Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture No. 02
Vectors: unit norm vector, Cauchy-Schwarz inequality, Radar application

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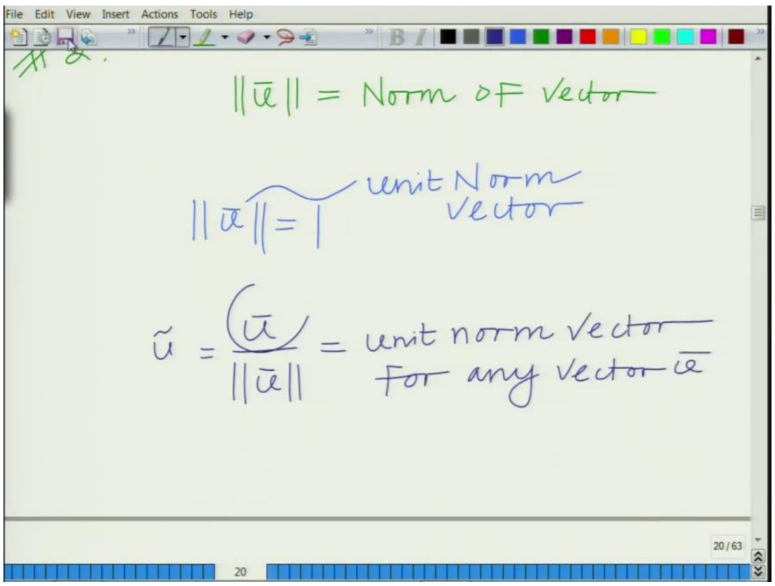

Hello, welcome to another module in this massive open online course on Applied Linear Algebra. So, we are looking at the norm of a vector and we have derived the norm of a vector and now let us continue our discussion. So, we have $||\bar{\boldsymbol{u}}||$ which is basically we say norm of the vector $\bar{\boldsymbol{u}}$.

And now we can also see some properties of norm. If norm of $||\bar{\boldsymbol{u}}|| = 1$, this is known as a unit norm. If norm of the vector that is length is 1, this is known as a unit norm vector. And any norm that is if you take any vector $\bar{\boldsymbol{u}}$ divided by $||\bar{\boldsymbol{u}}||$ this becomes a, that is if you define this vector $\tilde{\boldsymbol{u}}$ that is you take $\frac{\bar{\boldsymbol{u}}}{||\bar{\boldsymbol{u}}||}$, this becomes an unit norm vector for any vector $\bar{\boldsymbol{u}}$.

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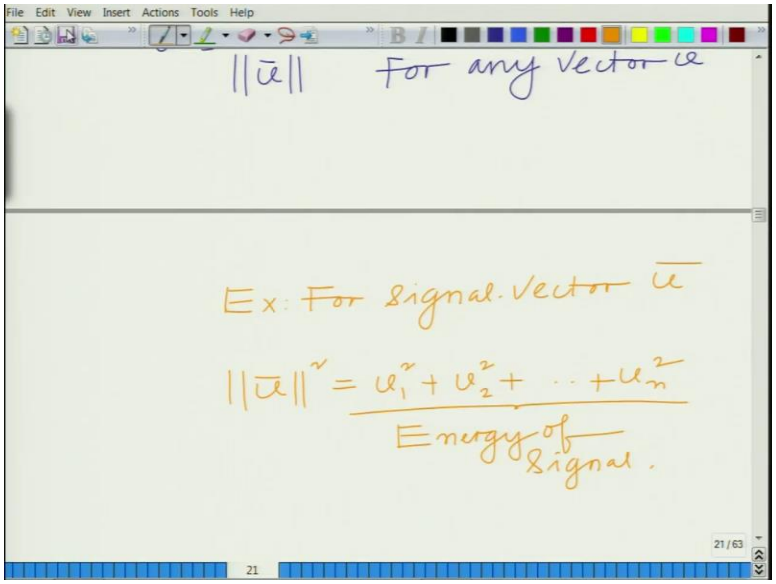

Now this norm has very interesting applications for a signal, for a signal vector. Example for a signal vector $\bar{\boldsymbol{u}}$, now $||\bar{\boldsymbol{u}}||^2$ equals, let us say we have a real signal, $||\bar{\boldsymbol{u}}||^2 = \sqrt{u_1^2 + u_2^2 + \cdots + u_m^2}$, this is basically the energy of the signal. So, the norm square of a signal vector denotes its energy.

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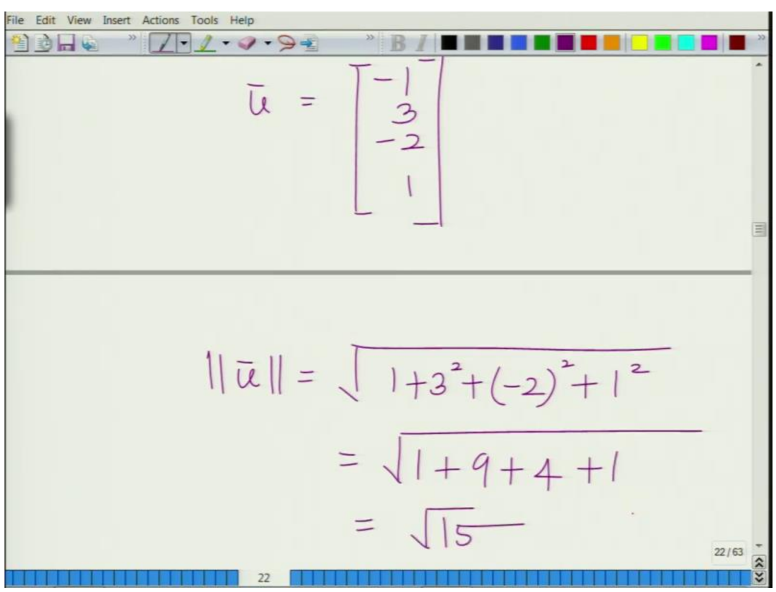

Let us look at a simple example, for instance, let us consider a vector $\bar{\boldsymbol{u}} = [-1,3,-2,1]$ and then we have $||\bar{\boldsymbol{u}}|| = \sqrt{1^2 + 3^2 + (-2)^2 + 1^2} = \sqrt{1 + 9 + 4 + 1} = \sqrt{15}$.

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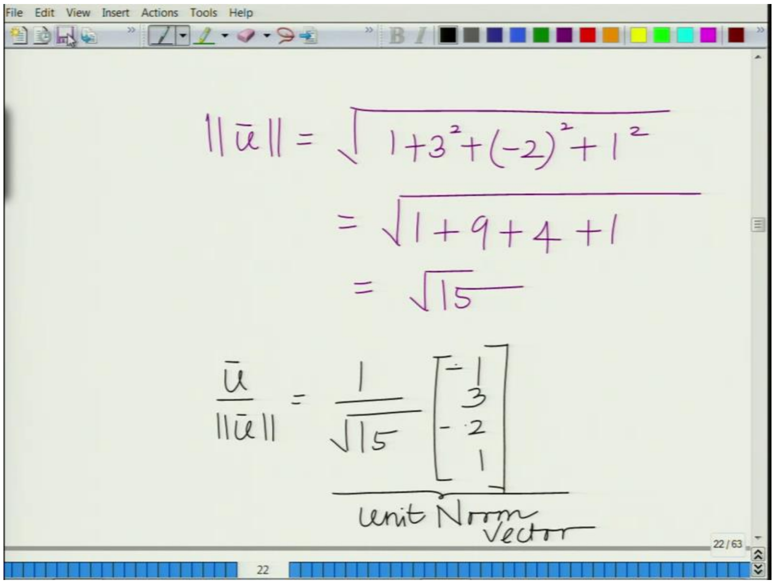

And the corresponding unit norm vector is given as

$$\frac{\bar{\boldsymbol{u}}}{||\bar{\boldsymbol{u}}||} = \frac{1}{\sqrt{15}} \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \end{bmatrix}.$$

Now this is basically your unit norm vector. So, that is basically you have computed the vector divided by its norm that makes the vector as a unit norm vector. Let us look at another very important application of the norm that is to find the distance between any two vectors and this is very important.

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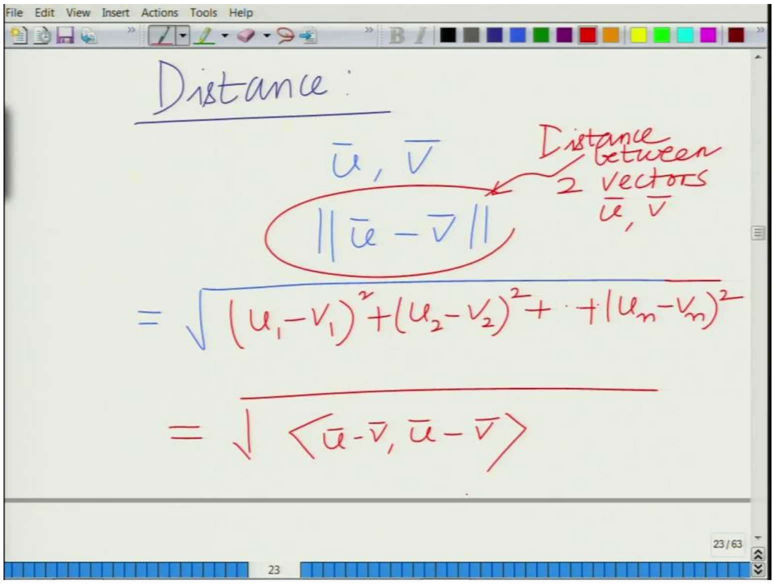

So, we have the distance, the notion of distance between two vectors that is where you have the vectors $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}$ that is these are the two points and the distance between them is defined as

$$||\bar{\boldsymbol{u}}-\bar{\boldsymbol{v}}|| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\cdots+(u_n-v_n)^2} = \sqrt{<\bar{\boldsymbol{u}}-\bar{\boldsymbol{v}}, \bar{\boldsymbol{u}}-\bar{\boldsymbol{v}}>}.$$

This basically gives the distance the distance between two vectors $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}$.

Let us now look at a very important property of the vectors related to the norm and which relates the norm of the vectors to the inner product, this is known as the Cauchy-Schwarz inequality, this is very important inequality, which has a large number of applications in Linear Algebra.

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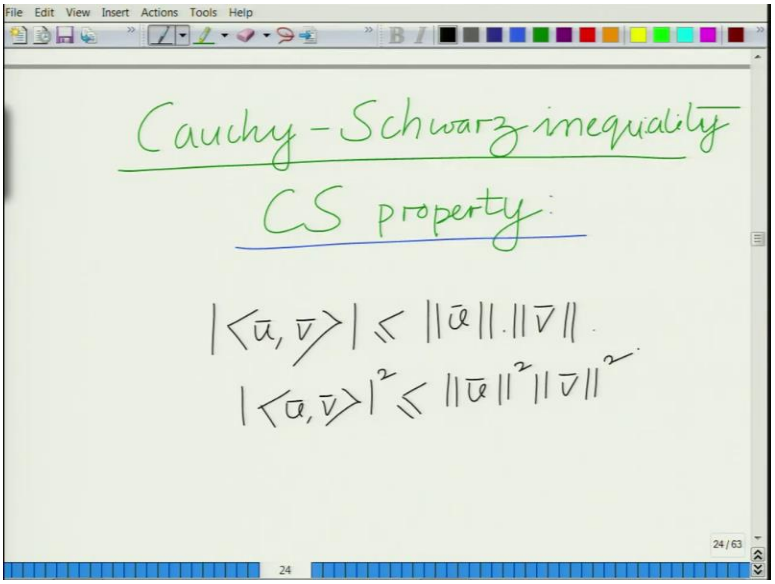

So, we have to look at Cauchy-Schwarz inequality, which is also called as the CS inequality or the CS property, or the Cauchy-Schwarz property, which basically relates the inner product to the norms. So, this simply states that the magnitude of the of two vectors $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}$ is less than equal to the product of their norms, that is

$$|< \bar{\boldsymbol{u}}, \bar{\boldsymbol{v}} >| \leq ||\bar{\boldsymbol{u}}||\,||\boldsymbol{v}||.$$

Or you can write this as

$$|< \bar{\boldsymbol{u}}, \bar{\boldsymbol{v}} >|^2 \leq ||\bar{\boldsymbol{u}}||^2\,||\boldsymbol{v}||^2.$$

This is a very interesting property, but it is for real vectors.

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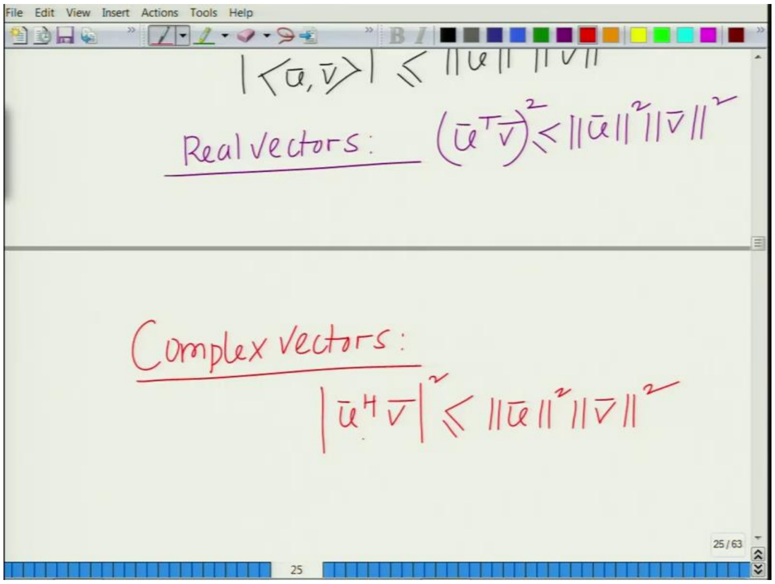
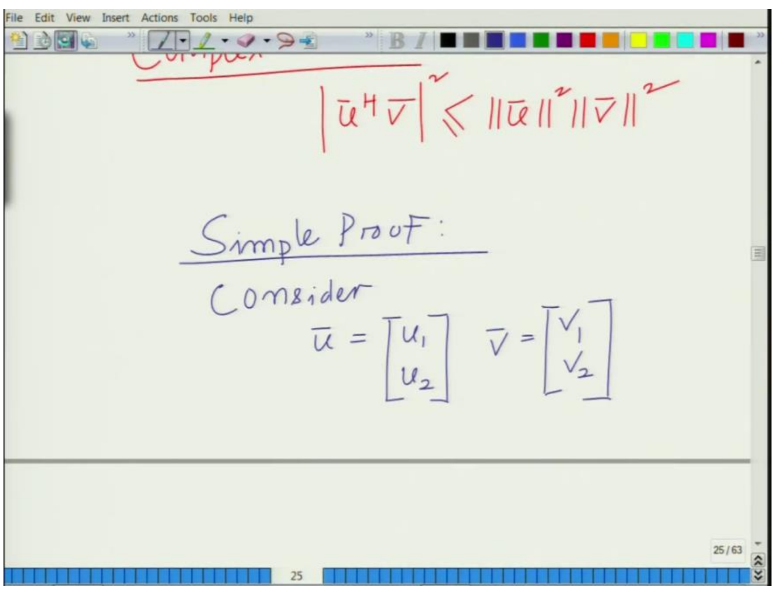


If you simplify this you can say

$$(\boldsymbol{u}^T\boldsymbol{v})^2 \leq ||\boldsymbol{u}||^2 ||\boldsymbol{v}||^2.$$

And for complex vectors one can say $(\boldsymbol{u}^H\boldsymbol{v})^2 \leq ||\boldsymbol{u}||^2 ||\boldsymbol{v}||^2$.

So, this is a very important property, which has a large number of applications.



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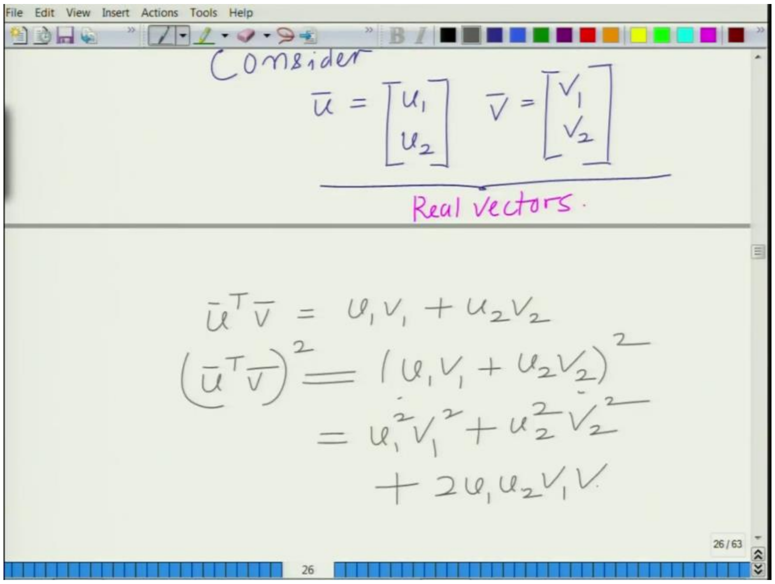
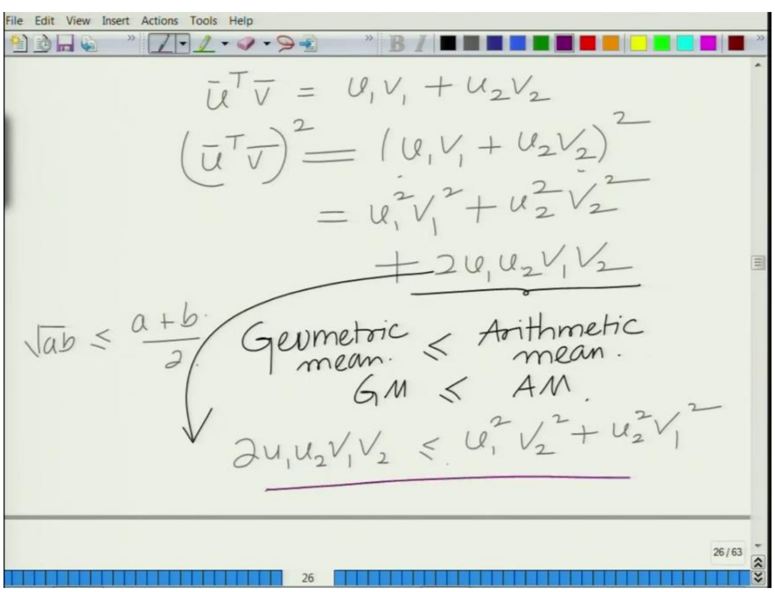


So, let us look at a simple proof, of course you can also look at a general proof. Let us just look at a simple proof of this for the two-dimensional case. Consider 2D real vectors and the general case is also fairly straight forward and then what we have is, since these are real vectors remember let us simplify this by assuming these are real vectors. Then we have

$$(\boldsymbol{u}^T \boldsymbol{v})^2 = (u_1 v_1 + u_2 v_2)^2 = u_1^2 v_1^2 + u_2^2 v_2^2 + 2 u_1 u_2 v_1 v_2.$$



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Now, if we use the inequality, that is, we use the geometric mean (GM) is less than or equal to the arithmetic mean (AM) that is GM less than or equal to AM. Then this reduces to

$$2u_1u_2v_1v_2 \leq u_1^2v_2^2 + u_2^2v_1^2.$$

So, geometric mean less than arithmetic mean simply states that if you are not familiar with that, that simply states the $\sqrt{ab} \leq \frac{a+b}{2}$. And that is essentially, if you look at it and clearly see that is essentially being used over here.

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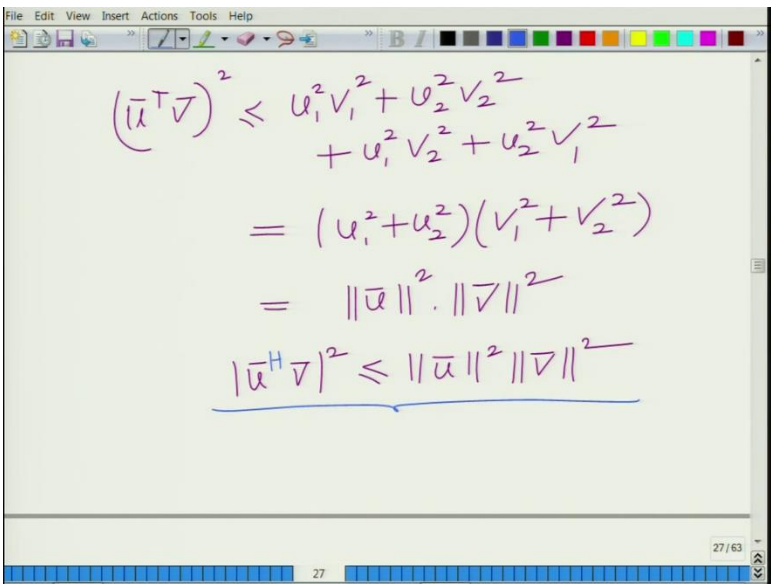

And therefore, now you can write

$$(\boldsymbol{u}^T\boldsymbol{v})^2 \leq u_1^2v_1^2 + u_2^2v_2^2 + u_1^2v_2^2 + u_2^2v_1^2 = (u_1^2 + u_2^2)(v_1^2 + v_2^2) = \left||\bar{\boldsymbol{u}}|\right|^2\left||\bar{\boldsymbol{v}}|\right|^2,$$

So that is essentially the Cauchy-Schwarz inequality. I think to cover both the general complex and real case one can simply write this

$$(\boldsymbol{u}^H\boldsymbol{v})^2 \leq \left||\bar{\boldsymbol{u}}|\right|^2\left||\bar{\boldsymbol{v}}|\right|^2.$$

So, this covers in fact both the real case and the complex case. And let us look at a simple example, so this is a very important example essentially as a large number of applications this Cauchy-Schwarz inequality. Let us just look at the simple example just using some 2 dimensions, some real vectors.

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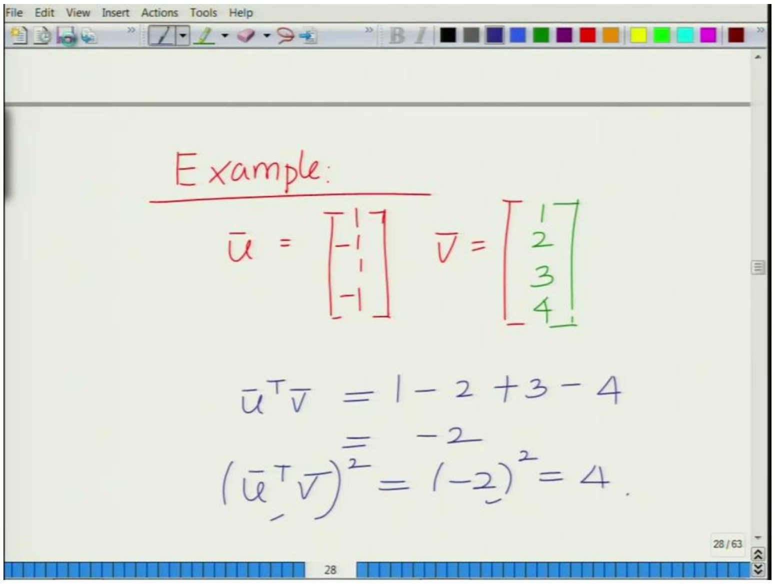

So, let us look, we have $\bar{\boldsymbol{u}} = [1, -1, 1, -1]^T$, and $\bar{\boldsymbol{v}} = [1, 2, 3, 4]^T$ these are some simple vectors. So, we have

$$\bar{\boldsymbol{u}}^T\bar{\boldsymbol{v}} = 1 - 2 + 3 - 4 = -2,$$

$$(\bar{\boldsymbol{u}}^T\bar{\boldsymbol{v}})^2 = (-2)^2 = 4.$$

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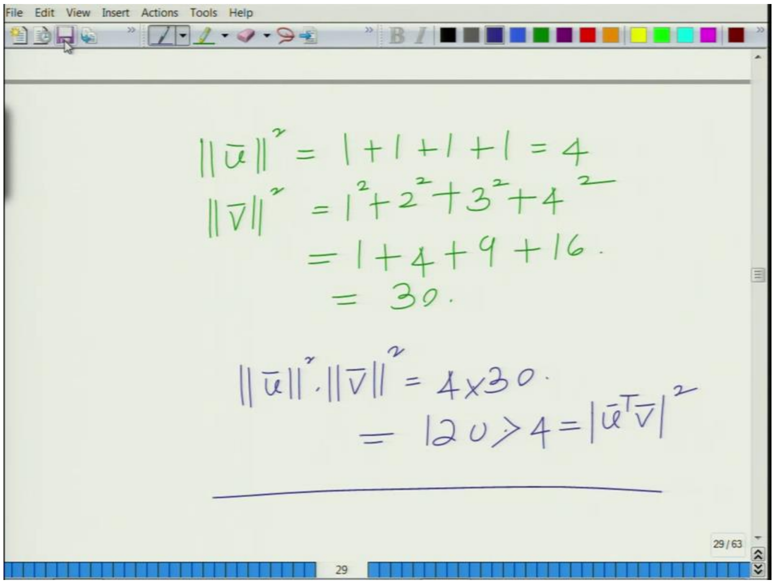

And now $||\overline{\boldsymbol{u}}||^2 = 1 + 1 + 1 + 1 = 4$, and $||\overline{\boldsymbol{v}}||^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30.$

And therefore, you can see $||\overline{\boldsymbol{u}}||^2||\overline{\boldsymbol{v}}||^2 = 4 \times 30 = 120$, which is much, which is greater than 4, which is equal to you can say $|\boldsymbol{u}^T\boldsymbol{v}|^2$. So, this is a simple example to illustrate this Cauchy-Schwarz inequality.

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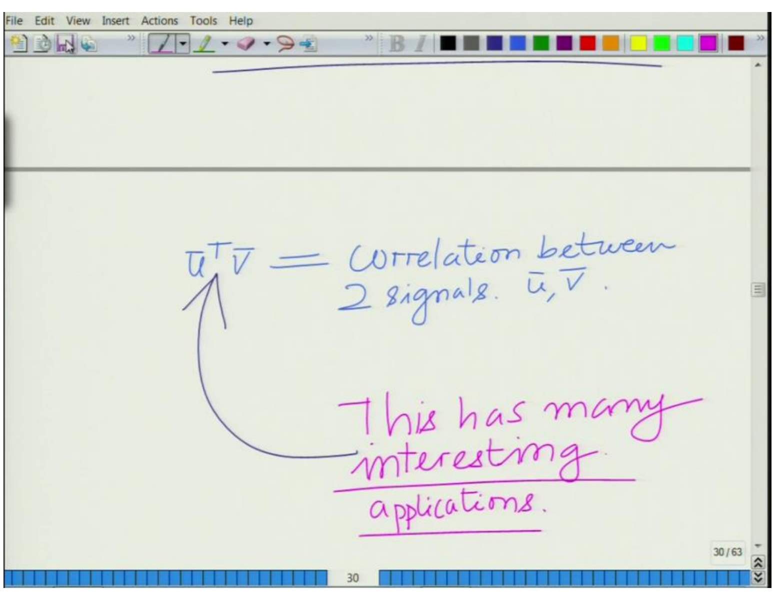

Now, this $\boldsymbol{u}^T\boldsymbol{v}$ this has very important connotations, so this $\boldsymbol{u}^T\boldsymbol{v}$ this is termed as the correlation between the two signals, and this has many practical applications, very interesting applications. I would say high-impact interesting applications.

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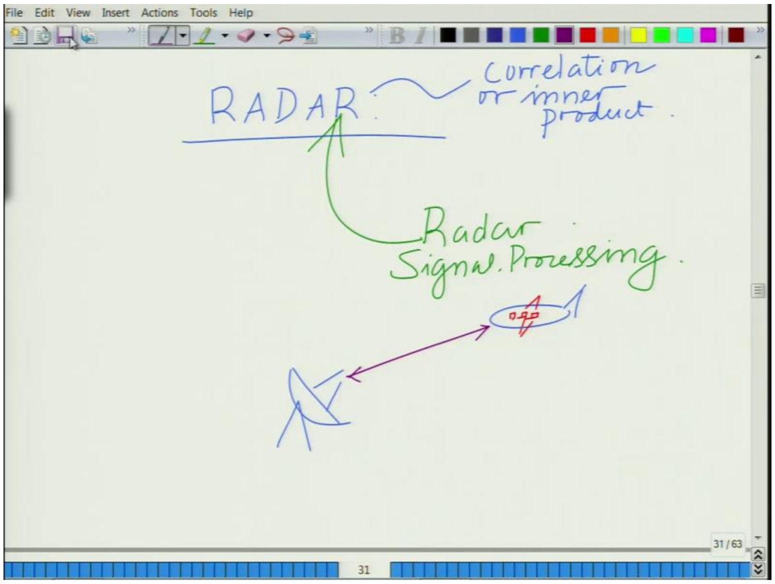

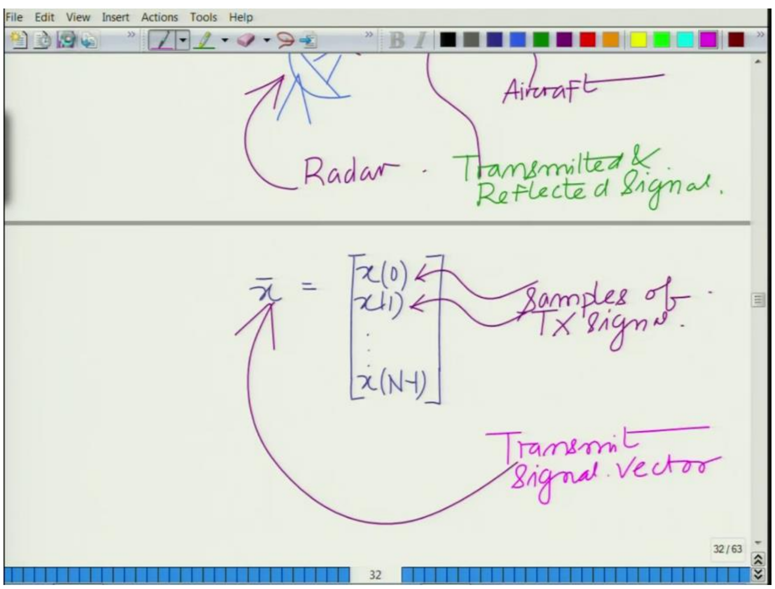

Let us look at a simple example application of correlation or inner product. Let us look at a simple application in radar. So we want to look at the application of correlation or basically your inner product in this RADAR in this field of RADAR. What is RADAR signal processing, all of you will know if you have a RADAR which is essentially trying to detect a plane.

Therefore, what is going to happen is the radar transmits a signal and then there is a reflection. So this is simple schematic diagram, so this is basically your radar and this is basically your object that you are trying to detect, that is, for instance an aircraft. And this is essentially your ground radar and this is basically transmitted and reflected signal. So, you have the transmitted and reflected signal.

So, let us say $\bar{\boldsymbol{x}}$, so naturally we consider a digital signal. So, let us say $\bar{\boldsymbol{x}}$ equals x_0, x_1, these are the samples of the transmitted signal. So, this is the transmit signal vector.

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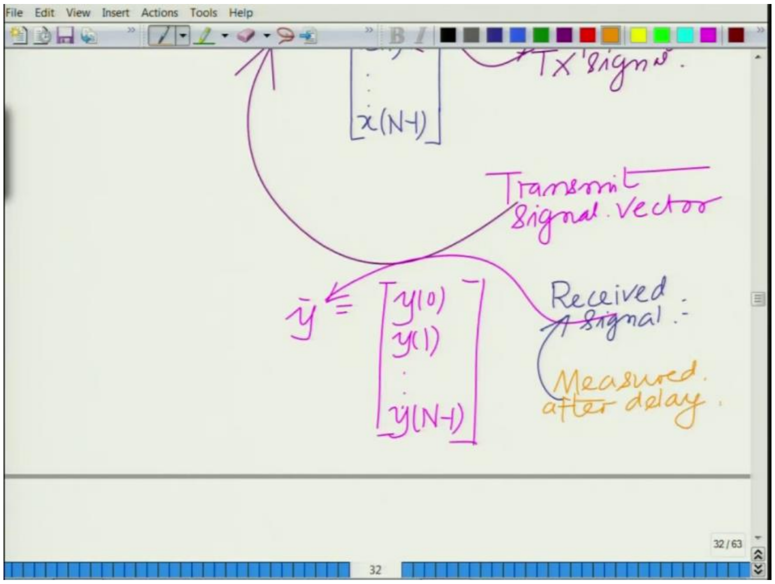

And let us say this is your $\bar{\boldsymbol{y}}$**,** which is a reflected signal. Again, you have the samples of the reflected signal or the received signal. You are transmitting a signal from the RADAR and you are measuring the response at the radar after a certain delay. So, this is your received signal which is measured after a certain delay.

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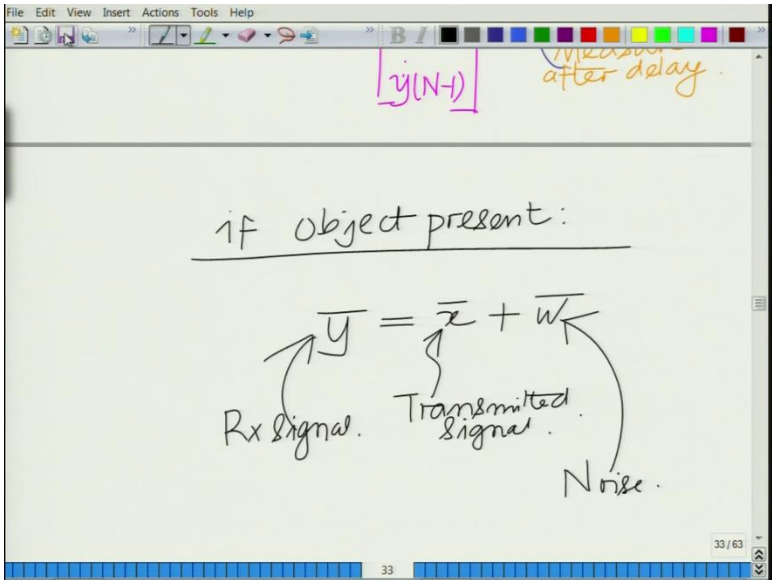

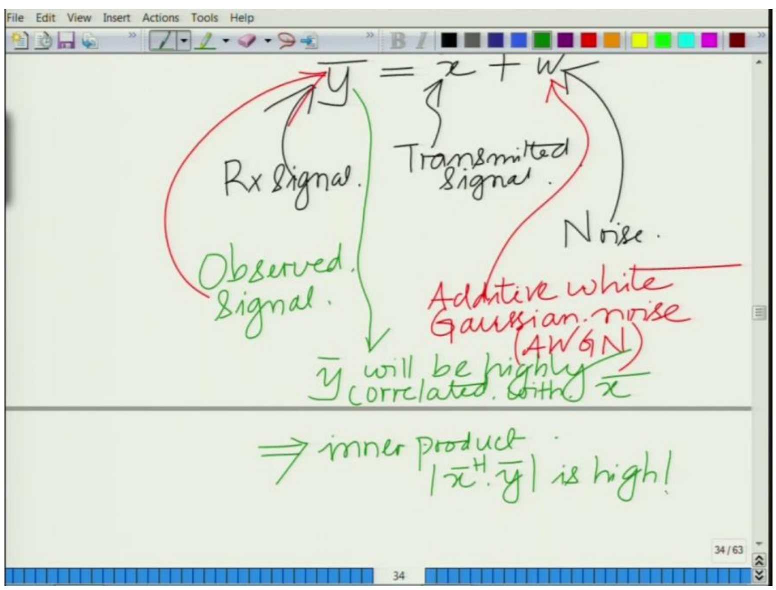

Now, the point here is that if there is an object such as an aircraft that is present then what is going to happen is that your received signal typically is $\bar{\boldsymbol{x}}$ plus the noise vector $\bar{\boldsymbol{w}}$. So, $\bar{\boldsymbol{x}}$ is your transmitted signal, $\bar{\boldsymbol{y}}$ is your received signal and this is simply the noise vector $\bar{\boldsymbol{w}}$. So, what it is saying is, of course there is going to be noise, remember because it is a practical communication system, so along with the signal you will have the noise.

So, what happens is that the transmitted signal $\bar{\boldsymbol{x}}$ is reflected and then of course you are going to have noise. So, the observed signal $\bar{\boldsymbol{y}}$ at the RADAR is going to be simply a copy of the transmitted signal plus the noise. Typically, the noise is assumed to be Gaussian this is also what is called as additive white Gaussian noise. So, although it is not very important you might understand this better, this is called as, this noise is additive, this is additive white Gaussian noise.

Therefore, in this case the observed signal $\bar{\boldsymbol{y}}$ will be highly correlated with $\bar{\boldsymbol{x}}$. If the object is present $\bar{\boldsymbol{y}}$ will be highly correlated with the transmitted signal $\bar{\boldsymbol{x}}$ or essentially that implies that the inner product, the magnitude of the inner product that is what we are calling as for instance we can call this as if these are complex signals in general $\bar{\boldsymbol{x}}^H\bar{\boldsymbol{y}}$ is very high because the copy of the signal is present in $\bar{\boldsymbol{x}}$, so the similarity between $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$ will be high, you get the point because $\bar{\boldsymbol{y}}$ is the observed signal, $\bar{\boldsymbol{x}}$ is reflected, so the similarity between $\bar{\boldsymbol{y}}$ and $\bar{\boldsymbol{x}}$ will be very high. On the other hand if the object is not present then it will simply be noise because there is no reflected signal.

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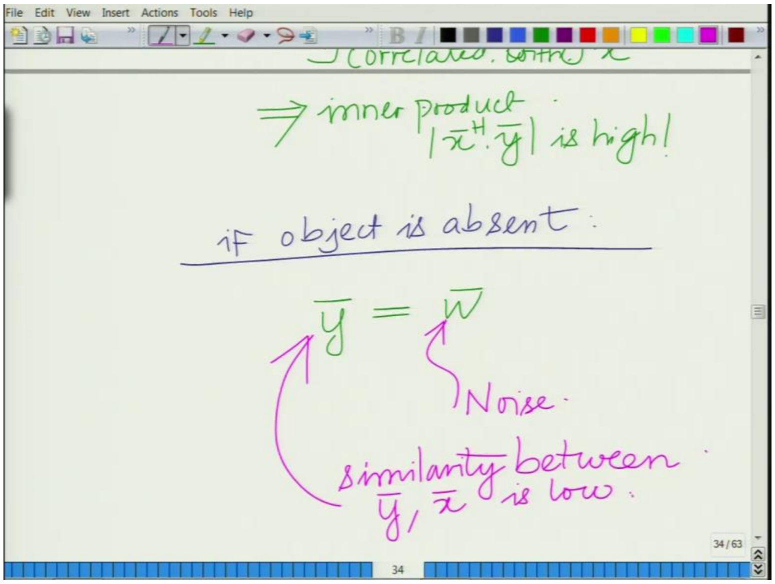

So, on the other hand if there is no object or if object is absent, then we are going to have $\bar{\boldsymbol{y}} = \bar{\boldsymbol{w}}$, that is, simply the noise. So, there is going to be no $\bar{\boldsymbol{x}}$, that is, there is going to be no reflected signal. So, in this case, because it is simply the noise, similarity between $\bar{\boldsymbol{y}}$, $\bar{\boldsymbol{x}}$ is low. Therefore, to determine the presence of target, this is called object or this is also known as target in the context of RADAR. So, to determine if a target is present or not? simply correlate. So, correlate the received signal with the transmitted signal, so what we do to determine the presence of target, simply correlate the observed signal with the transmitted signal, if this correlation is high, then the target is present, if the correlation is low, then the target is absent.

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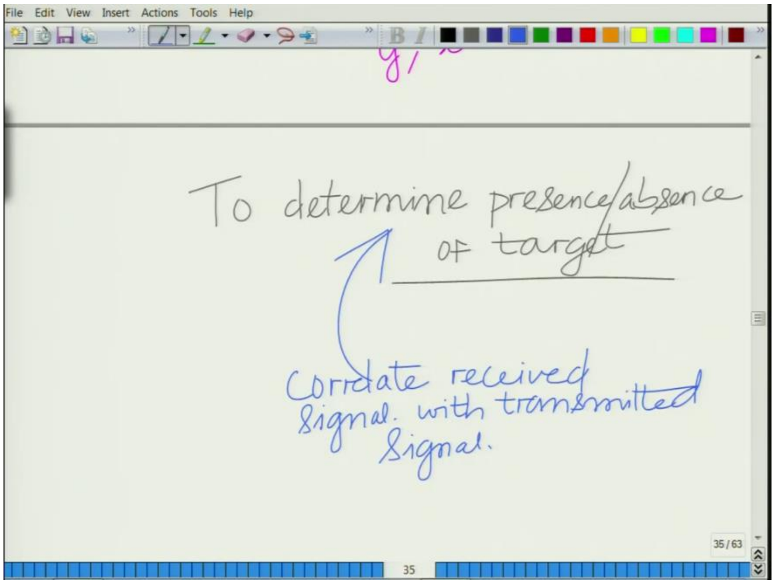

So, to determine presence of target we correlate received signal with the transmitted signal.

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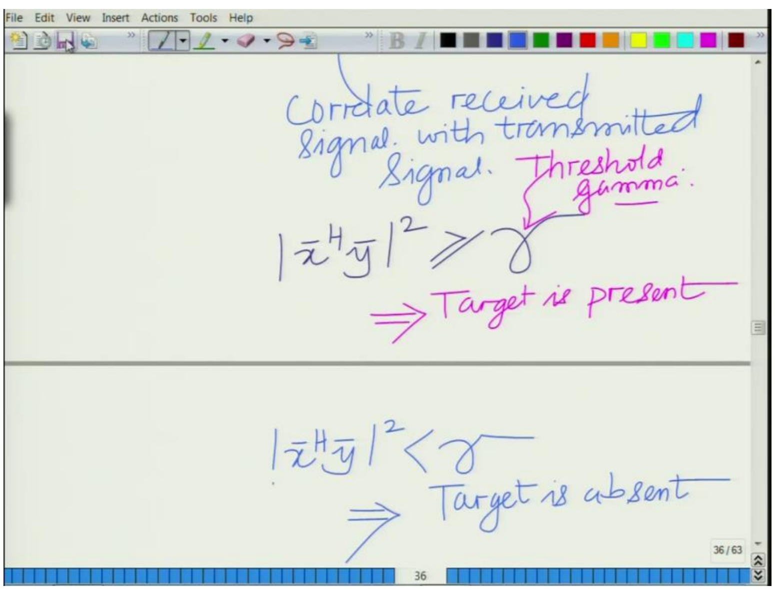

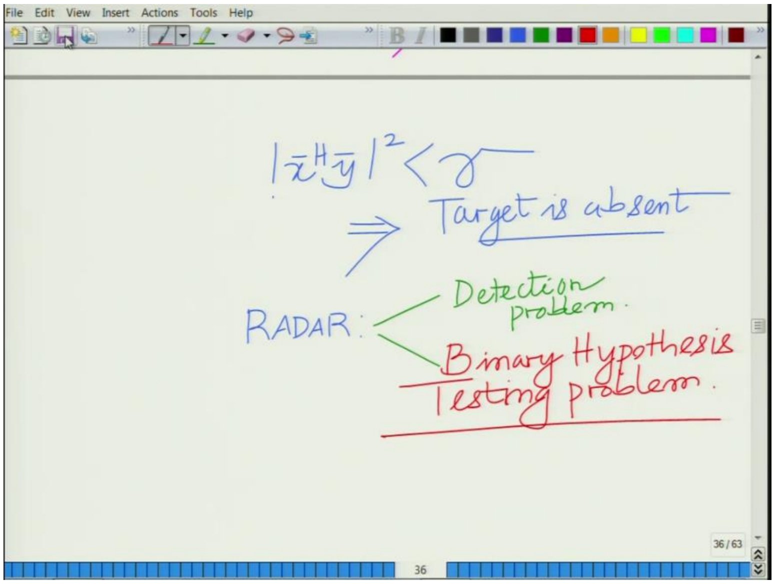

So, what we do is, we look at the inner product that is $|\bar{\boldsymbol{x}}^H \, \bar{\boldsymbol{y}}\,|^2$ and we compared it with the threshold γ. So, we have a significant, because remember there is noise, so we have to have a threshold, so we compare it with the threshold, the threshold is γ, so if it is greater than threshold this implies that target is present, else implies target is absent.

This is also known as a hypothesis, a detection problem or this is essentially a RADAR problem. This is essentially one of the most fundamental problem this is also known as a detection problem or this is also known as hypothesis testing. In fact, that is there are two hypotheses, the target is present or absent and you are testing each hypothesis. So, this is also known as the hypothesis testing, and more specifically a binary hypothesis testing problem.

So, another interesting application is this as already told you the similarity between two signals or images.

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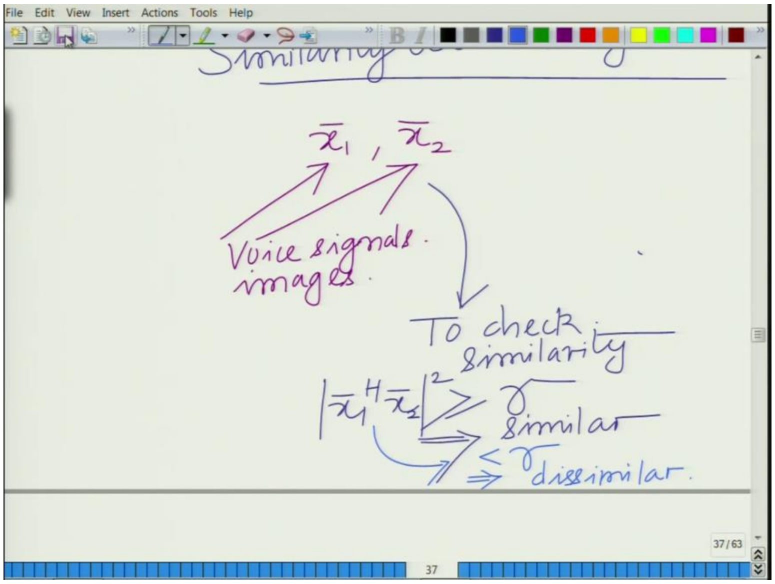

So, we can also use it to determine for instance you have two signals $\bar{\boldsymbol{x}}_1$ and $\bar{\boldsymbol{x}}_2$. Let us say these are two images or audio signals, so on and so forth, I mean signals can be anything, voice signals images. To determine the similarity, to determine, to see if they are similar, to check the similarity you again take the inner product that is magnitude or essentially the correlation $|x_1^H x_2|^2 \leq \gamma$, this implies, that they are similar, if the same thing is less than γ this implies, they are dissimilar. So, that is a very interesting property.

So, this property of the inner product or the correlation between these two, in fact, this is the correlation between these two data vectors or the correlation between these two signals has several key interesting applications and this is a very important concept. We are going to see another important application of this in the context of wireless communication in the next module. Thank you very much.