Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture – 33
Multiple Signal Classification (MUSIC) Algorithm System Model

Hello, welcome to another module in this massive open online course. So, in this module, let us look at yet another very interesting application of linear algebra, analysis linear systems.

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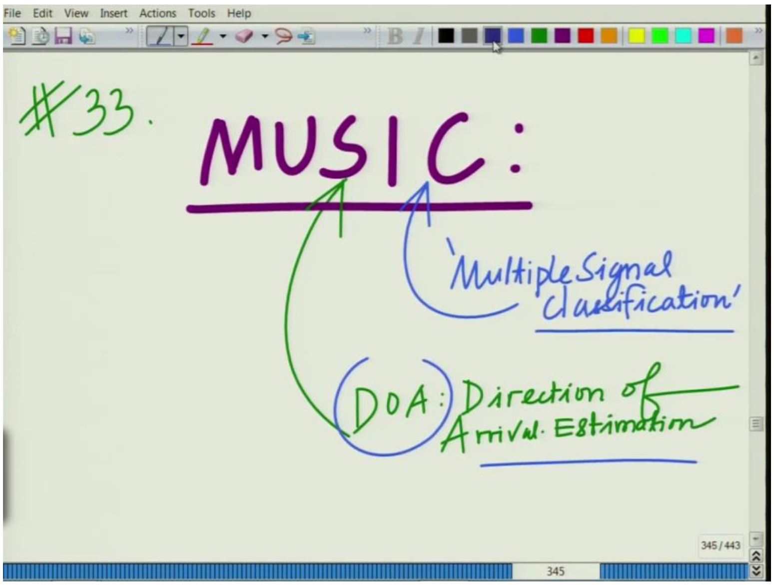

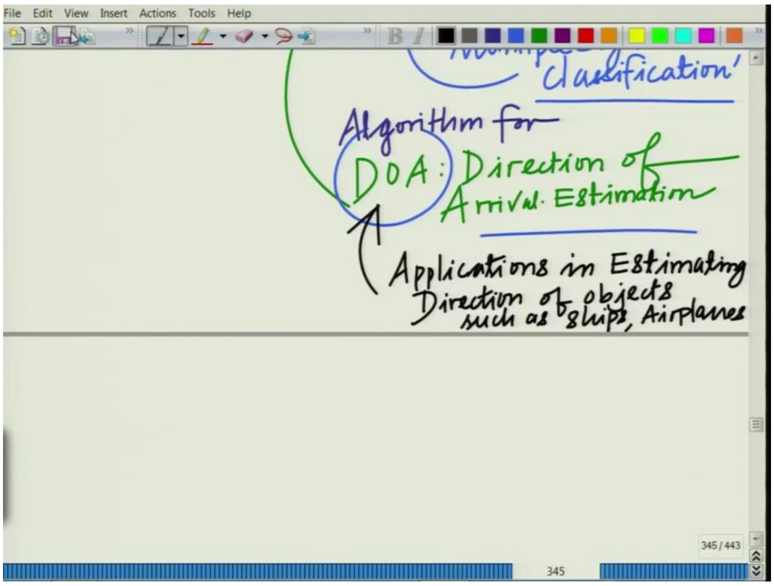

And that is a very powerful algorithm which is known as music algorithm. And do not worry, this is not music in the traditional sense, although in the sense it is also related to signal processing the music here is an acronym, which essentially stands for Multiple Signal Classification.

So, this music essentially stands for Multiple, this stands for Multiple Signal Classification. And what is the music algorithm used for? This is used basically for DoA or Direction of Arrival Estimation which has naturally significant applications in practice that is to estimate the direction of arrival of a particular signal, this is termed as DoA, that is the Direction of Arrival.

And the music is essentially an algorithm. Music is basically an algorithm for direction of arrival estimation that is to estimate the direction of arrival of a signal and you can see, naturally this can have, this has significant applications in practice to estimate the direction wherein a signal is coming from, and hence the direction of the object which is emitting that signal for instance be it either a ship or an aircraft and naturally you can see there are several applications for instance, the tracking of civilian aircraft or in the tracking of fighter jets and so on and so forth, so it has a lot of applications. So, this has a lot of applications in estimating the location of objects or applications, what we call as ranging. Or we can simply say, estimating the direction, objects such as for instance ships or airplanes, etc.

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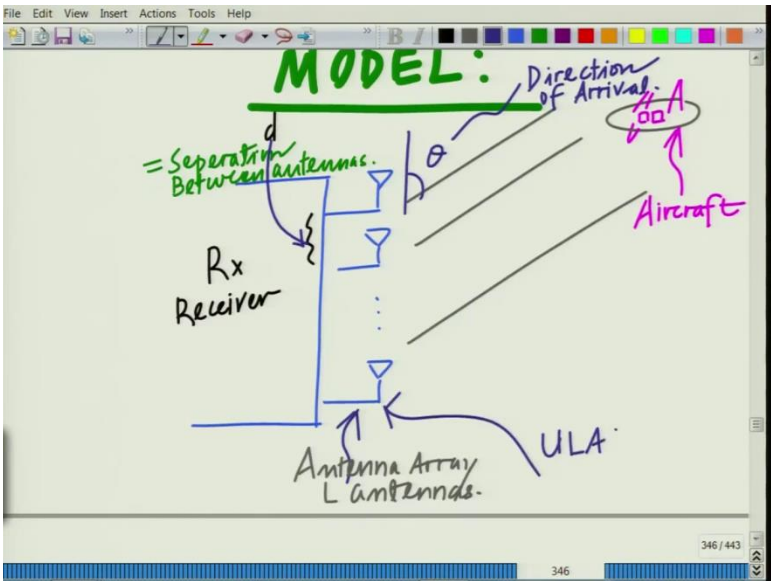

Now first let us develop the model for because the model is rather interesting and involved. So, let us spend some time on developing the model for this music technique. So, what is the model for this music technique? And the model can be described as follows and this involves multiple antennas. So, to estimate the direction of arrival, naturally, you need multiple antennas so that you can differentiate the signals arriving from different directions, and let us say there is an object such as for instance an aircraft that is, so this is my cartoon of an aircraft that is emitting these signals which are coming at a direction of theta. So, this is essentially your direction of arrival. Right.

This is essentially your antenna array with L antennas, this is the receiver, and an important parameter here let us say we have L antennas in this antenna array, an important parameter here is the distance between the antennas, this is termed as d, d is the separation between, so d here is essentially the d this is basically the separation between, this is essentially the separation between the antennas and what we are saying is this d that is this separation between the antennas is fixed that is these different antennas are spaced at a… This is an array of antennas, a linear array and the spacing between the antennas is uniform that is d and therefore, this is also known as a uniform linear array okay. So, more specifically this is known as ULA.

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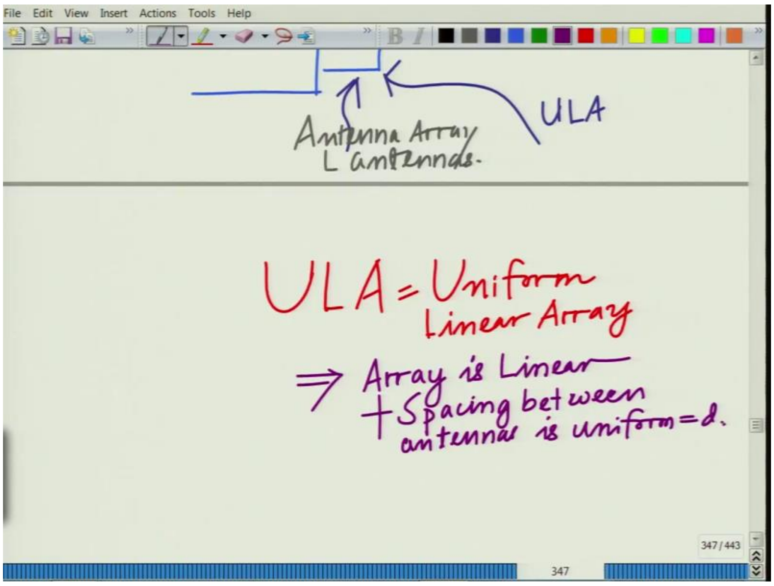

ULA stands for and this is very important concept in signal processing, ULA stands for a Uniform Linear Array implies, which essentially implies that array is linear plus spacing between antennas is uniform which is equal to d, the spacing between the antenna antennas is equal to d.

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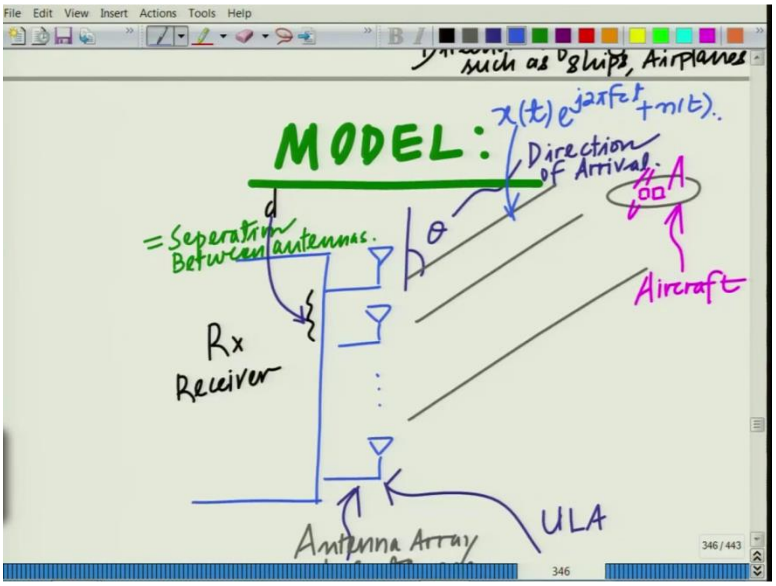

And now, let us consider the signal that is arriving at the first antenna to be X t e raise to minus j 2 pi fct that is the signal that is arriving here to be X t e raise to or x t you can say x t e raise to j 2 pi fct plus of course, there will be the noise. So, let me draw this again just to illustrate this clearly.

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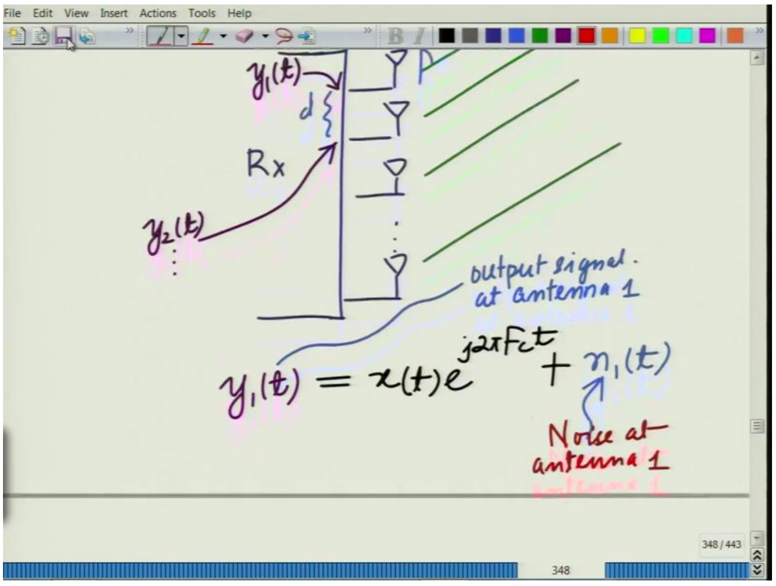

So, you have the antenna array and you have the signal that is arriving at an angle of theta and this is the distance between the antenna elements d and this is the signal x t, you can say this is x t into e raise to j 2 pi. This is the signal at the first antenna element that is x t into e raise to j 2 pi fct. So, we have y t So, the signal at the first antenna is we can write this as y t or y 1 t if I recall that as the signal arriving as the output signal at the first antenna. So, this is essentially your y 1 t this will be your the second antenna y 2 t and so on. And I can therefore write y 1 t equals x t into e raise to j 2 pi f c t plus the noise alright. Or the noise that is n 1 t that is the noise at the output of the first antenna. So, this is output signal, this is the output signal at antenna 1 and this n 1 t naturally this is the noise at antenna 1, this is the noise at antenna 1. Very good.

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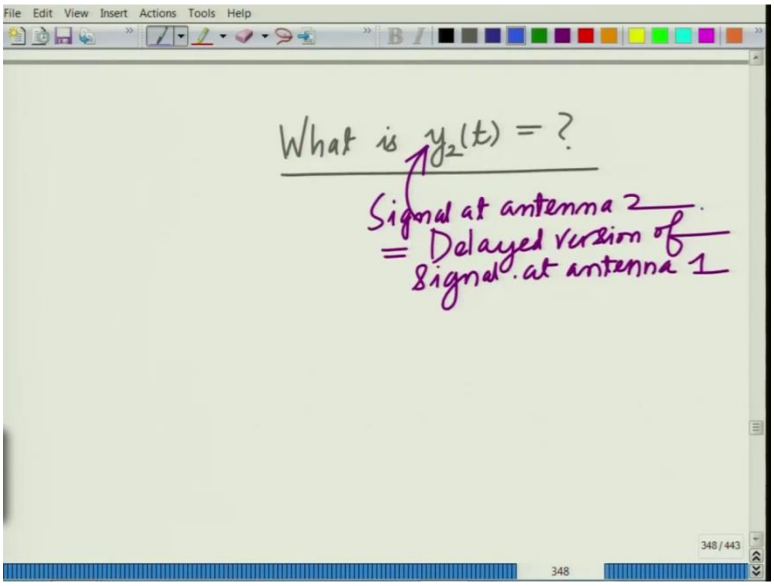

And now we ask the question, what is the signal what is the signal y 2 t which is the output of antenna 2. And naturally you can see the signal at antenna 2 will be related to antenna 1, and in fact the signal at antenna 2 is going to be a delayed version of the signal at antenna 1 and how do we capture this in our model? So, you can see the interesting observation that signal at antenna 2 is our delayed version is a delayed version of the signal at antenna 1 and we would like to capture this, how do you capture this, how do you capture this phenomenon?

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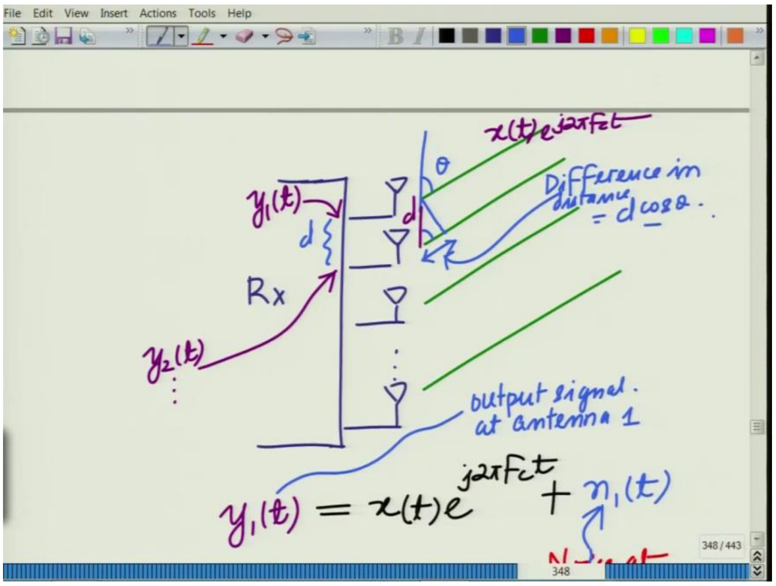

And the idea is very simple, if you look at the difference in the distance right. So, this is basically your d and if you look at this is angle theta and therefore, this distance if you look at the difference in the propagation distance, difference in distance this is equal to d cosine theta, this is the difference in propagation distance.

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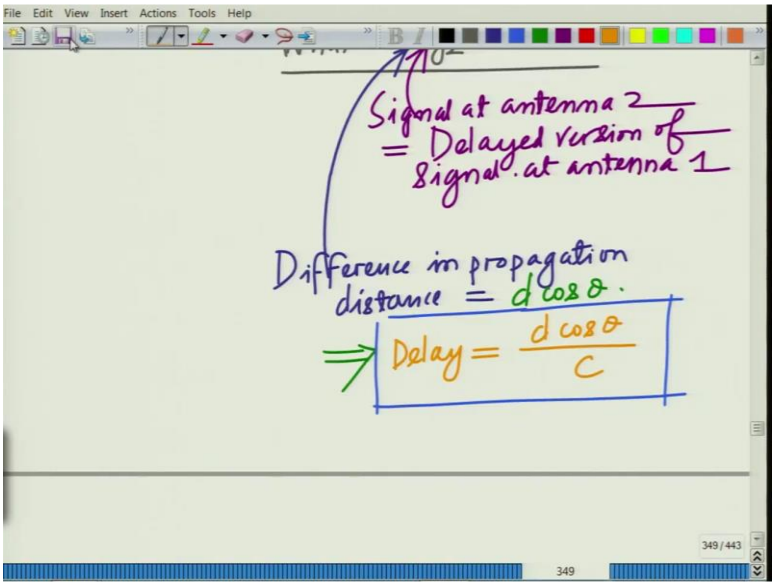

So, the difference in propagation distance at antenna 2 we have, difference in propagation distance. The difference in propagation distance is d cosine theta which implies the delay this is interesting, the delay is distance divided by velocity so this is equal to d cosine theta divided by C. So, at the signal at the second antenna at the second antenna, the signal is delayed by this quantity d cosine theta over C where d is essentially if you remember this is the inter antenna spacing, theta is the angle of arrival right, remember that is the direction of arrival.

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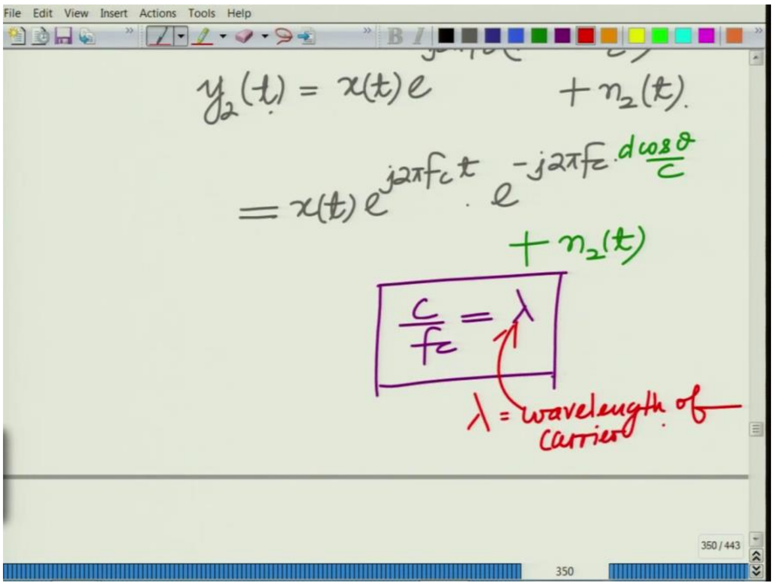

Now therefore, the signal y 2 t is going to be is going to be the same signal x t times e raise to j 2 pi f c, and now the delay is captured by t minus the delay which is essentially d cosine theta divided by c plus of course the noise is going to be independent right because this is the noise at a different antenna. And now I can write this as x t e raise to j 2 pi f c t times e raise to minus j 2 pi f c d cosine theta over C plus n 2 t. Now, use the property C by f c is equal to lambda right. What is lambda? Lambda is the wavelength of the carrier implies this becomes.

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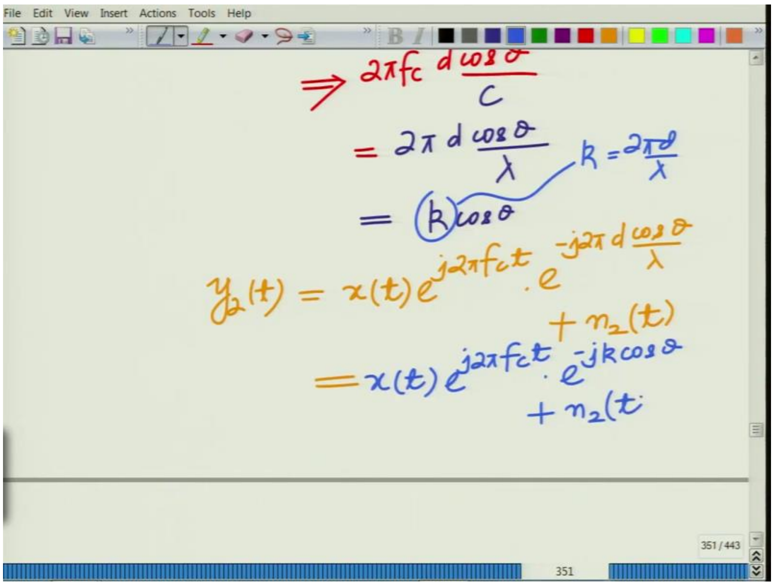

Implies if you look at this, you have the 2 pi f c d cosine theta over C, I can write this as, well d cosine theta over C, C by f c lambda so I can write this as 2 pi d cosine theta over lambda over 2 pi d over lambda cosine theta which I can write as k some constant k times cosine theta with k equals 2 pi d over lambda. And therefore, now if you look at this, this becomes y 2 t equals y 2 t equals x t e raise to j 2 pi f c t times e raise to minus j 2 pi f c d cosine theta over 2 pi f c 2 pi d cosine theta over lambda plus n 2 t which I am writing as which I am going to write as x t e to the power of j 2 pi f c t e raise to minus j times k cosine theta plus n 2 of t.

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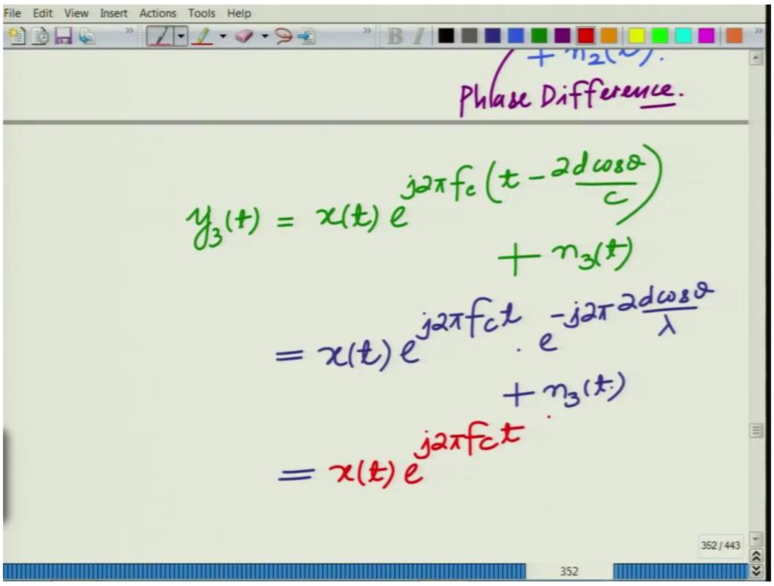

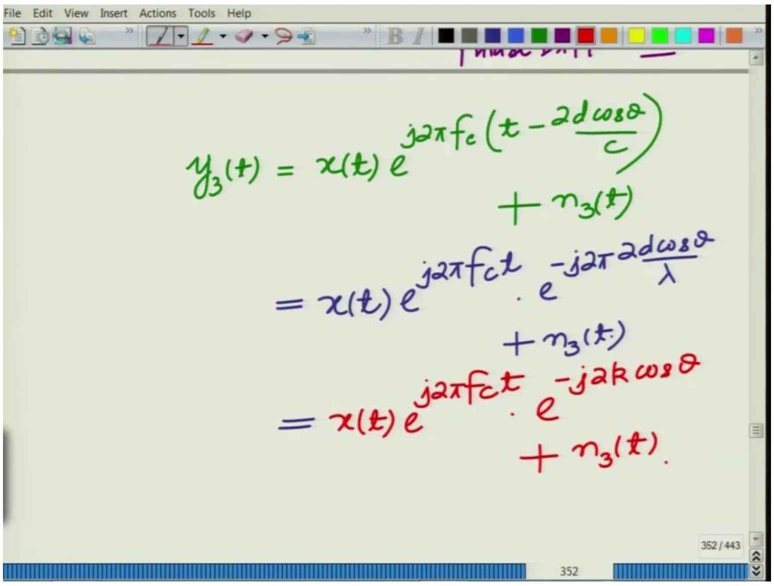

And similarly, so essentially, we are using the property that where you have the signal received at the second antenna is a delayed version of the signal received at the first antenna.

And in fact, if we look at this that delay depends on the angle theta because it depends on cosine theta, and it in turn translates into a phase difference.

So, there is a phase difference you can clearly see that in the received signal, so there is a phase difference right. So, there is a phase difference, now similarly you will have y 3 t this is equal to x t e raise to j 2 pi f c d minus, now the delay is going to be 2 d cosine theta divided by C plus n 3 t, which is equal to which is equal to x t e raise j 2 pi f c t e raise to minus j which will be e raise to minus j 2 pi 2 d cosine theta over lambda plus n 3 t which is essentially equal to x t e raise to j 2 pi f c t times e raise to minus j 2 pi, e raise 2 minus j 2 k right this is 2 d cosine theta over lambda. So, this will be 2 k cosine theta plus n 3 t and so on.

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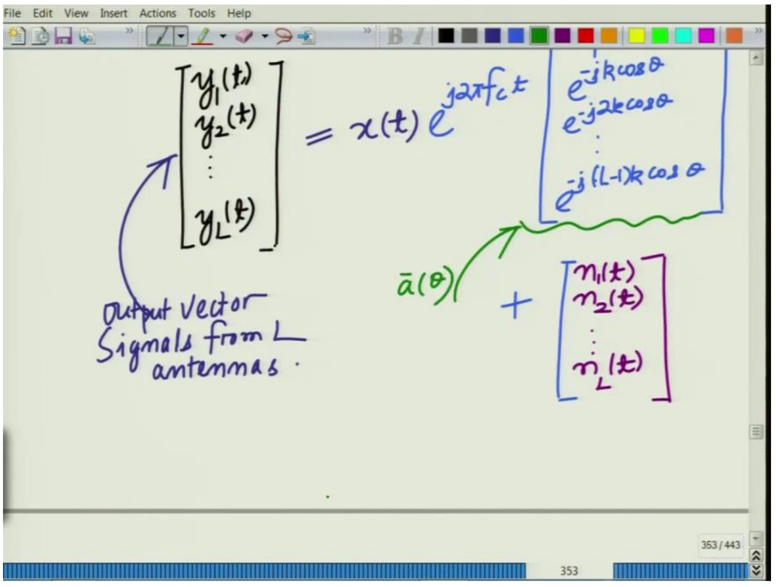

And therefore, now one can summarize this model by writing y 1 t, y 2 t so, on up to y L t that is the received vector right, you can think of this as the output vector comprising all the signals from the L antennas, output vector of signals from L antennas, this is equal to x t e raise to j 2 pi f c t times, you will have the vector of phases 1 e raise to minus j k cosine theta, e raise 2 minus j 2 k cosine theta e raise 2 minus j L minus 1 k cosine theta, alright. So, plus the noise vector which is you have your the usual n 1 t, n 2 t and n of L of T, noise samples for the different antennas. And therefore, if you now look at this, the interesting thing about this is this I can write this as my vector a bar of theta. So, I can write this as the vector a bar of theta.

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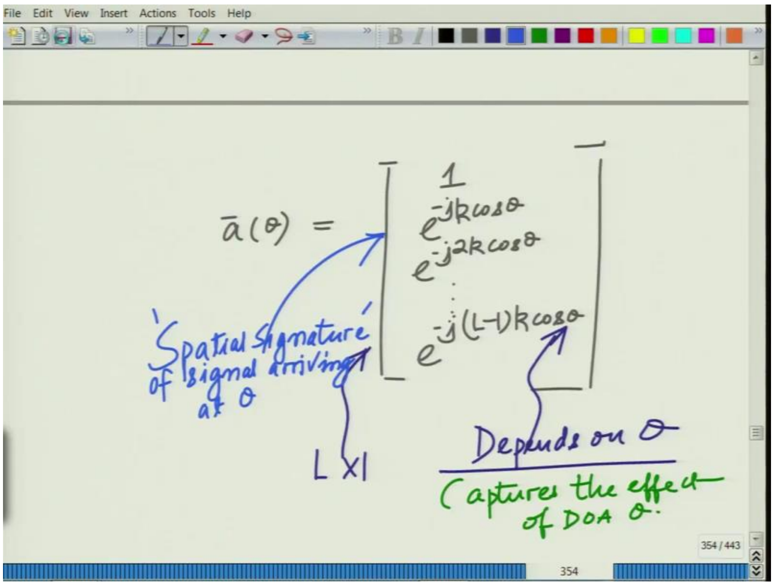

And you can see this a bar of theta this has an interesting structure, let me just write this again over here, this has a graded phase constant phase offset of e raise to minus j k cosine theta with respect to e raise to minus j k cosine theta e raise to minus j 2 k cosine theta, e raise to minus j L minus 1. So, this depends on theta as you can see, this is an L cross 1 vector and this is it depends on theta.

So, it captures that you can clearly see it captures the effect, it captures the effect of the direction of arrival theta right, it is it is the signature of a signal that is arriving with a direction theta at the array, and this property can be used to estimate the direction of the direction of arrival of the signal at the receiver.

So, this is known as the area response vector which is a function of theta or this is also known as the array steering vector with respect to the direction of theta or the spatial signature of the array corresponding to the direction theta. So, the interesting thing about this is, this captures this is the spatial signature right, it is a signature. This is the spatial signature of a signal arriving at a direction theta this is also known as.

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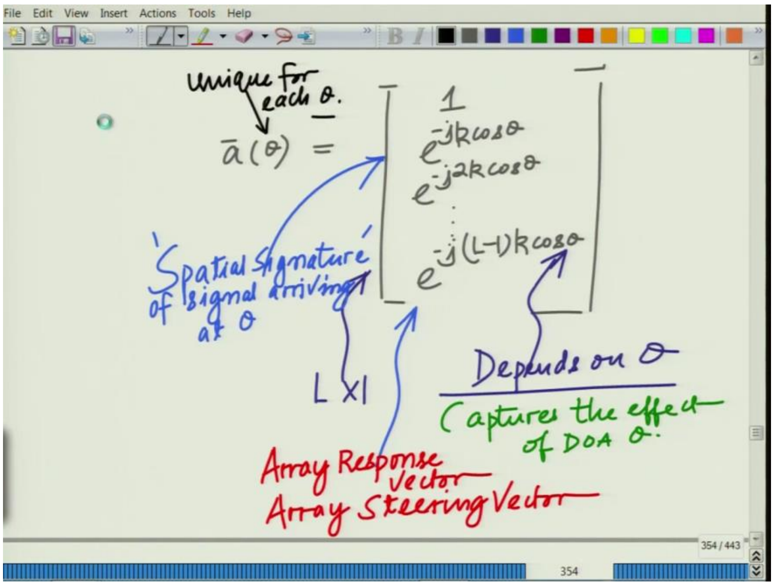

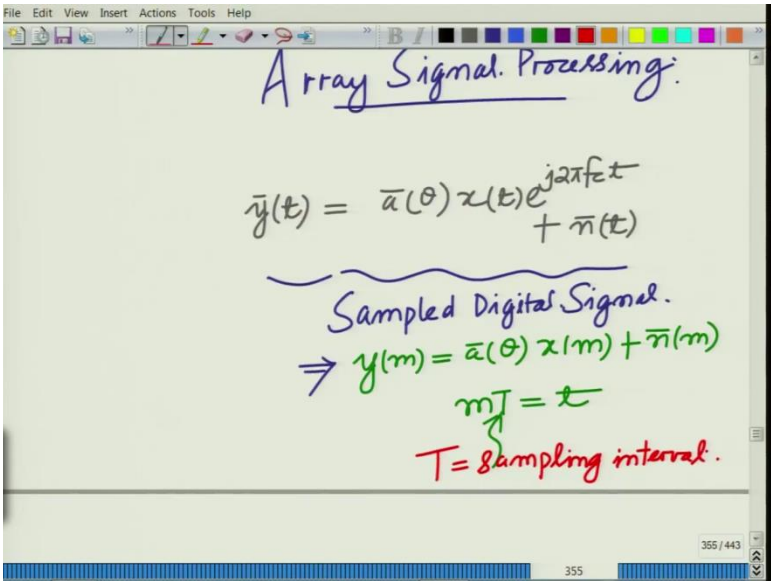

So, this a bar theta this is also known as the Array response vector or this is also known as the Array steering vector and this has an important role to play in array processing. So, array response vector this is very useful in the entire area of array processing and MUSIC is an algorithm for the same, array signal processing right, the broad area of array signal processing of which MUSIC belongs is one of the algorithms that can be used for array signal processing that is signal processing using arrays, which was in fact a revolutionary step forward moving from signal processing using a single antenna that is we have a single antenna receiving the signal and what has been discovered, in fact as we are going to see in this music algorithm that once you use an array of received signals and process the signals collectively, then that can lead to much better significantly improved performance.

And of course, newer applications for instance such as direction of arrival, determining the direction of arrival, which you can clearly see is not possible with a single antenna, because if you have a single antenna with a uniform response throughout at all angles? It is not it is not possible to determine the direction of arrival using the signal received at the single antenna. On the other hand, when you have these multiple antennas, you can clearly see there is a signature corresponding to the angle of arrival theta of the signal and this can be used to determine, this can be used to determine the direction of arrival of the signal as we are going to see in the MUSIC algorithm. So, this is the array steering vector and this is unique for theta right, this is unique for each theta that is the interesting property. This is unique for each theta okay.

And therefore, now I can write this as y bar t equals a bar t x t e raise to j 2 pi f c t plus n bar t. Now, once we sample this now the sampled version of this, sampled version the digital signal corresponding to sampled digital signal can be written as y of m equals a bar theta times x of m plus n bar of m where you can think of m that is if you have the sampling duration m T, this is at t equals you can think of this although we will not dwell on this, T equals the sampling interval which is 1 over F s that is the sampling frequency.

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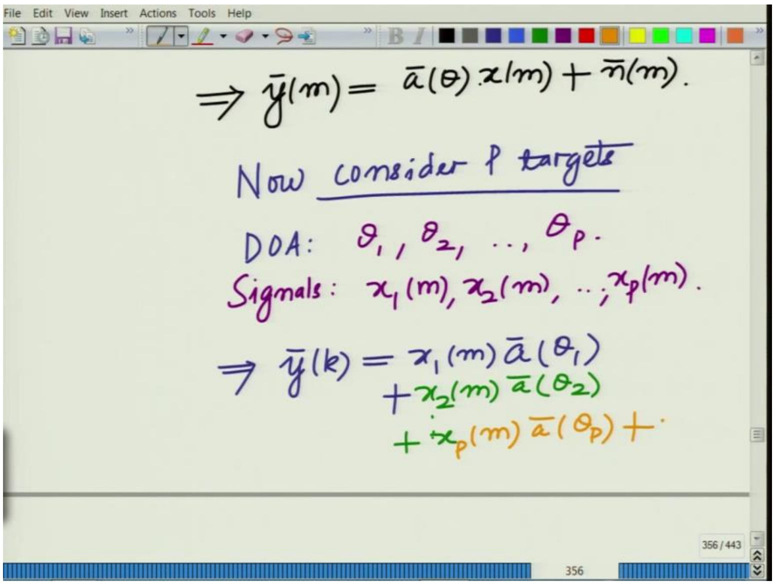

Which is not very important, but the point is you can write this as this is the model that is y bar of m let me make it make this correction. This is a vector y bar of m equals a bar of theta into x m plus n bar of m, and now if you have multiple signals, if you have P targets now extends, now let us consider P targets and this becomes with the signals and DoAs of the P targets can be theta 1 theta 2 up the theta T and the signals can be x 1 m, x 2 m so on, x p m and this implies that the multiple signal model will be the superposition of all of these that is x 1 of m a bar of theta 1 plus x 2 of m plus x p of m a bar theta p plus n bar.

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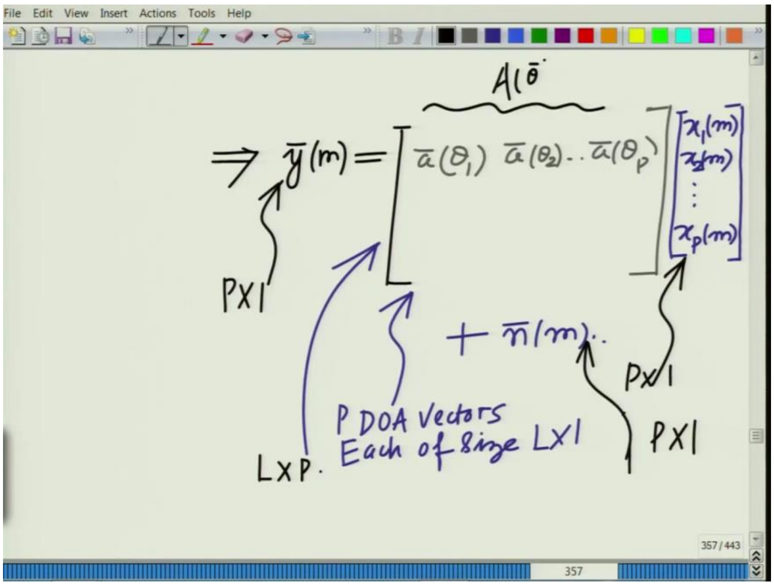

Which I can write as the matrix y, which I can now write as using the matrix notation y bar of k or rather y bar of m equals a bar theta 1 a bar theta 2 so on, a bar theta p times x 1 k, x 2 k that is the symbols times x 1 m, x 2 m, x p m plus n bar m, this is the vector this is the matrix representation as usual, this is the matrix you have p DoA vectors each of size L cross 1 so this will be matrix will be of size L cross p matrix, this is a P cross 1 vector, this is whenever you write a linear model it is good to do a check always do a check to make sure that all the dimensions are correct right, the dimensions should match. Output is p vector, output is a vector of size P, input also has to be a size vector. In fact, this is your matrix I can now call this as the matrix A theta bar and therefore, I can and I can call this as your vector x bar of m.

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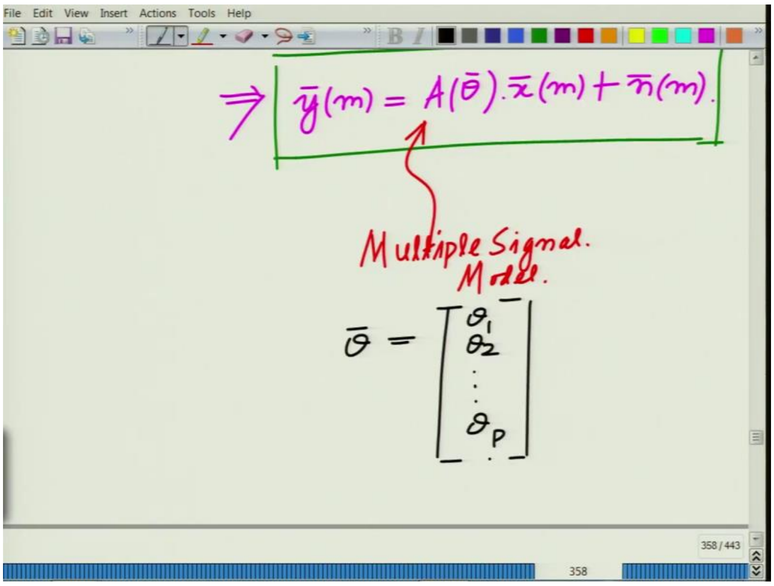

And therefore, I can write this model as implies y bar of m, now with the multiple signal this becomes a theta bar times x bar m plus n bar m and this is now your multiple signal model. So, this is basically the multiple signal model, this is basically your multiple signal model. And now we want to we want to classify this implies we want to the classification is basically data mining these directions of arrival theta 1 theta 2 up to theta p right. So, these are directions of arrival of the P signals, we want to determine these DoA's right.

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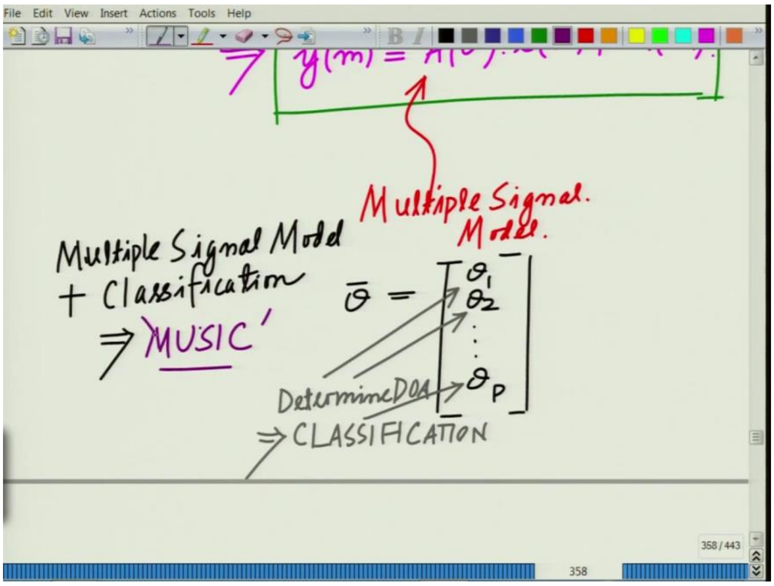

So, we want to determine these determine and this is nothing but the classification, this is essentially what is termed as the classification part. So, essentially you have multiple signal model so multiple signal model which is given above plus the classification aspect and this essentially gives rise to your algorithm that we are going to talk about, which is the music that is multiple signal classification. How do you use this multiple signal? This array antenna array comprising of these array response vectors of the P targets right corresponding to the P directions of arrival use this multiple signal model and actually classify the signals that is determined these directions of arrival theta 1, theta 2 up to theta p.

Theta p this is essentially your music algorithm that is the multiple signal classification which we will explore in the next model. So, this is the system model. Starting with this we are going to explore or discover the music algorithm. Thank you very much.