + a_1 s + a_0}$$

Now, this is the transfer function of the system. The characteristic equation you can obtain by equating the denominator polynomial of the transform function equal to 0. So, you will set this equal to 0, so this will be nothing but the characteristic equation of the system.

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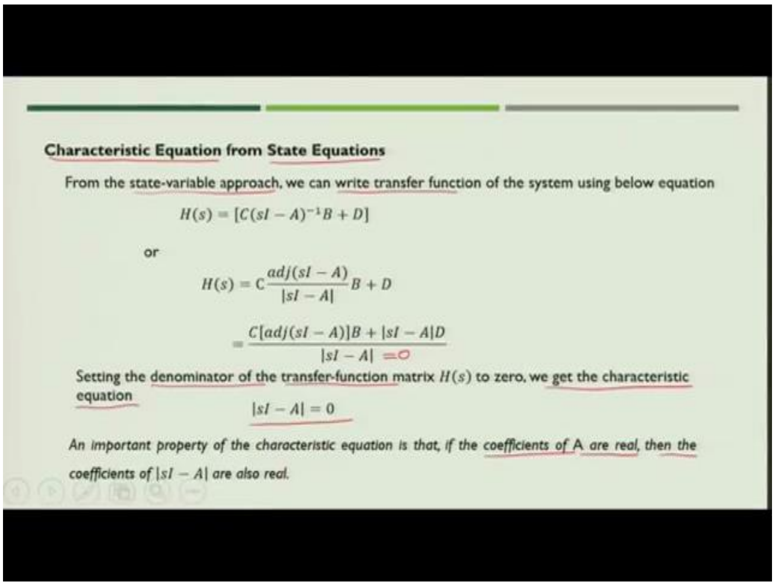

Now the third case, when you need to find out the characteristic equation from State equation. So, when you, we were discussing about the State variable approach we found that the relationship between State variable coefficient matrices and the transfer function is as follows. So, the transfer function from the State variable we can get $H(s) = [C(sI - A)^{-1}B + D]$.

Now, this can be written as

$$H(s) = \mathrm{C}\frac{adj(sI - A)}{|sI - A|}B + D$$

$$= \frac{C[adj(sI - A)]B + |sI - A|D}{|sI - A|}$$

Now, the denominator of the transfer function if you set equal 0, we get the characteristic equation of the system. So, that means that $|sI - A| = 0$ will give you the characteristic equation of the system.

Now, the important property of characteristic equation is that if the coefficients of A that is the coefficient matrix. So, coefficient of A are real then the coefficient of $|sI - A|$ will also be real.

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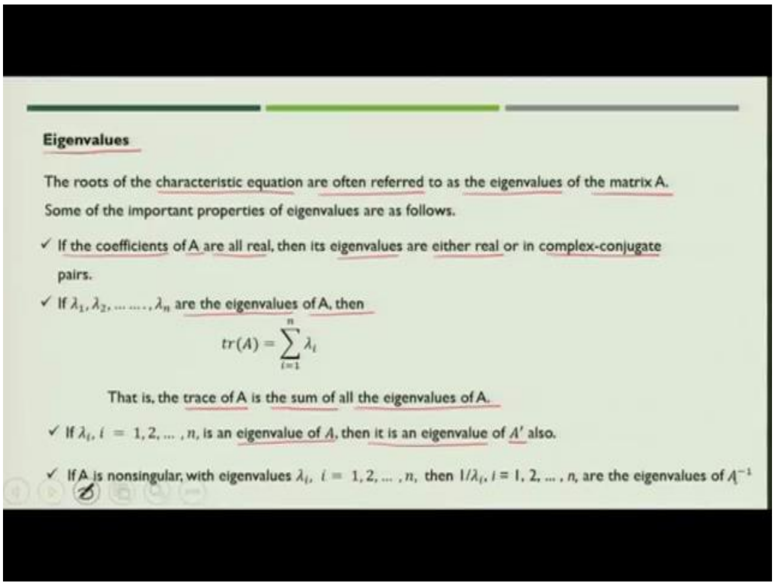

Now, let us try to understand what eigenvalues is. So, the roots of the characteristic equation which we just discuss are refer to as the eigenvalues of the matrix A. So, what are the important properties of eigenvalues? So, if the coefficients of A are all real then its eigenvalues would be either real or in complex conjugate pairs.

Now, suppose if $\lambda_1, \lambda_2, \dots\dots, \lambda_n$ are the eigenvalues of A then trace of A can be written as,

$$tr(A) = \sum_{i=1}^{n} \lambda_i$$

What does it mean? That means the trace of A matrix is the sum of all the eigenvalues of A. Now, if λ_i is the eigenvalue of A that is $i = 1, 2, \dots, n$. If there are n eigenvalues so you will have the eigenvalues ranging from $\lambda_1, \lambda_2, \dots\dots, \lambda_n$.

Now, if this is an eigenvalue of A, then it is an eigenvalue of A' also. So, if you take the transpose of the matrix the eigenvalues will remain same. Now, if you see that the matrix A is nonsingular, and it has the eigenvalues $\lambda_i,\ i = 1, 2, \ldots, n,$ then $1/\lambda_i$, i = 1, 2, ... , n, are the eigenvalues of A^{-1}. So, this is important property when we use in our various analysis. We must keep in mind that when we calculate the eigenvalues of A the reciprocal of those eigenvalues will be the eigenvalues for A^{-1}.

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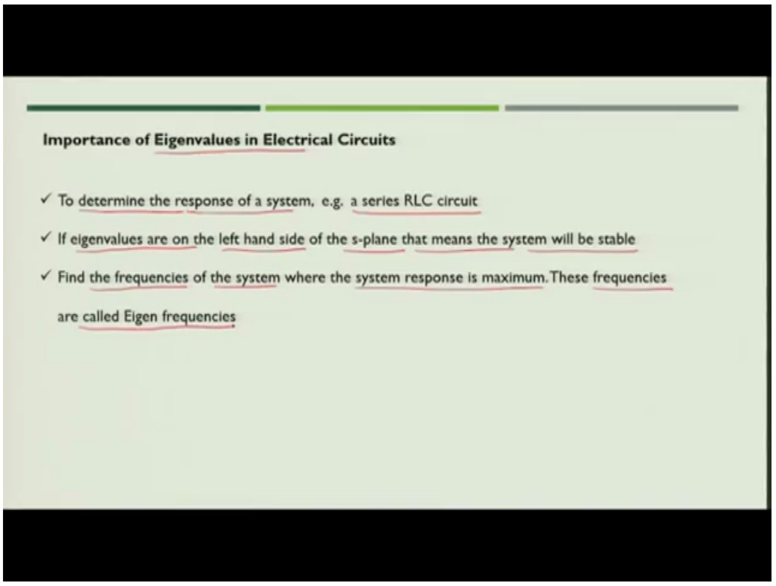

Now, what is the importance of eigenvalues? These eigenvalues help in determining the response of the system say if you have a series analyses circuit. So, if you converted into State variable firm and find the eigenvalues these eigenvalues will help us in finding out the response of the system. If the eigenvalues are on the left inside of the s-plane that means if they have the real component negative. That means that the system is stable.

Now, eigenvalues can also be used in finding the frequencies of the system where the system response is maximum. In that case, these frequencies are called as Eigen frequencies.

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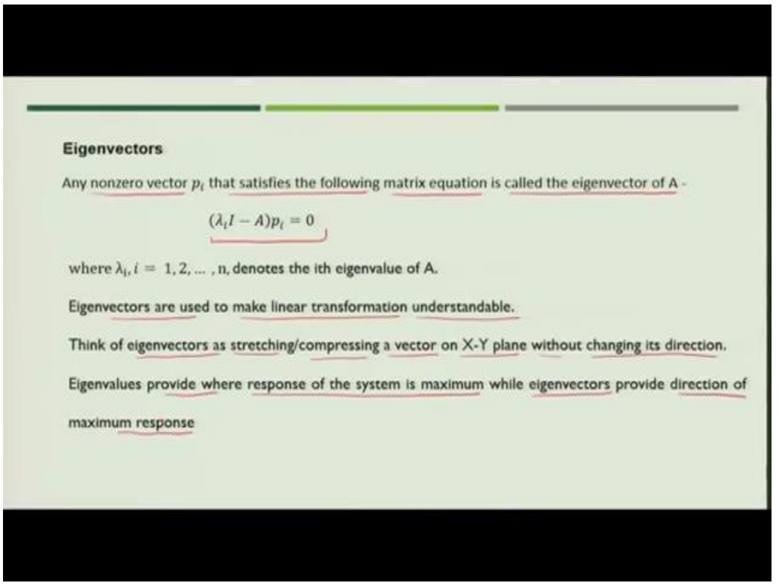

Now, let us talk about the eigenvectors. So, suppose if there is a nonzero vector say p_i that satisfy the following matrix equation. So, what is the condition? So,

$$(\lambda_i I - A)p_i = 0$$

If it is satisfied, then p_i is considered to be the eigenvector of matrix A.

Now, what is the use of eigenvectors? Eigenvectors are used to make the linear transformation understandable that means the eigenvectors if you stretch and compress the vector on XY plane than the direction of the vector in XY plane is not going to change. So, that means that if you have the eigenvector you just scale up and down the vector without changing the direction of that vector.

Now, eigenvalues provide where the response of the system is maximum and eigenvectors provide the direction of that maximum response. So, this is how we correlate the eigenvalues and the eigenvectors.

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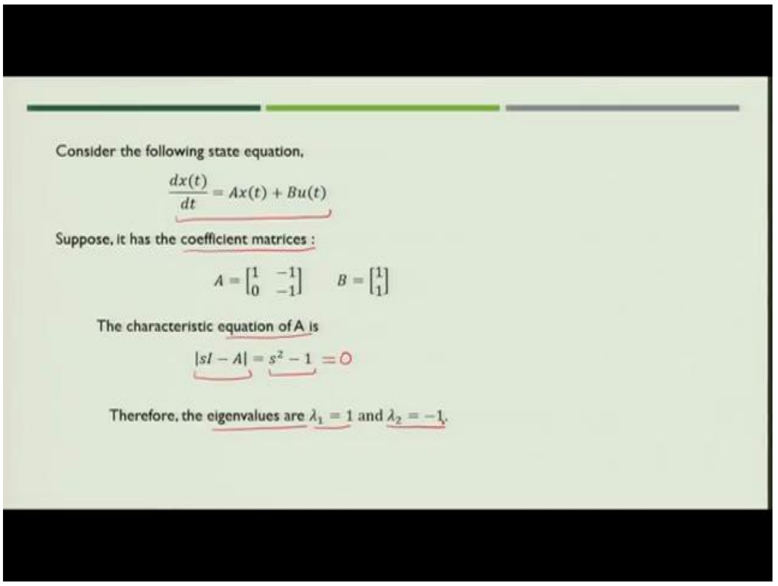

Now, let us take one example to understand the eigenvalues and the eigenvectors. Now, let us consider the state equation we have that is $\frac{dx(t)}{dt} = Ax(t) + Bu(t)$. Now, suppose if it has the coefficient matrices,

$$A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The characteristic equation for A will be

$$|sI - A| = s^2 - 1$$

So, what you need to do? You need to equate it to 0 then if you solve, then you get the eigenvalues as $\lambda_1 = 1$ and $\lambda_2 = -1$.

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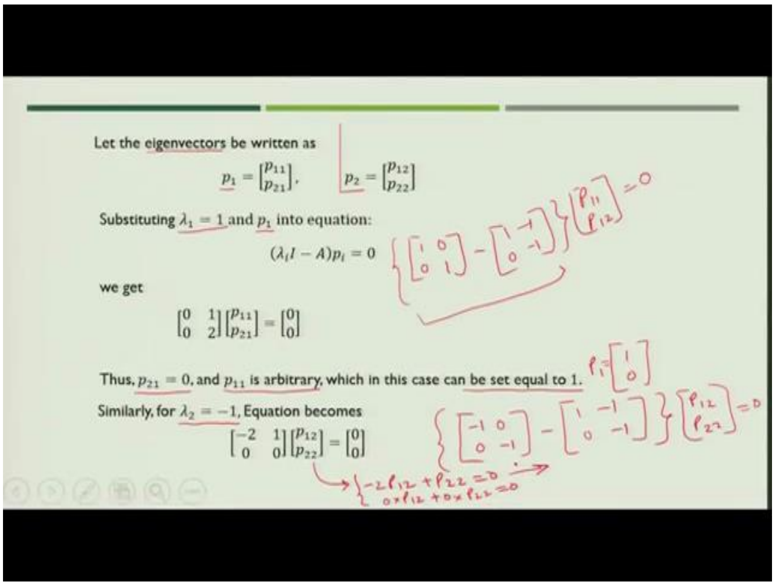

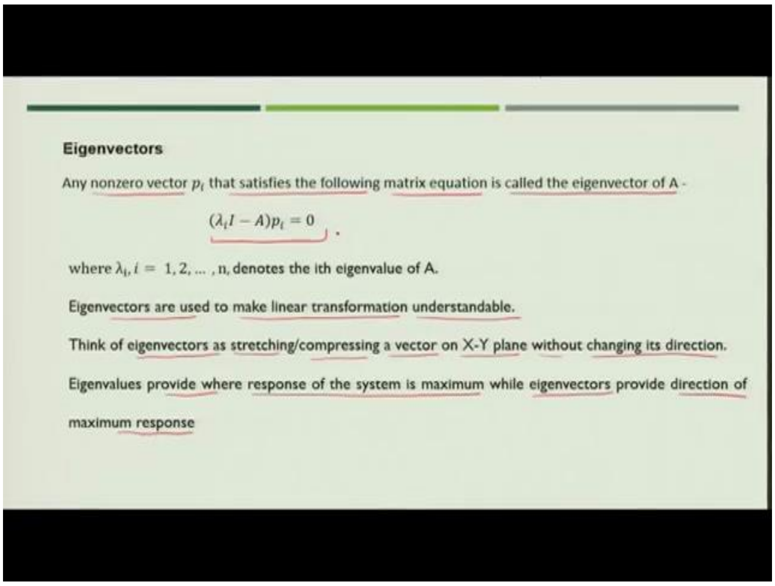

Now, let us assume that the there are two eigenvectors attached to each eigenvalue. So, let us assume that there are two eigenvectors

$$p_1 = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}, \quad p_2 = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}$$

Now, let us first consider $\lambda_1 = 1$ and then now put $\lambda_1 = 1$ and p_1 into the characteristic equation for the eigenvectors. So, when you put that value what we get?

$$(\lambda_i I - A)p_i = 0$$

So, the value you will get is,

$$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $p_{21} = 0$, and p_{11} is arbitrary, which in this case can be set equal to 1.

Similarly, for $\lambda_2 = -1$,Equation becomes.

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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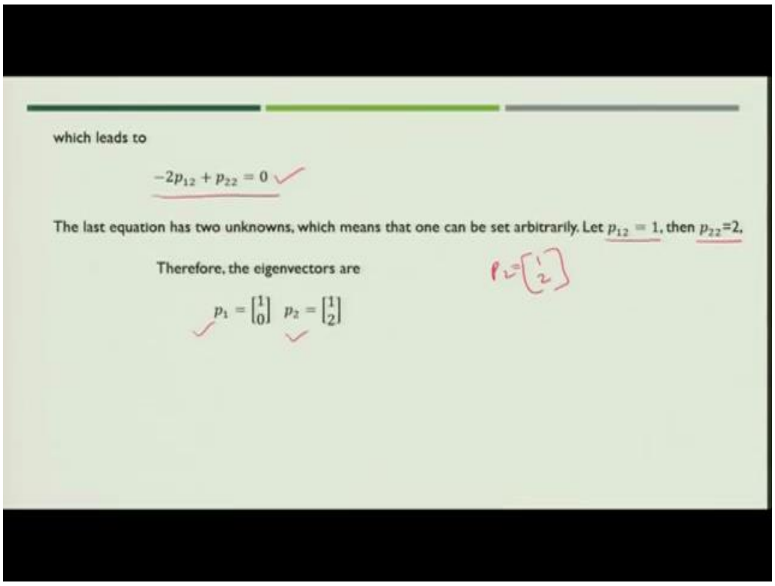

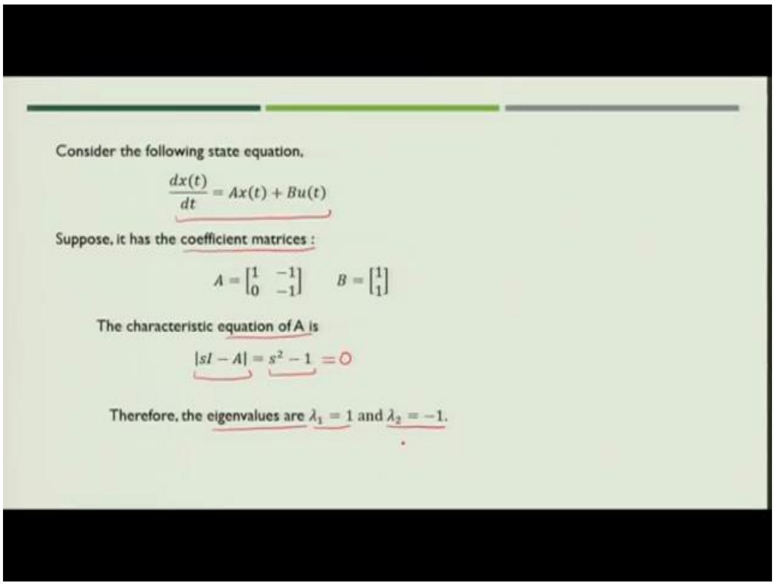

So, the second equation does not have any meaning, so we are left with only one equation that is $-2p_{12} + p_{22} = 0$. Now, we have only one equation and two unknowns, so what we can do? We will set the value of one and find out the value of second. So, let us assume $p_{12} = 1$, then p_{22}=2. So, in that case

$$p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad p_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So, now you have got the eigenvectors as p_1 and p_2 which are corresponding to the eigenvalue that is λ_1 and λ_2.

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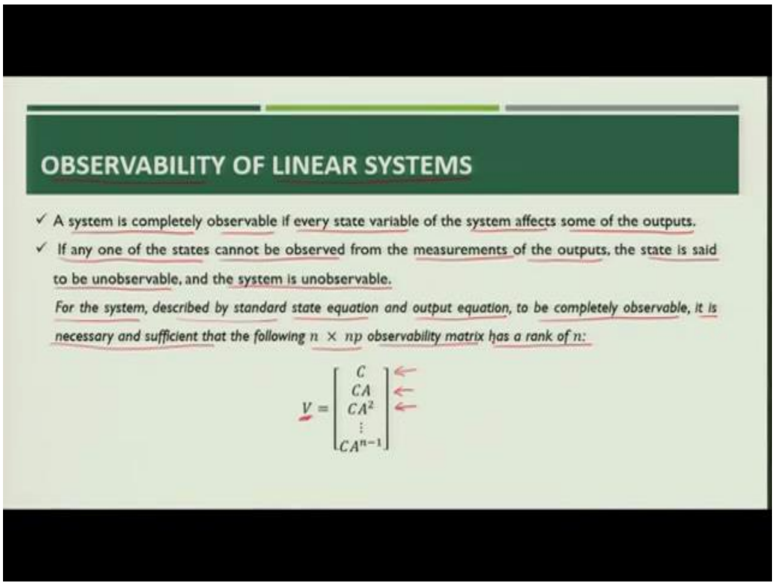

Now, let us try to understand another concept called concept of controllability. Now, the process is said to be completely controllable if every state variable of the process can be controlled to reach a certain objective infinite time. So, if you repeat the same objective infinite times, you will always get the control output provided you have some unconstrained controlled connected to the system.

So, let us say this is control is the input that is u t. So, in that case the process is said to be completely controllable if every state variable of the process can be controlled to reach a certain object at multiple times. Now, if any one of this state variables is independent of the control that is u t there would be no way of driving this state variable because the State variable is independent of control.

So, in that case there will be no way of driving that particular state variable to a desired state infinite times by means of any control effort. Because this state variable is not controllable by the input $u(t)$. So, in that case, the as long as there is at least one uncontrollable state the system is said to be not completely controllable, or we just call it as uncontrollable.

Now, the concept of an controllability given here refers to the states and is referred to as state controllability. Because you are defining controllability in terms of state it is called as state controllability. Now, controllability can also be defined for the outputs of the system. Here in this case, we considered states for the controllability. When you consider output for the controllability we call it as output controllability condition.

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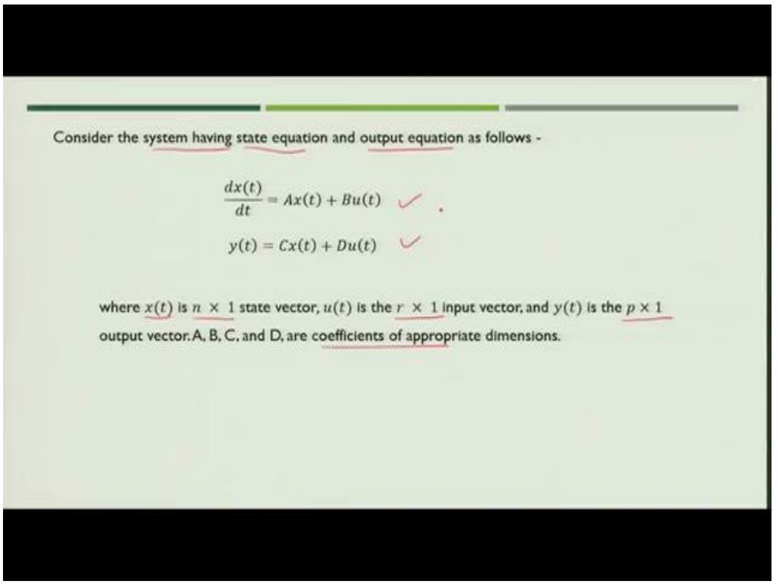

Now, let us consider the system, we have state equation and output equation that is the standard state and output equation given as follows that is

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Now, $x(t)$ is $n \times 1$ state vector, $u(t)$ is the $r \times 1$ input vector, and $y(t)$ is the $p \times 1$ output vector. A, B, C, and D are coefficients of appropriate dimensions. Now, we will use these two equations to find out the controllability condition.

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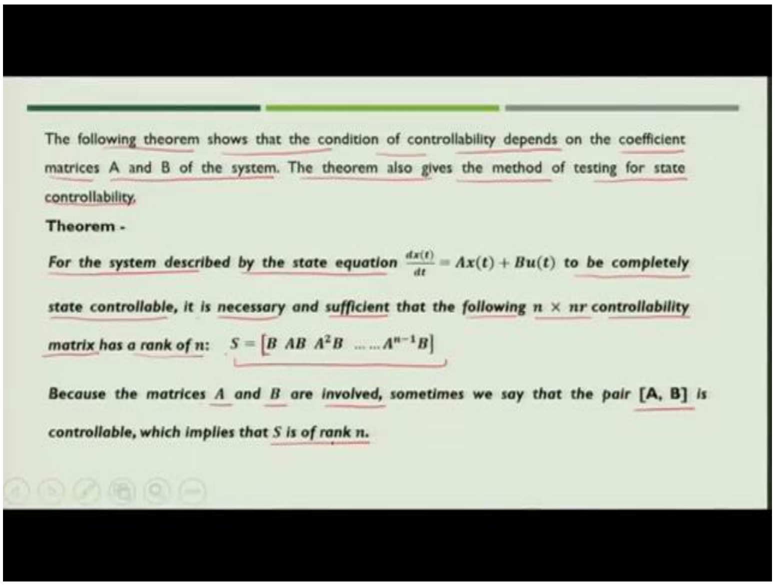

Now, the following theorem shows that the condition of controllability depends on the coefficient matrices A and B of the system. And we are particularly talking about the state controllability that is why A and B Matrices are being considered for the controllability analysis. Now, theorem also gives the method of testing for the state controllability. What is the theorem? Theorem says that for the system described by the state equation that is $\frac{dx(t)}{dt} = Ax(t) + Bu(t)$.

So, this system to be completely state controllable that is necessary and sufficient condition is that the following the matrix that is S which is having $n \times n_r$ dimension, the controllability matrix which we have augmented matrix containing B and A matrix as component. So, the condition is necessary and sufficient that the following S matrix has the rank equal to n. So, what is the controllability matrix S?

So, controllability matrix $\boldsymbol{S} = [B \ \ AB \ \ A^2B \ \ \dots\dots A^{n-1}B]$. Then we find out the rank of this matrix and if the rank of this matrix is not equal to n that shows that the System State are not controllable. Now, since the matrices A and B are involved in this particular case, sometimes we also say that pair AB is controllable which implies that the S that is augmented matrix for controllability has the rank equal to n.

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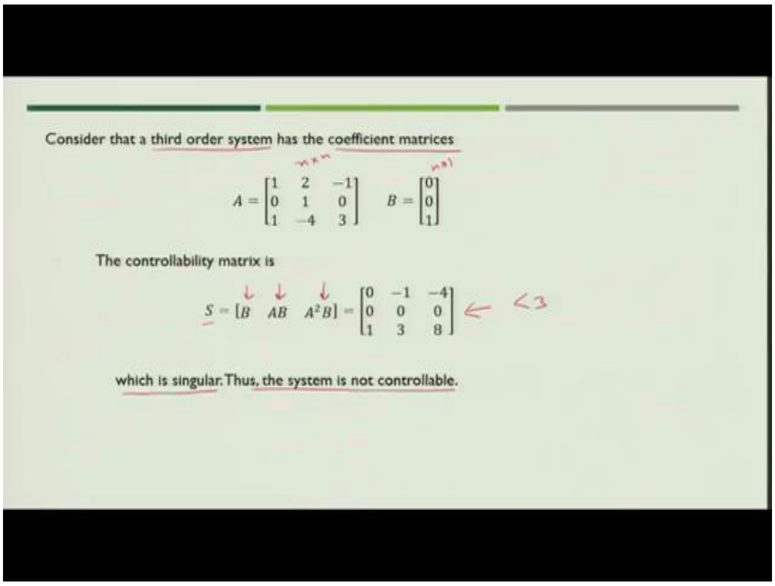

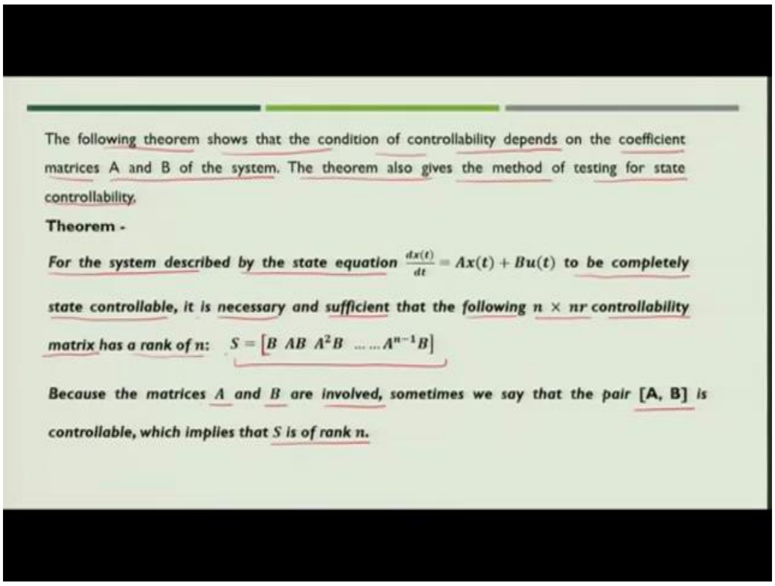

Now, let us take one example so that you can understand how we find the State controllability condition. Now, let us consider that there is a third order system which has the coefficient matrices as

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now, since we have A is n x n and this is n cross 1. So, the matrix n in this case is 3. So, the controllability matrix will now have 3 columns. So, what we will do? We will first compile the S matrix. How we will compile? We will find

$$S = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & -1 & -4 \\ 0 & 0 & 0 \\ 1 & 3 & 8 \end{bmatrix}$$

Now, if you look this particular matrix closely you will see that the second column is, second row is having all elements is 0. That means that the rank of matrix S is less than 3 and since it is also singular means there will be a case when the rank of the matrix will be less than 3. So, therefore this system is not state controllable. So, with this you can get an idea that how to find the controllability of the system.

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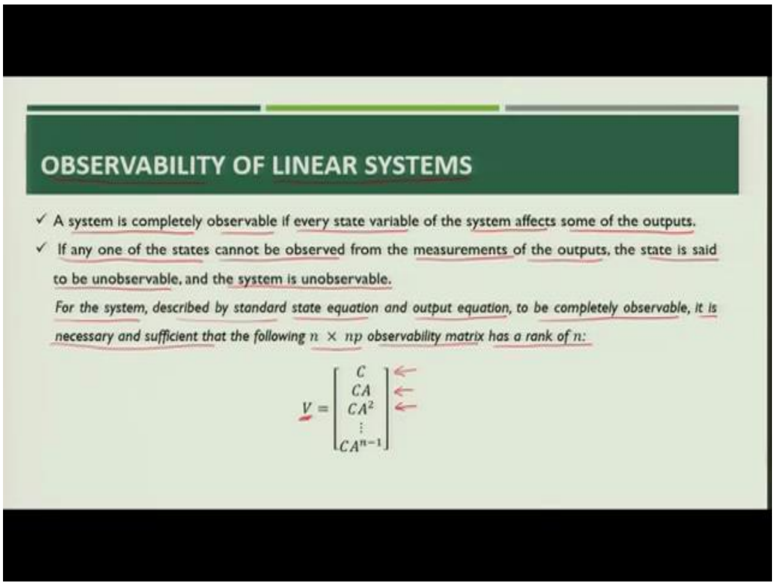

Now, let us understand another concept called observability of the linear system. Now, the system is completely observable if every state variable of the system affects some of the outputs. That means that if anyone of the states cannot be observed from the measurements that is the output measurements, the state is said to be unobservable and therefore the system will be unobservable.

What is the criteria to find out the observability of the system? So, for the system described by the standard equation, state equation and the output equation can be completely observable. It is necessary and sufficient that the following matrix V that is n cross np matrix observability matrix as the rank equal to n. How we compile the observability matrix? We compile by

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

So, when you compile this matrix you get the observability matrix which will be n into $n \times n_p$. Now, when you find the rank of this matrix and this rank is less than n, the system is not observable. Otherwise, when the rank is equal to n you will say the system is observable.

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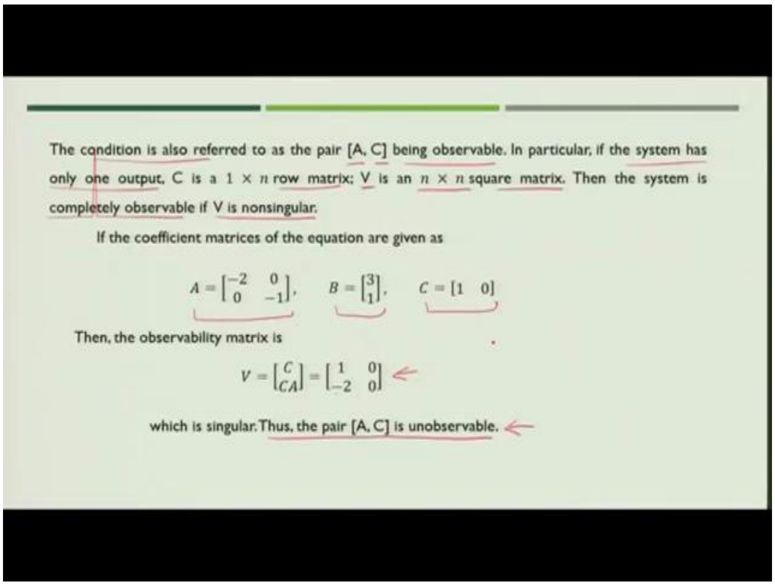

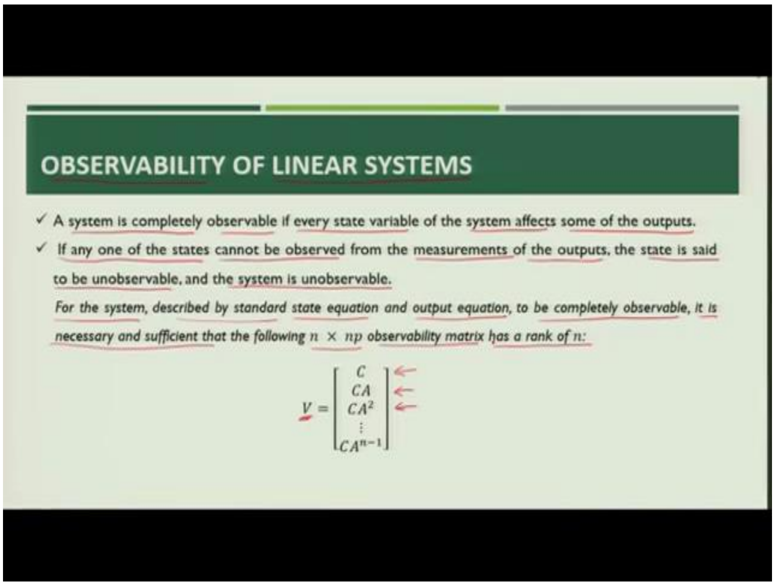

Now, let us understand with the help of the example. So, the condition that is containing A and C both matrices, we generally call it as the pair that is [A, C] is also observable.

If the system has only one output that C is a $1 \times n$ row matrix; V is an $n \times n$ square matrix. In that case, the system is completely observable if V is nonsingular.

So, let us try to understand how to find out whether the system is observable or not. Let us consider the coefficient matrices that is

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Now, we have n equal to 2 so now the observability matrix will have two rows that is

$$V = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

Now, if you look this matrix, you will see that the second column has all elements as 0 means this matrix is also singular. In that case, we will say that the system is not observable, or we can say that the matrix pair that is A, C is unobservable. So, using the augmented matrices that is V, you can find out the observability condition of linear a system.

So, with this we can close our today’s discussion in which we discuss about the controllability and observability aspect of the State variable system. So, we close this particular topic where we discuss about the State variable analysis. In the next session we will start our last topic that is the analogous system. Thank you.