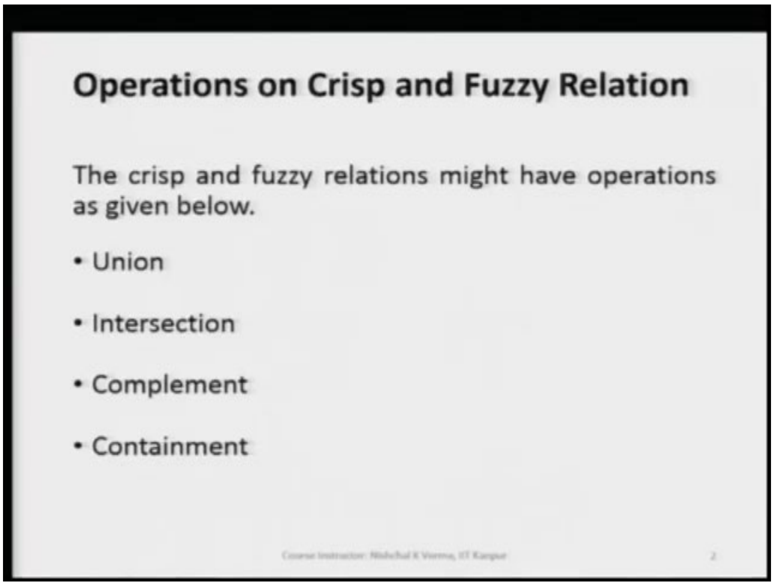
Fuzzy Sets, Logic and Systems and Applications
Prof. Nishchal K. Verma
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture – 30
Operations on Crisp and Fuzzy Relations

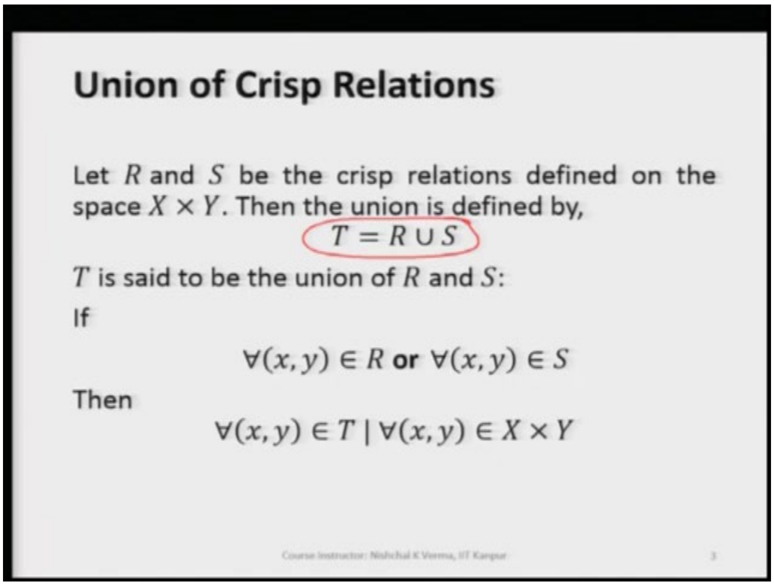
Welcome to lecture number 30, Fuzzy Sets, Logic and Systems and Applications. So, in this lecture we will discuss the Operations on Crisp and Fuzzy Relations. And of course as I already mentioned in my previous lecture that fuzzy relation set is also a fuzzy set.

(Refer Slide Time 00:50)



So, in the previous class I mentioned that related to these fuzzy relation set, we have the operations as union, intersection, complement and containment.

(Refer Slide Time 00:58)

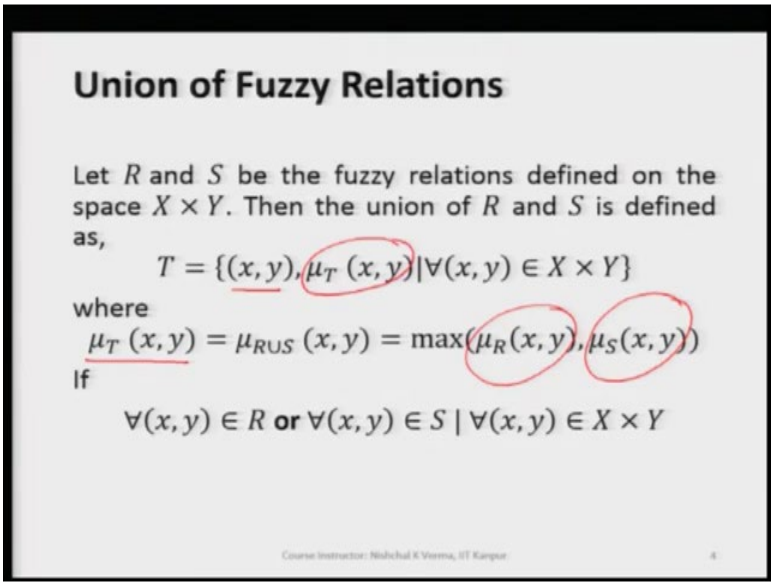


And, let us first understand, what is the union of relations? So, let us first take the union of crisp set. So, if we have two crisp sets R and S and these sets are the relation sets. So, we have in other words we can say, the R and S be the crisp relations defined on the space $X \times Y$. And, then the union is defined by another crisp set that is $T = R \cup S$, you can see here and this T is said to be the union of R and S.

And, here this T set will contain the ordered pairs as elements in this set. So, for every ordered paired $\forall (x, y) \in R \; or \; \forall (x, y) \in S$. So, then what is happening to the elements of the set T. So, here also for every (x, y), that is the ordered pairs, that is belonging into fuzzy relations T such that for every ordered pair (x, y) is belonging into $X \times Y$.

So, this was basically for the crisp relation sets R and S and then we had out of the union, we have another fuzzy set which is T, which is coming out of the $R \cup S$.

(Refer Slide Time 02:52)



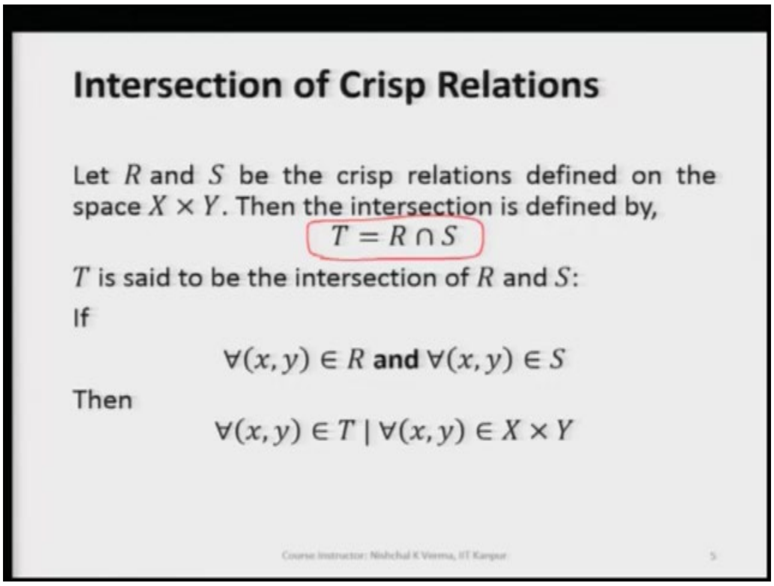
Now, let us discuss the union of fuzzy relations. So, as we have seen earlier also that there is a difference in crisp and fuzzy relation. And, what is that difference? Difference here is that here in fuzzy relation, and as of course, I have already told you, that fuzzy relation is a fuzzy set finally. So, the fuzzy relations set will have it is corresponding membership values.

So, that is, what is the difference here other than this there is no difference. So, in fuzzy relations set, we will have apart from the ordered pairs we will have the corresponding membership values. Here we have the ordered pair and then we have the $\mu_T(x, y)$ means, the corresponding membership value. So, this way we have the union of fuzzy relations and these are the fuzzy relations sets. And, as I have already mentioned that these R and S basically fuzzy relations, but these are fuzzy sets.

Now, how to get this $\mu_T(x, y)$ means μ_T of the ordered pair is the corresponding membership value? So, how to get that in case of union? So, in case of union you see here $\mu_T(x, y) = \max(\mu_R(x, y), \mu_S(x. y))$. So, this means what? This means, you see here we take the corresponding membership value, which is present in the fuzzy relation set R and then fuzzy relation set S.

And, when we are taking the union of it the R and S, then we take the max of these two membership values and then we term this as the $\mu_T(x, y)$. So, this way we have find the corresponding membership value corresponding to the order pair. And, of course, it is needless to mention here that for every $(x, y) \in S$ such that for every $(x, y) \in X \times Y$. And, this is because R and S are already defined in this space capital $X \times Y$, because R and S are the fuzzy relations set.

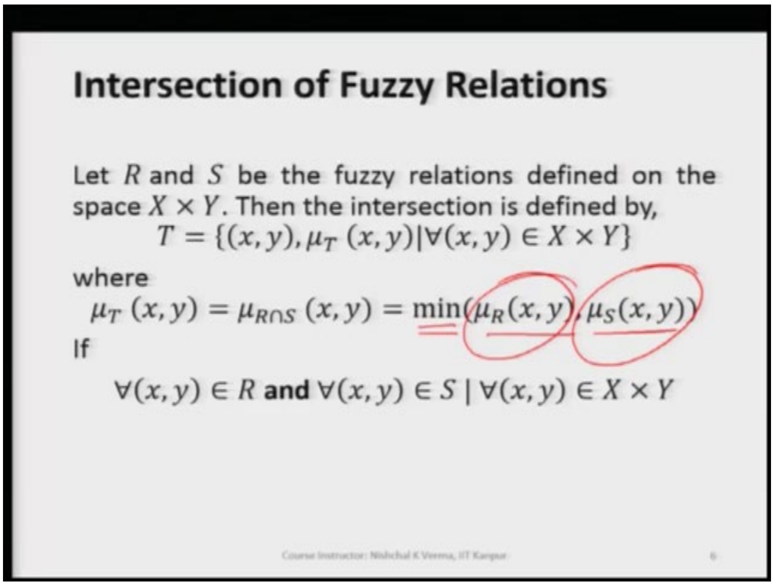
(Refer Slide Time 05:35)



So, when we talk of the intersection of crisp relations? We will go similar on similar lines you see that the when we talk of crisp relations? So, you see here for crisp relation you take the intersection and when you take the intersection, you take only the common elements right.

So, T said to be the intersection of R and S if for every x, y, that is belonging into R and belonging into S. And, the resulting set here will be the x, y that will belong into the T, that is the ordered paired element, which will belong into the T, which is the outcome of the intersection of R and S.

(Refer Slide Time 06:30)



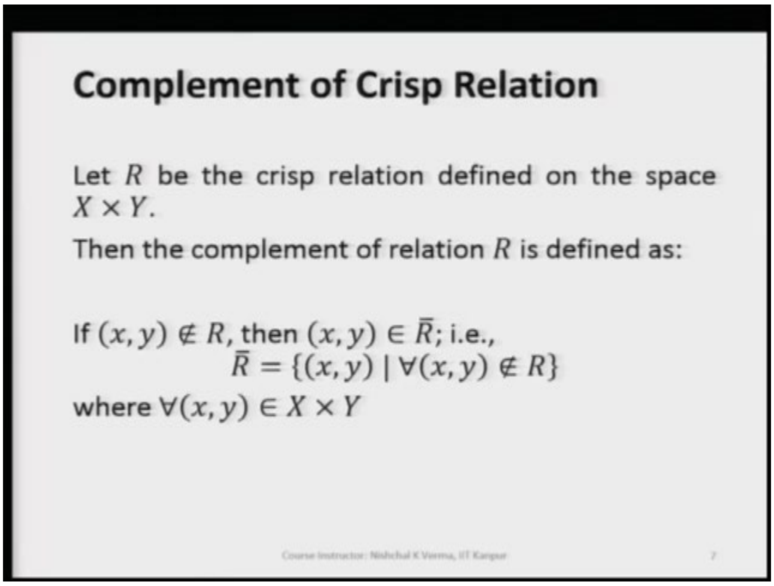
Now, let us understand the intersection of fuzzy relations with respect to here again the fuzzy relations we have to have an additional term here in the set. So, T set here we will have the associated membership values, which was not there in the crisp set. And, you know why I have already explained a couple of times that here in a fuzzy set we have to have this $\mu_T(x, y)$ or the associated membership values, associate and otherwise you know we may not be aware we may not be knowing as to with what membership value a particular element is adjusting in the fuzzy set.

So, let us now quickly define this. So, T here is the fuzzy set and here the element will be the x, y which is ordered pair. And, then it is corresponding membership value $\mu_T(x, y)$ and let us see as to how we can compute $\mu_T(x, y)$. So, here instead of max as in the previous case where we were taking union here we since we are having we are interested in finding the intersection. So, we take min of the two membership values.

So, when we are taking the intersection of fuzzy relations we use the min criteria. So, when we take the min of these two membership values. The resulting value will be termed as $\mu_T(x, y)$. And, which is coming because of the intersection of R and S fuzzy relation set.

So, it is again needless to mention here that for every x, y, which is belonging into R and S and then this x and x, y will also be coming from $x \times\ y$. So, $R\ S$ both are the fuzzy relation set and this is again drawn from $X \times\ Y$, which is the Cartesian product space.

(Refer Slide Time 09:00)

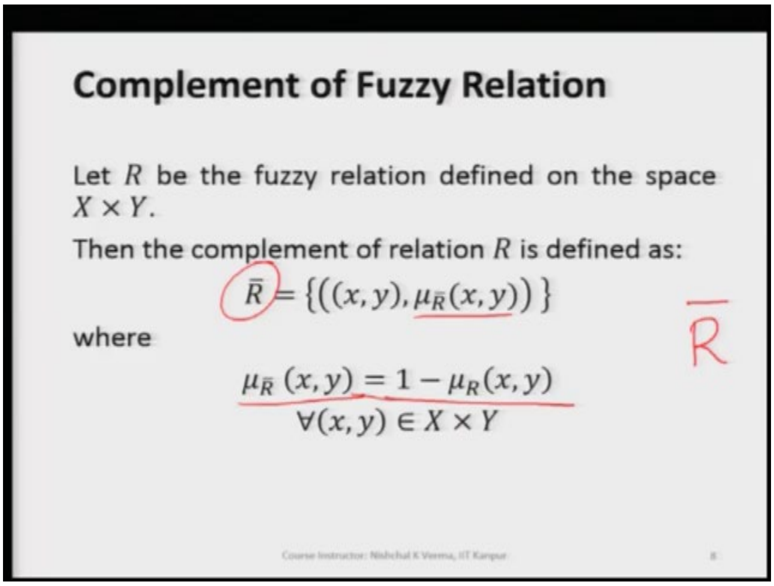


So, now coming to complement of crisp relation. So, as we have seen union and intersection here, the complement of crisp relation we'll first discuss and then we will go to the complement of fuzzy relations. So, as we have already seen in the case of crisp set, how to find the complement of any crisp set. Here also since we are discussing about the crisp relations. So, crisp relation again is a crisp set.

So, let R be the crisp relation defined on the space $X \times Y$, then the complement of relation capital R is defined as, if the ordered pair x, y ordered paired element, which is not belonging into R, then x comma y will belong into the $\bar{R}$; $\bar{R}$ is the complement set.

And, this complement of relation R will be basically the collection of all the ordered pairs elements like x, y, such that for every x, y is not belonging into the set R, that is the crisp relation set R that we have taken. So, it is very easy to understand that we will include all the ordered pairs, which are not there in R. And, all the ordered pairs means the all the ordered pairs that are existing in the Cartesian product space. So, this way we have understood the complement of crisp relation. Now, let us go to the complement of a fuzzy relation.

(Refer Slide Time 10:52)

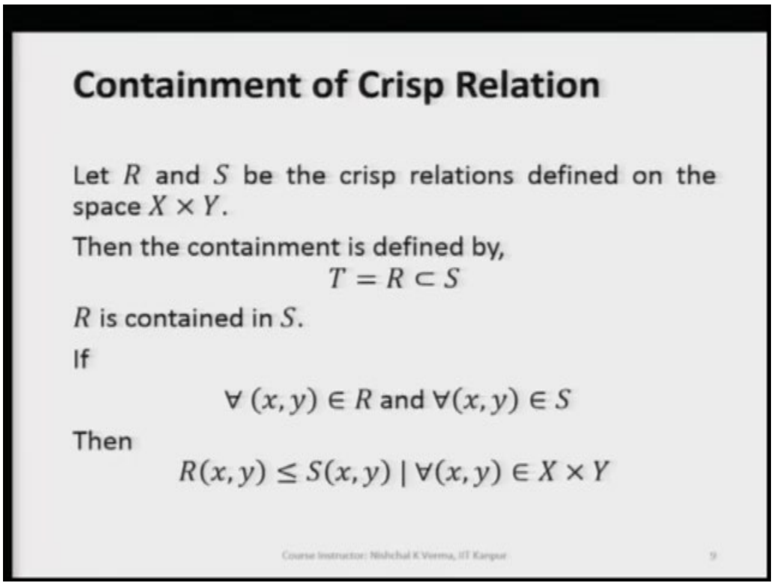


So, as I have already mentioned that here the difference is that we include the membership values along with the ordered pair elements. So, here we have $\mu_{\bar{R}}(x, y)$. So, this is complement of relation R is represented by $\bar{R}$, which is here you see. So, this is basically the collection of these equal to the set which is collection of all the ordered pair elements.

And, these elements are those elements which are not existing in the set that we have taken, but these are existing in the Cartesian product space, $X \times Y$. And, this along with the membership values. So, how to find this membership value? $\mu_{\bar{R}}(x, y)$ see here. So, this very easy we here take a very basic complement, otherwise you can take other complements also like we have done in previous lectures.

So, we are discussing only the basic complement here which is $1 - \mu_R(x, y)$. So, if we apply this we will get we will compute the $\mu_{\bar{R}}(x, y)$. So, this way it is very easy to compute the complement of fuzzy relation.

(Refer Slide Time 12:21)

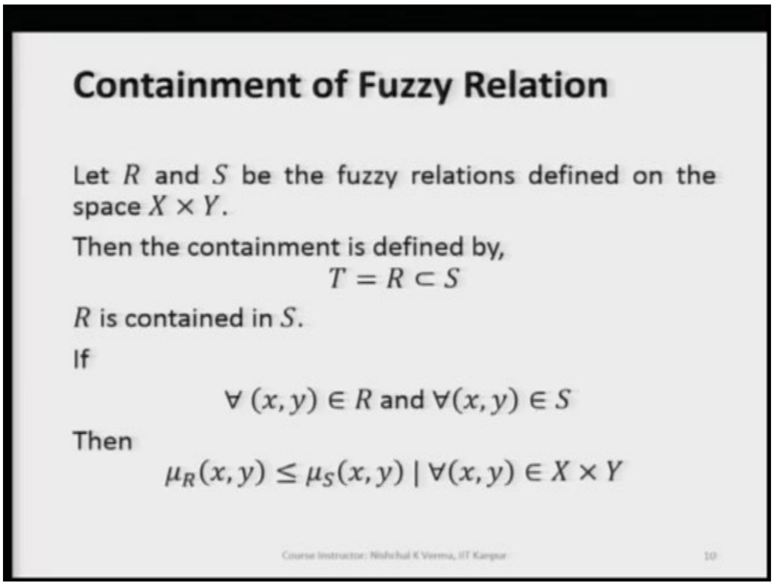


Now, coming over to the containment of crisp relation. So, if we have any two fuzzy sets $R\ and\ S$. So, here let us first before we move to fuzzy sets let us first understand the crisp relation, then we see the transition from crisp relation to fuzzy relation.

So, if we have $R\ and\ S$ as crisp relation set. So, then the containment is defined by the set, which is let us say it $T = R \subset S$. So, R is contained in S if for every x, y, which is the ordered pair element belonging into $R(x, y)$ again belonging into capital S.

So, this way then $R(x, y) \leq S(x, y) | \forall (x, y) \in X \times Y$. So, let us now move to the containment of fuzzy relation.

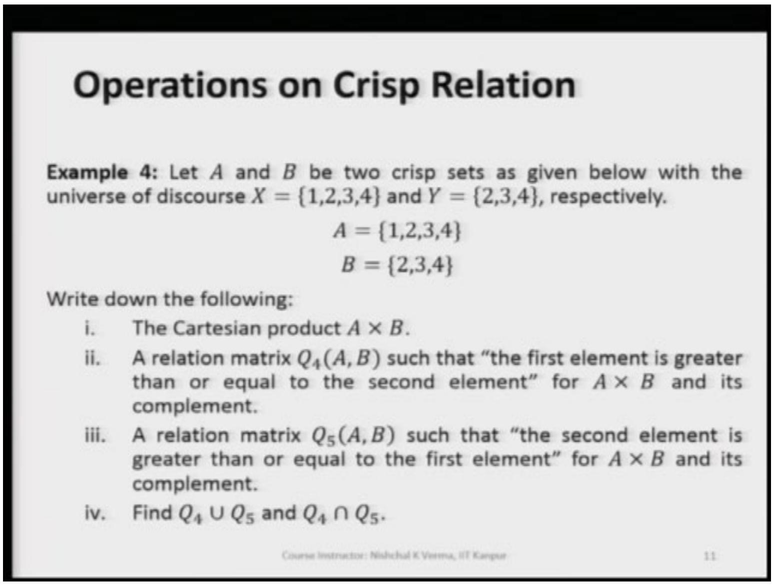
(Refer Slide Time 13:40)



So, here as I just mentioned initially, so if we have R and S as fuzzy relation sets how to represent the containment here. So, containment if we have the containment right like R is contained in S. So, that is possible only when if for every ordered pair elements x, y is belonging into R and S.

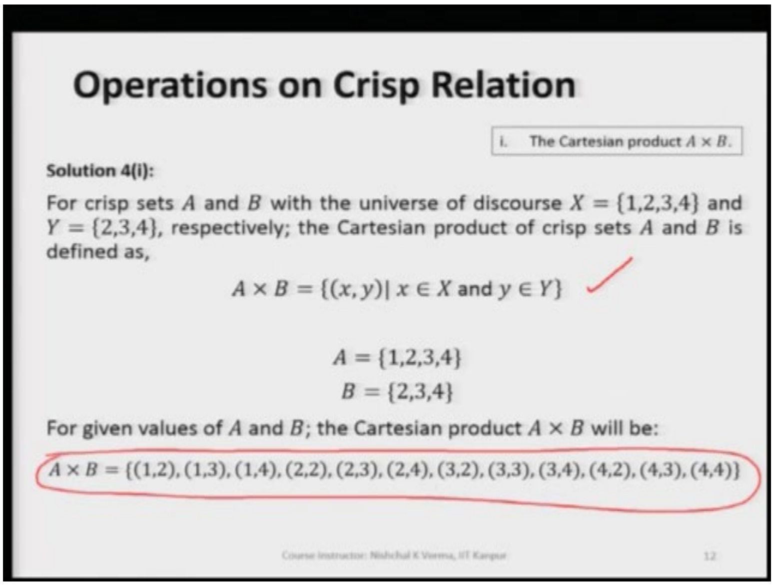
And then $\mu_R(x, y) \leq \mu_S(x, y) | \forall (x, y) \in X \times Y$.

(Refer Slide Time 14:29)



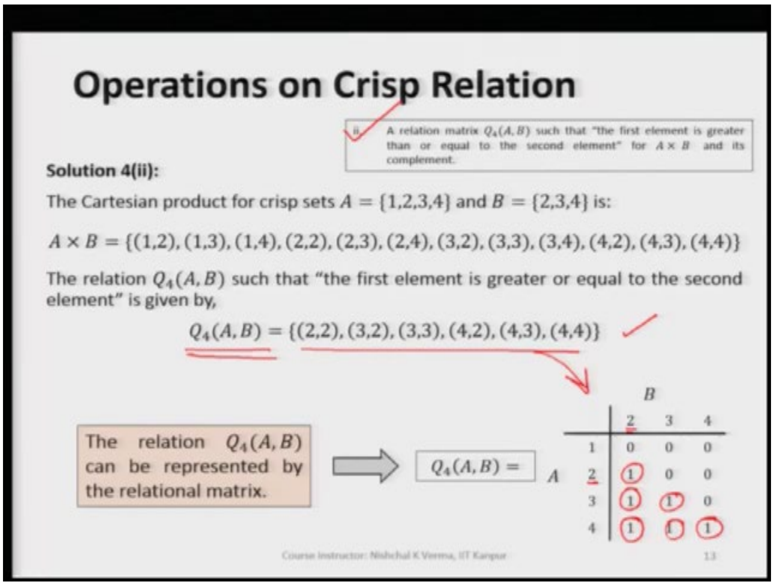
So, this can be very well understood by taking one example here. And, in this example here we have taken two crisp sets A and B. So, let us first understand that here we have two crisp sets A and B and the universe of discourse of both the sets like X and Y are also the same. So, let us first point the Cartesian product of A and B that means, $A \times B$.

(Refer Slide Time 15:00)



So, as we have already seen that how we can get the Cartesian product of crisp set A and B? So, we can very easily find the Cartesian product of two crisp sets A and B, you can see here. So, we have all of these elements. So, very easy to quickly find and this way when we have found this then now let us go to the relation matrix.

(Refer Slide Time 15:26)



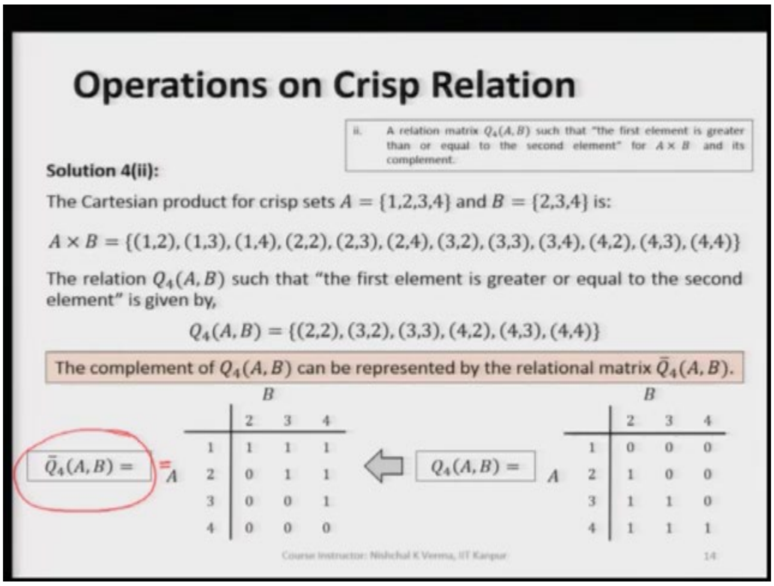
So, let us put some condition as I have already mentioned in case of crisp sets in case of crisp relations. So, here we have the complete population we have the ordered pair elements in $A \times B$. So, now, let us put some condition here and the condition that we are putting here is the first element is greater or equal to the second element.

So, if we put this condition here, that is this case and let this be represented by Q_4. So, Q here is co relation so $Q_4(A, B)$ here. So, this represents the relation in crisp set A, B. So, we have collected here all those elements which follow the condition that has been stated here, like the first element is greater or equal to the second element like $(2, 2)$; $(3, 2)$; $(3, 3)$; $(4, 2)$; $(4, 3)$; $(4, 4)$ all these have been included.

And, the same can be represented by the relational matrix. So, we can see that we have few 1s and few 0s. So, the elements that are existing the pairs that are existing here like, from A 2 as the element and from B 2 also as the element both are forming the ordered pair in $Q_4(A, B)$.

So, that is why this is existing here. So, that is why one has been put here as one of the elements of the relation matrix. Similarly, here $(3, 2)$ is also existing, then $(4, 2)$ is also existing and then $(3, 3)$ is existing, $(4, 3)$ is existing and then $(4, 4)$ is also existing in the $Q_4(A, B)$ set no other elements are existing. So, that is why other elements have been put as 0.

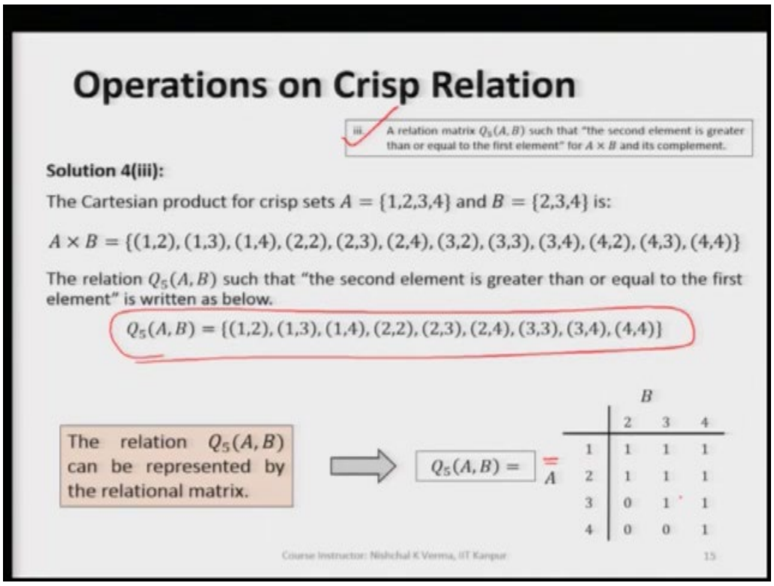
(Refer Slide Time 17:27)



Now, when we have this $Q_4(A,B)$ since this is a crisp set as the relation set. So, if we are interested in finding the complement of this relation set, we can quickly see as to how we can find that. So, if it is the complement relation. So, we represent this by the $\overline{Q_4}$ you can see here.

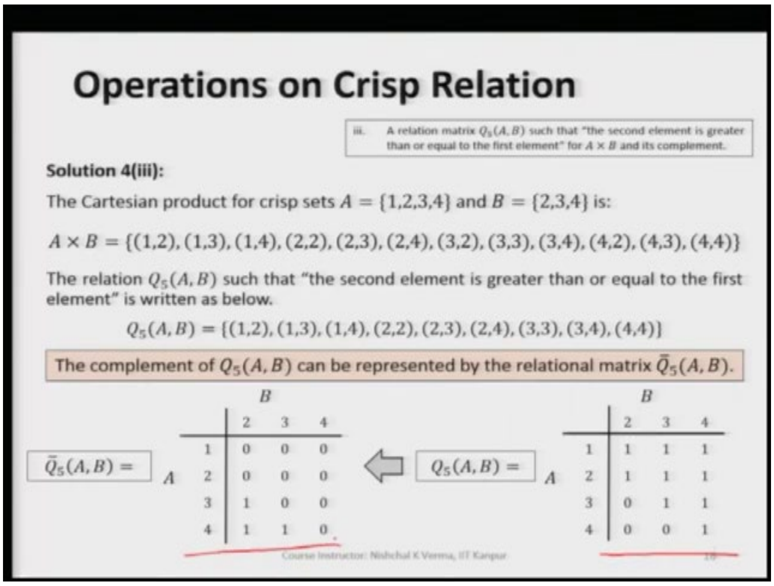
So, this is $\overline{Q_4}(A,B)$ and this is equal to you know A relational representation here, we see that we change 1 into 0 and 0 into 1 means, those elements which were not present in $Q_4(A,B)$ are present here in this set $\overline{Q_4}(A,B)$. So, this way we find the complement of a crisp relation.

(Refer Slide Time 18:24)



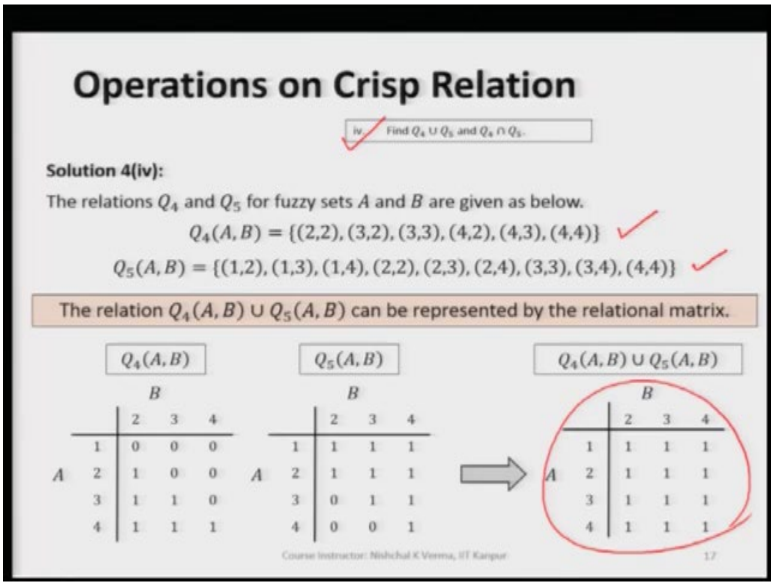
Now, another relation set that is $Q_5(A,B)$ such that the second element is greater than or equal to the first element. So, on the same lines we can find this set here you can just try. And, then we find the relational matrix here as I have described in the previous case.

(Refer Slide Time 18:45)



And, if we are interested in finding the complement we can quickly get the complement by just changing 1 to 0 and 0 to 1. So, this way the complement is found.

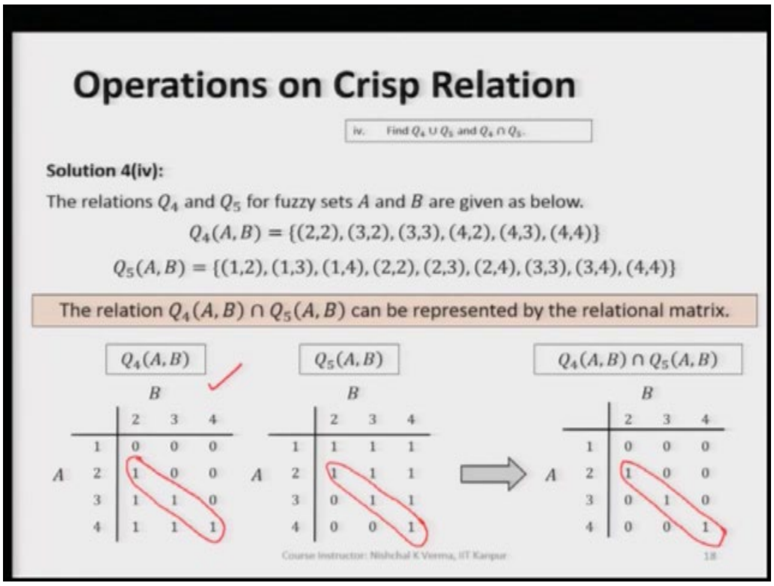
(Refer Slide Time 18:57)



Now, as the fourth case here, we are interested in finding the union of Q_4 and Q_5 and then the intersection of Q_4 and Q_5. So, we have Q_4 here Q_4 relation set the crisp relation set and then we have Q_5 as the another crisp relation set. So, we have two crisp religion sets. Let us now find the union first.

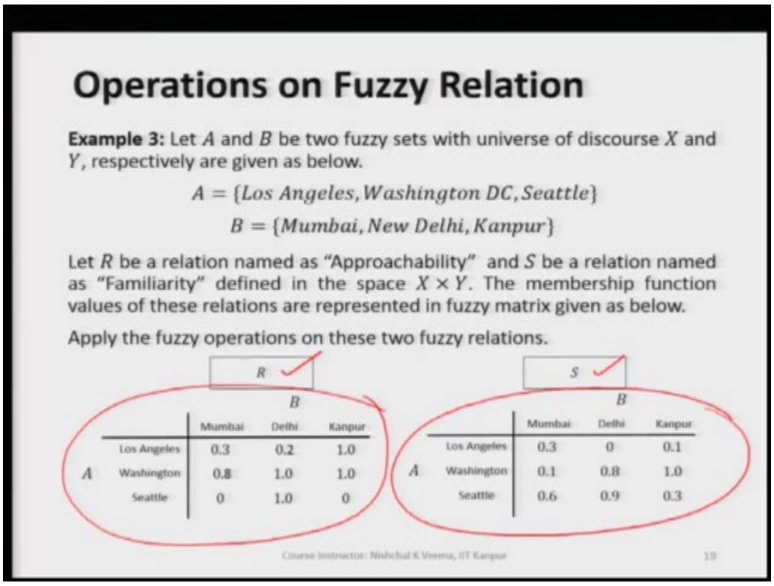
So, when we take the union you see here, we again see that we have those elements which are present in the set $Q_4(A, B)$. And, again those elements which are present in $Q_5(A, B)$ you see. So, all those elements have are being accounted. So, this way all the elements are present here. So, $Q_4 Q_5$ we look at the relational matrix and then we keep all once, which are in both the relation sets that are $Q_4 Q_5$.

(Refer Slide Time 20:08)



Now, when we are taking the intersection, so in intersection we only take the common 1s. So, if we see here in $Q_4(A,B)$ these 1s are present in $Q_4(A,B)$ and this 1s are also present in $Q_5(A,B)$. No other 1s are present in both that relations matrix Q_4 and Q_5. So, that is why only these are kept. So, this is how we get the crisp relation operations done.

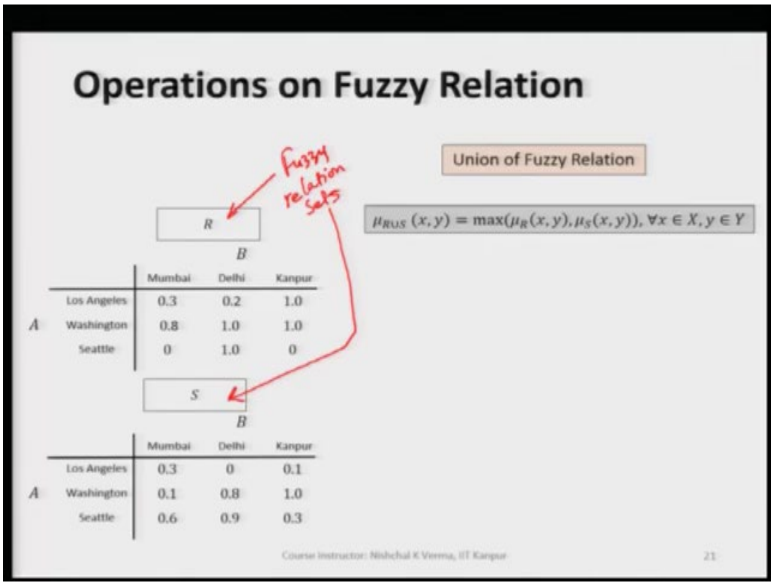
(Refer Slide Time 20:48)



And, similarly if we are interested applying the operations on fuzzy relation set. So, we can also do that and as we have already seen that, we have fuzzy relations set like this if we have an R fuzzy relation set and another fuzzy relation set S. So, we can represent this R fuzzy relation set like this.

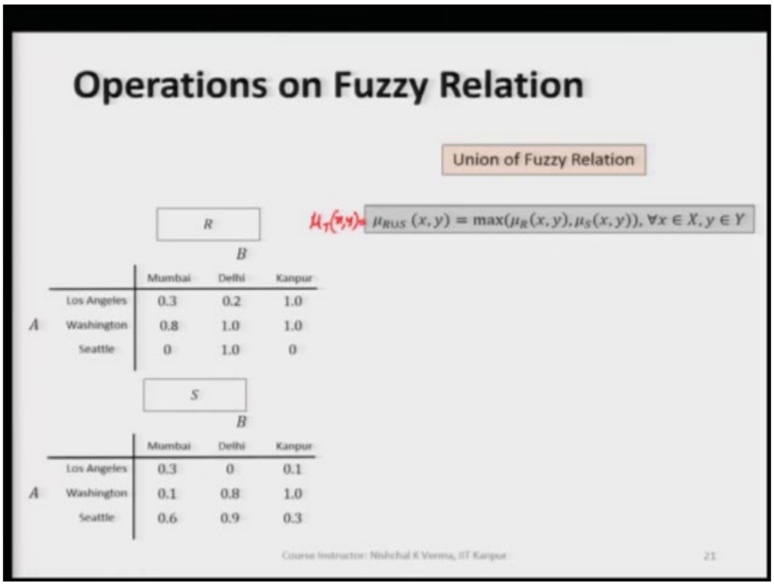
And, S fuzzy relation set like this and if we are interested in the applying the operations on in the fuzzy relations these R and S sets. So, let us now see as to how we can move ahead.

(Refer Slide Time 21:32)



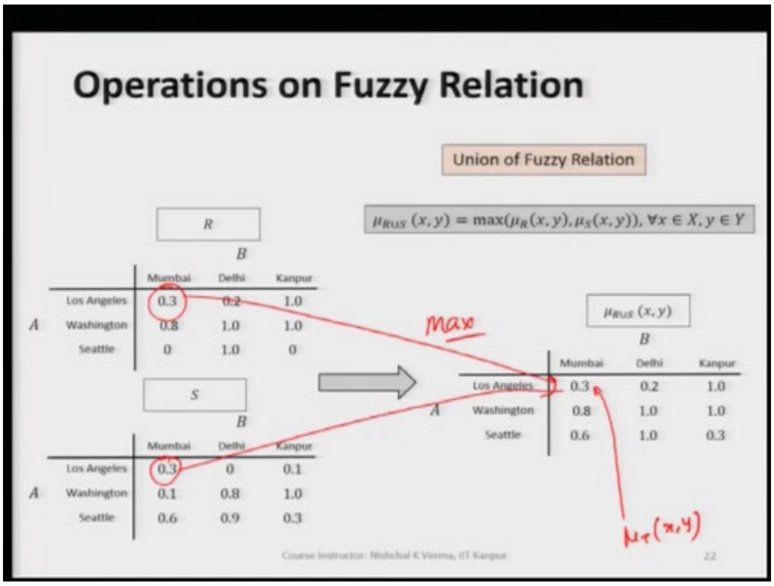
So, here basically our intention is to find the union of these two fuzzy relations. So, R is the fuzzy relation I am writing here set and S also. So, both of these are the fuzzy relation sets. Now, please recall as to how we can find the union of these two fuzzy relation sets. So, if you recall we see that we have fuzzy relation set, which is out of the union of the 2 R and S.

(Refer Slide Time 22:16)



And, the membership values we represent this by $\mu_T(x, y)$. So, this way we represent.

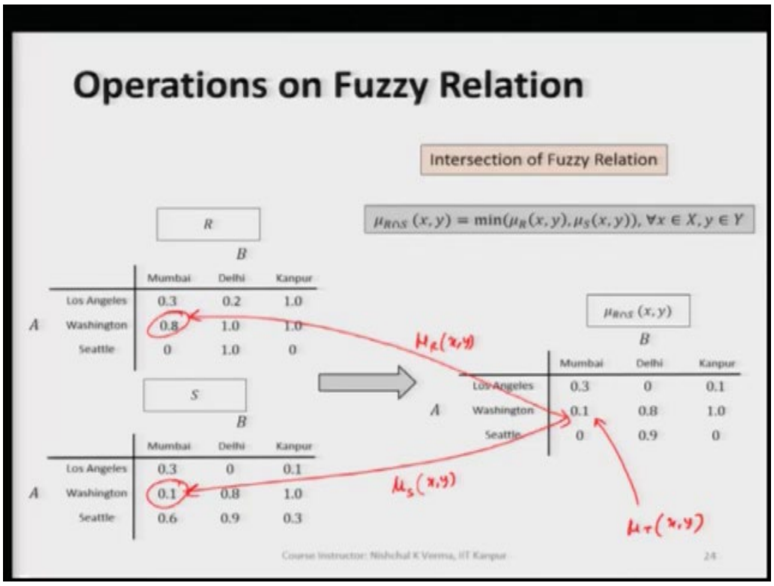
(Refer Slide Time 22:24)



And, when we are taking the union here. So, in union what we do, we recall that we have applied the max criteria and we take the max of the membership values corresponding to the ordered paired elements. So, here we see that when we take the union. So, we see that we have 0.3 here we have 0.3 here. And, if we take the union we use the max. So, here we apply max criteria. And, this way we get the $\mu_T(x, y)$.

So, all the elements of mu T can be computed by applying max very quickly. And, all these have been associated along with the generic variable values or the ordered pair elements like x, y in the resulting fuzzy religion set T.

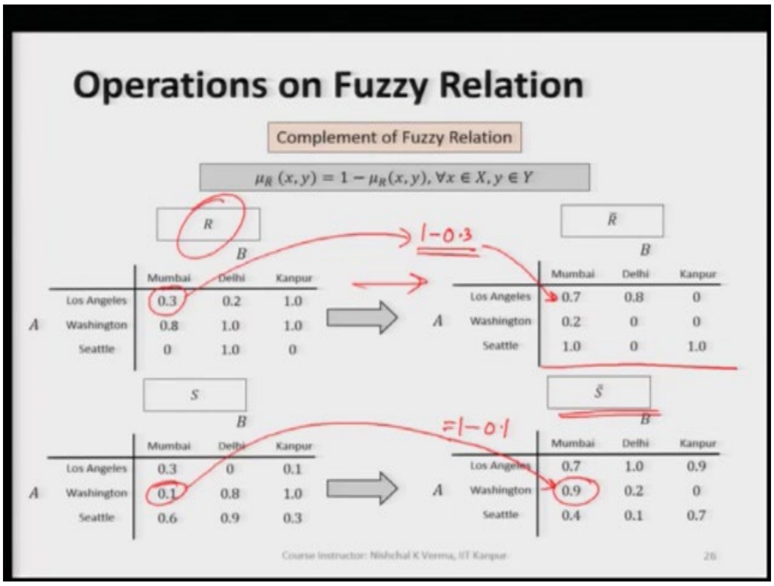
(Refer Slide Time 23:29)



And, so if this was the union when we take the intersection? So, in intersection instead of max we take the minimum and when we take minimum let us say we take this element and this element and corresponding to the ordered pair values we take min we are getting here 0.1.

So, this is nothing but the mu is as $\mu_S(x, y)$ and then this is $\mu_R(x, y)$ and this is here is $\mu_T(x, y)$. So, all these corresponding values can be very easily computed and this way we find the intersection of two fuzzy relations.

(Refer Slide Time 24:17)

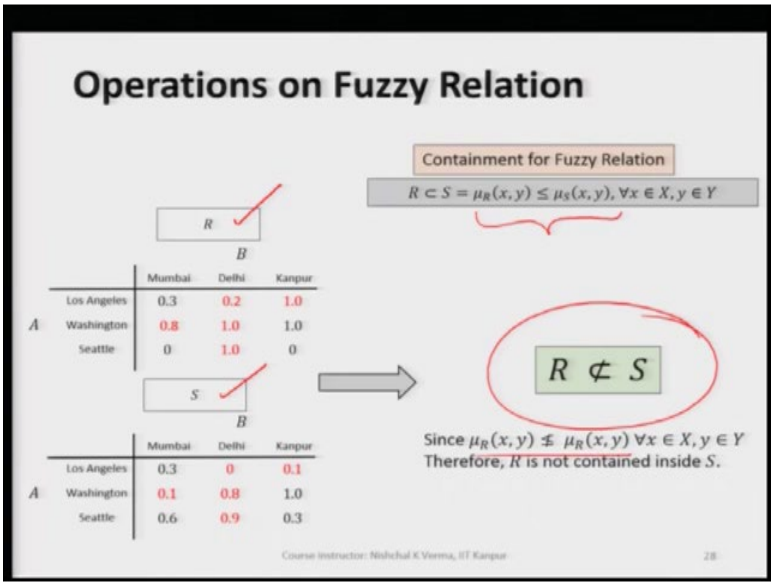


And, then when it comes to complement of fuzzy relation then we apply this criteria we simply subtract the corresponding membership values from 1, which is again the basic complement. And, if you wish you can apply any other complements that I have already taught in the previous lectures. So, if you want to get the complement of fuzzy relation capital R set. So, you can quickly write the fuzzy relation R set here.

And, then how to get this $\bar{R}$ is just a subtract all the corresponding elements here corresponding elements means you see these are the membership values and these membership values are subtracted from 1. So, when you subtract this value from 1 so that means, that we are subtracting 0.3 from 1 and this is going to give us the value which is 0.7. So, likewise all other values of the complement of fuzzy relation set we get and this way we managed to get the complement of any fuzzy relation set.

Similarly, we can get the complement of fuzzy relation set S and which is represented by $\bar{S}$. Here also if we see let us say we take 0.1 and the corresponding the element in $\bar{S}$ will be 0.9, because if I subtract 0.1 from 1 we are going to get 0.9. So, this is how we get the corresponding a membership value which is the membership value of the complement of a fuzzy relations set.

(Refer Slide Time 26:13)



So, this is how we get the these operations applied on fuzzy relations and now when it comes to the containment for fuzzy relations. So, similarly if fuzzy relation set $R \subset S$ then you see if this is the case then $\mu_R(x, y) \leq \mu_S(x, y)$. And, this is for every $x \in X$ and $y \in Y$. So, if we have two sets you see here. So, with these two fuzzy relation sets that we have if we put this condition here, so we find that R is not a subset of S, because this condition is not satisfied; that means, all the membership values of set R is not less than or equal to the membership values of relation set S. So, that is why we can say that $R \not\subset S$. So, this is mentioned over here. So, therefore, we can say R is not contained in S.

(Refer Slide Time 27:28)

So, this way we have seen that we have understood operations they complement intersection union, containment with respect to crisp sets and fuzzy sets in today's lecture and not only fuzzy sets, but I would say we have studied these operations on the fuzzy relation set of course, the fuzzy relation set is also a fuzzy set. We would like to stop here and in the next lecture, we will discuss the following the projection of fuzzy relation, cylindrical extension of projection, properties of fuzzy relations.

Thank you.