**Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning**
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**Lecture – 38**
**LMMSE Application: Wireless Channel Estimation and Example**

Hello, welcome to another module in this massive open online course. So, we are looking at the principle of LMMSE estimation or the linear minimum means squared error estimator essentially which is applicable for any kind of an input-output system that is for a nonlinear input output system and can also as a special case would be a linear input-output system.

And we have derived in the previous module what the LMMSE structure would be for a linear IO system such as for instance either when you are talking about the receiver for a multiple input multiple output wireless communication system or when you are trying to determine the regressor for instance the regression regression coefficients.

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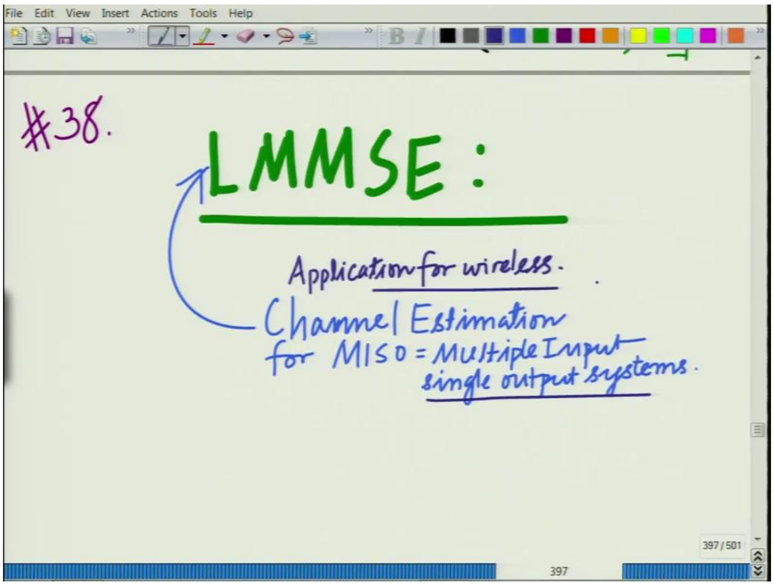


Now, let us again continue on the along the same lines and look at another application of LMMSE that is the Linear Minimum Mean Squared Error principle and this time in the context of channel estimation remember that is we want to look at an application channel estimation for your MISO that is multi multiple input multiple input single output systems, channel input for Channel estimation for MISO systems. And of course, we can say this is an application in the context of wireless communication application of your LMMSE principle for wireless communication systems.

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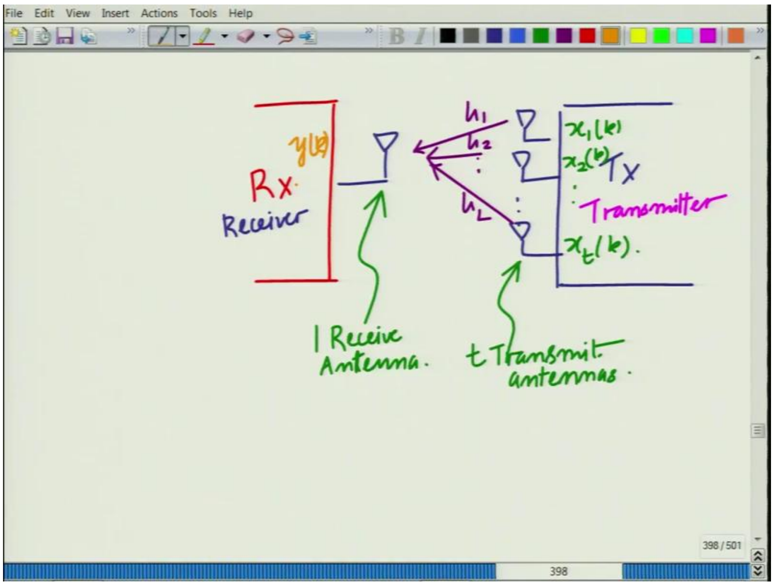


And if you remember what is a MISO wireless communication system MISO stands for multiple input single output. So, you have essentially what you have is a single transmit antenna, I mean a single receive antenna that is the single output and you have multiple transmit antennas. So this is your receiver, this is your transmitter and you have multiple transmit antennas and therefore, you have the corresponding channel coefficients given by h 1e h 2 up to h L and the transmit symbols are x 1 k, x 2 k, I can denote this by x t of k.

Let us say we have t transmit antennas and let us say you have 1 receive antenna, this is a MISO system and we denote the single output symbol, we now denote it by y of k. So, you have transmit antennas that has multiple transmit antennas or multiple inputs and you have 1 receive antenna that is a single output and therefore this becomes a MISO system. So just so that you can recall so these are essentially your multiple inputs and this is your single output.

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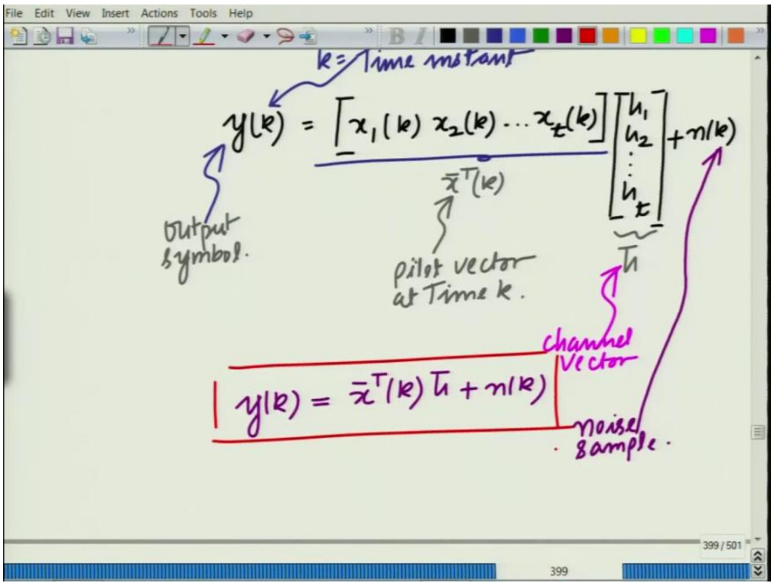


And together this essentially implies that you have a multiple input single output that is essentially your MISO or your MISO system and the output symbol in this MISO system that can be expressed as y of k equals x 1 of k, x 2 of k, x t of k times h 1, h 2 up to h t plus n of k. So this is essentially your what is this, so this k this denotes the time instant. So, this k equals a particular time instant. This is the output at time k, this is the pilot vector, you can denote this as x bar t, this is the pilot vector at time k and this is your well known h bar which is essentially your channel vector.

And coming now to this, this is your noise sample. And therefore, I can write this succinctly, I can write this as y k equals x bar transpose k times h bar plus k, this is the model, this is the IO model at time k. And now, let us assume we have the transmission of L pilot vectors over L time instants we call them x 1 bar, x bar 1, x bar 2 up to x bar L, these are the L pilot vectors transmitted over L time instant.

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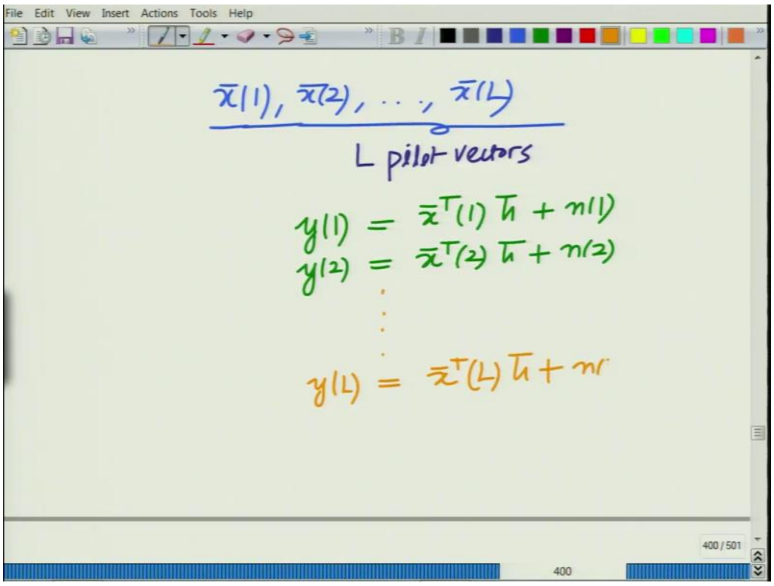


So you have x bar 1, these are your L pilot vectors. So, essentially if you write the model corresponding to that you will have y 1 equal to x bar transpose 1 h bar plus n 1, y 2 equal to x bar transpose 2 into h bar plus n 2 so on and so forth, y L equal to x bar transpose L into h bar plus n L.

Now writing this as a vector, so write as vector in fact, that is what we want to do because this is remember course on applied linear algebra, we want to use compact the presentation of all these quantities in vectors and matrices and directly do the manipulations in terms of vectors and matrices.

And that remember is the essence of this course or essence of this massive open online course on applications of linear algebra, how to use vectors, matrices and the properties of these vectors and matrices efficiently and do the manipulations directly in terms of vectors and matrices, rather than writing these things in terms of these different scalar quantities or vector quantities. So that is essentially the idea of the entire idea behind this course.

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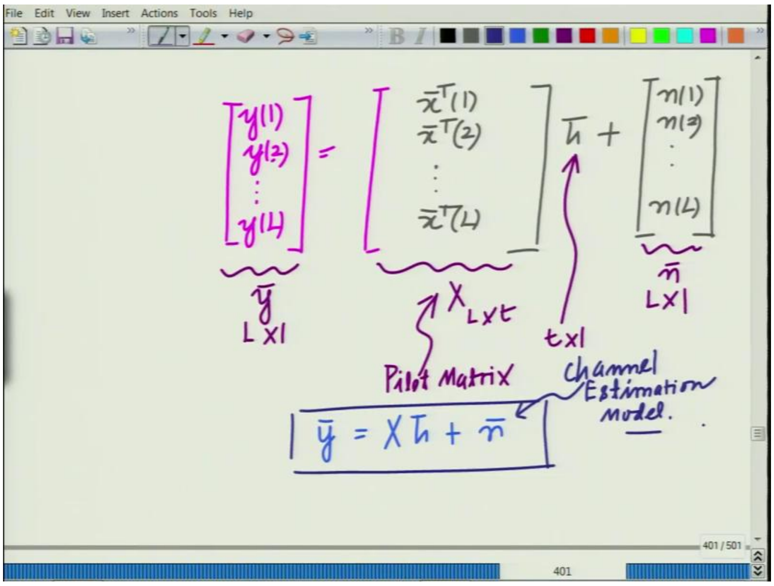


So, now, you write this as a vector. In fact, by this time, you have to be very comfortable making these manipulations and it should directly occur to you how to represent this in a compact form, x bar transpose 1, x bar transpose 2, x bar transpose L times this is a matrix times h bar plus n 1, n 2 times n L which is essentially your noise vector. And now, if you look at this, this is your y bar which is an L cross 1 vector, this is your pilot matrix x, which is L cross t, this is your h bar which is T cross 1 and this is your noise vector n bar which is once again L cross 1 and therefore and this is your pilot matrix.

So, these things you have to become very comfortable and start thinking in terms of vectors and matrices. So, this can be written as y equal to x H plus n bar and this is essentially this is your channel estimation model, this is your channel estimation model. And what we have seen is that here we have previously considered the least squares estimate if you remember that is minimize norm of y minus x h bar square.

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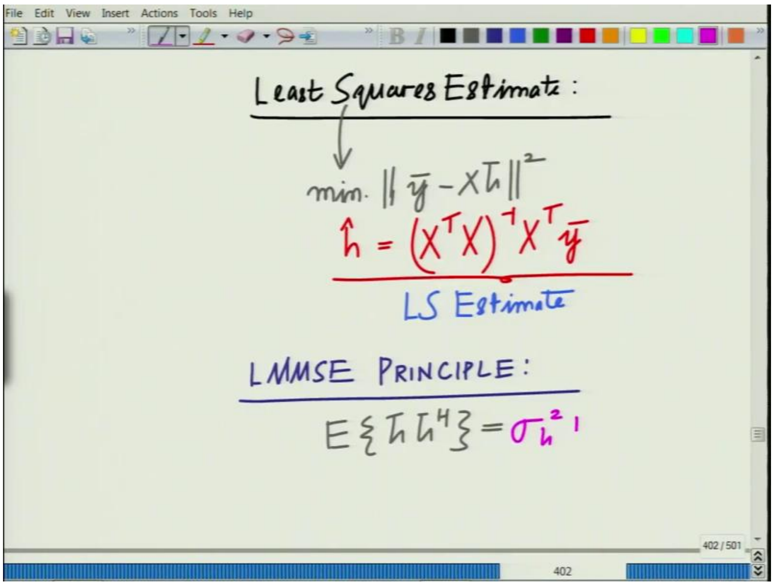


If I might remind you what the least squares estimate, what the least squares estimate does is you essentially have minimize norm of y minus x h bar square and we know the least square estimate this is given h equal to x transpose x or x Hermitian x depending on if x is real or complex, x transpose x inverse x transpose times y bar and this is essentially what we call as the LS estimate. Now, let us look at what is the LMMSE estimate. Now, we let us look at the illustration of the LMMSE principle. Let us look at an illustration of the what is the LMMSE principle?

For the LMMSE principle you remember, we need the covariance as in the cross covariance, this is important so we need prior this is known as the prior information. This is important, this is what distinguishes the LMMSE from the LS, so the least squares, we do not need the prior information we do not need either the covariance of the a vector h bar that is being estimated or we do not even need the noise covariance. All we need is the knowledge I mean ideally speaking, even that is not I mean, if you even that is not needed, but essentially if you want it to be optimal, 1 simply needs to know the fact that the noise samples are independent identically distributed, so do not worry about that.

But essentially the point is in the LMMSE you need to construct the covariance and the cross covariance, and for that you need the covariance of the parameter vector and also the covariance of the noise. So essentially, this is an important aspect of the LMMSE. So, you have expected value of h bar h bar hermitian this is equal to sigma h square times identity assuming the parameter components of the parameter vector are independent.

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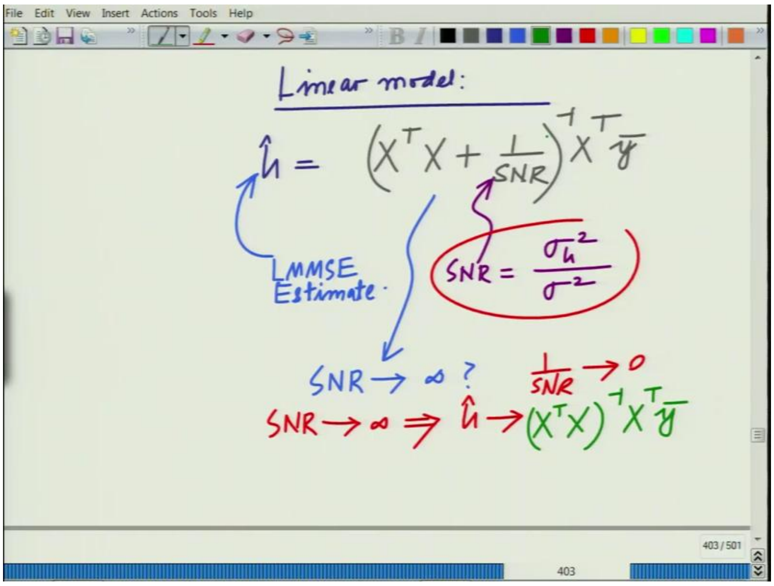


And we have a very simple model, noise samples are also independent identically distributed n bar n bar hermitian equals sigma square identity. And also just go without saying to make the model simple, let us assume that these are expected value of h bar that is mean that is the mean of the channel vector is 0.

And we know from the linear model from the linear model results for the linear model, we know that the channel estimate is given as simply your x Hermitian x plus 1 over SNR inverse x Hermitian y bar, you can make it transpose if you have real matrices let us just make it transpose for simplicity so 1 SNR.

So, the point is here SNR this quantity SNR here is defined as sigma h square divided by sigma square because the parameter vector being estimated is h, and the components of H are variance or power sigma square H square, signal signal power sigma H square to the parameter power divided by sigma square noise variance that is essentially your SNR. And therefore, this is a very simple and elegant expression and in fact, now let us observe something very interesting about this. So this is your LMMSE estimate.

Now in fact, if you see this let us ask the question what happens as SNR tends to infinity. As SNR tends to infinity you can see something very interesting happens, SNR tends to infinity this implies that h hat tends to so 1 over SNR tends to 0. So, this implies h hat tends to X transpose X inverse X transpose y bar because 1 over SNR I am sorry, there has to be an identity matrix over here so 1 over SNR times identity, so 1 over SNR tends to 0 so h transpose this is nothing but the LS estimate and this is a very interesting property.

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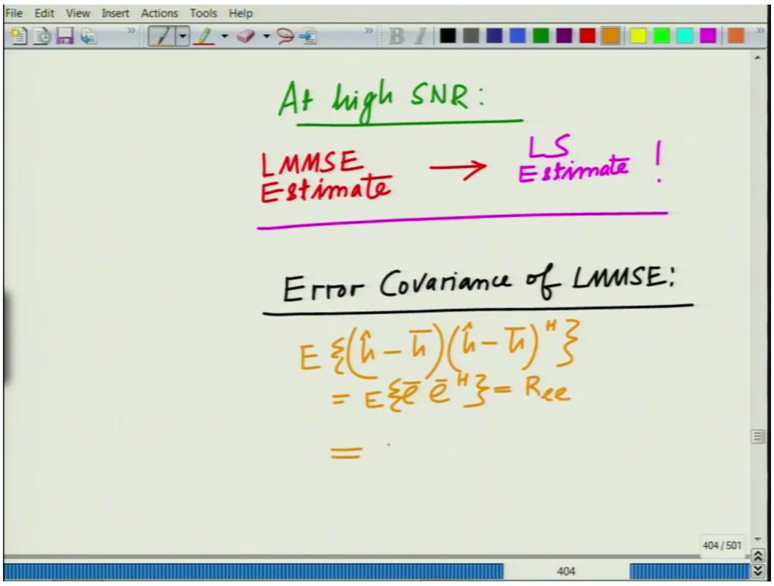


So at high SNR, the least squares estimate tends to the LMMSE estimate So, this is a very very interesting property of what you are observing, SNR least squares estimate that is your estimate, I am sorry at high SNR the LMC estimate tends to the least squares estimate that is what we have to write.

Sorry, I just wrote it the other way around. So, at high SNR, the LMMSE estimate tends to the, this is a very interesting property. At higher SNR the LMSSE estimate linear minimum mean squared error estimate tends to the LMS tends to the LS that is the least square estimate which is a very interesting property.

Let us look at what happens to the error covariance what happens now, let us ask the question, what is the error covariance of LMMSE? Now, error covariance of LMMSE you can see this is expected value of h hat minus h bar times h hat minus h bar that is Error Hermitian expected value which is essentially your R e e.

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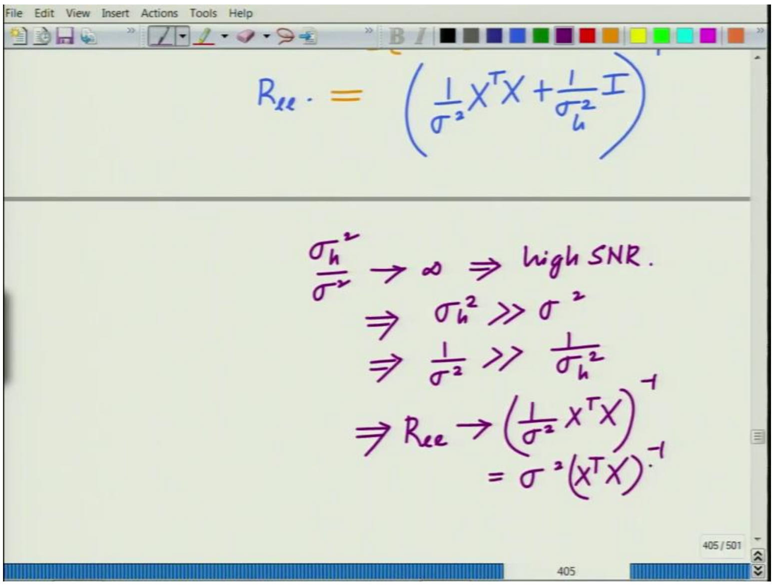


And this is equal to, we have seen once again for the linear model this is once again 1 over epsilon but in this case epsilon sigma square times X transpose X plus 1 over gamma which in this case sigma square times identity inverse and this is your R e e, this is your error covariance matrix.

Now once again as sigma square, now let us look at there are 2 cases for this, sigma square over sigma h square, sigma square over sigma h square or sigma h square sigma square over sigma h square tends to infinity implies once again high SNR, which means sigma h square is very high implies sigma square is much larger than sigma square, which implies that 1 over sigma square X transpose X is much so 1 over sigma h square is much larger than sigma square.

So, 1 over sigma square is much larger than 1 over sigma h square which means, square 1 over sigma square is much larger than 1 over sigma square which implies R e e because 1 over sigma square X transpose X dominates which implies R e e tends to 1 over sigma square X transpose X inverse which is equal to sigma square X transpose X inverse which is nothing but the error covariance of the LS estimate which is nothing but the error covariance or the LS estimate.

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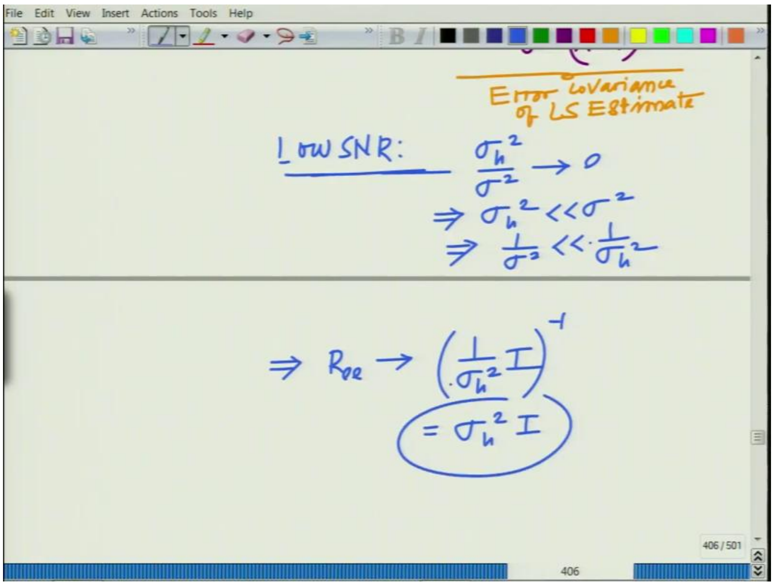


On the other hand at low SNR something interesting happens. What happens at low SNR at low SNR we have sigma square over sigma square tends to 0, this implies that sigma square is much smaller than sigma square which implies 1 over sigma square much smaller than 1 over sigma h square, which implies now that R e e tends to 1 over sigma square identity inverse which is nothing but sigma square identity. And if you look at this, this is nothing but the prior covariance.

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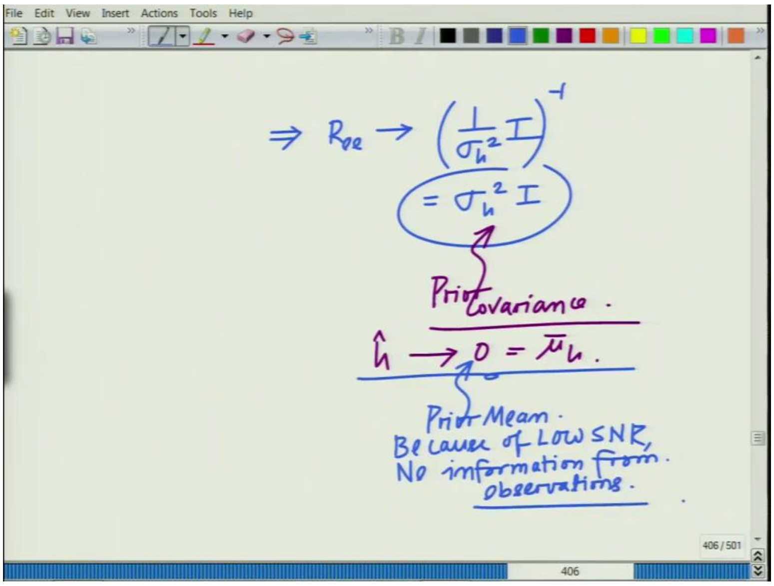


So this is nothing but the prior covariance, which is essentially the covariance before making any observations, before even making any observations, so what they are saying is because the signal to noise power ratio is low that is sigma square is very high. So, making the observations do not does not add any additional information. So, you are again the uncertainty in the parameter vector h bar is the same as what you had started with that is sigma square times identity that is error covariance.

In fact, the estimate of this h bar in this case as you probably already know is that it simply reduces to the prior estimate which is as if you had no observations and what is the best estimate of h when you have no observations that is nothing but the mean. So in fact, h hat tends to 0 which is equal to Mu bar h.

So, at low SNR h bar tends to 0 that is the prior mean as if you have no observations because of the low SNR observations are not adding any information. This is but nothing but the prior mean or mean which essentially the insight is no information, no information or no useful information. So, it does not add because of the SNR is very low, SNR is very low. So, the observations do not add any useful information to this estimation process that is essentially what it was.

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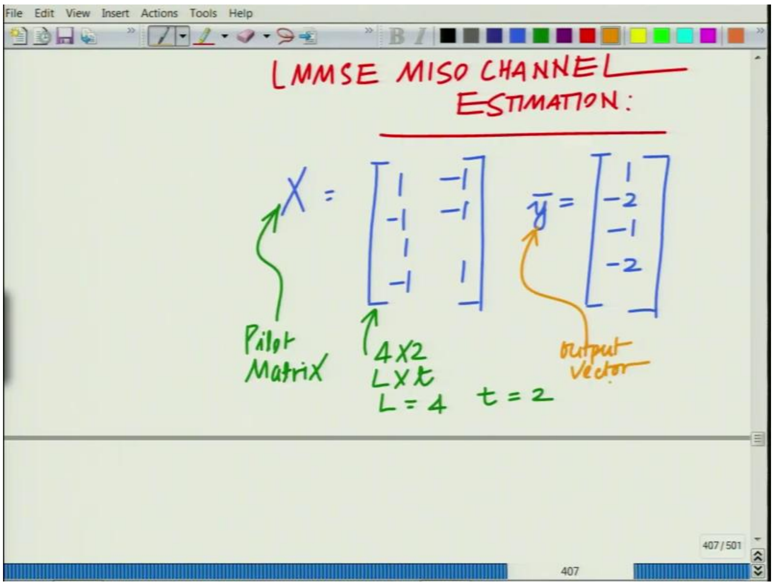


Let us look at a simple example to understand this. So, let us look at a simple example for the LMMSE. LMMSE MISO channel estimation. So, what happens in this case? Let us look at a simple example, an example for MISO channel estimation LMMSE, we have the pilot matrix x this is equal to 1, minus 1, 1, minus 1, minus 1, minus 1, 1, 1 and then we have, the we have the output vector 1 minus 2 minus 1 minus 2. So, this is your pilot matrix, this is pilot matrix, this is 4 cross 2, which you remember is L cross t. So, L equal to 4 number of transmit antennas t equal to 2. And this is essentially your output vector, this is essentially your this is essentially your output vector.

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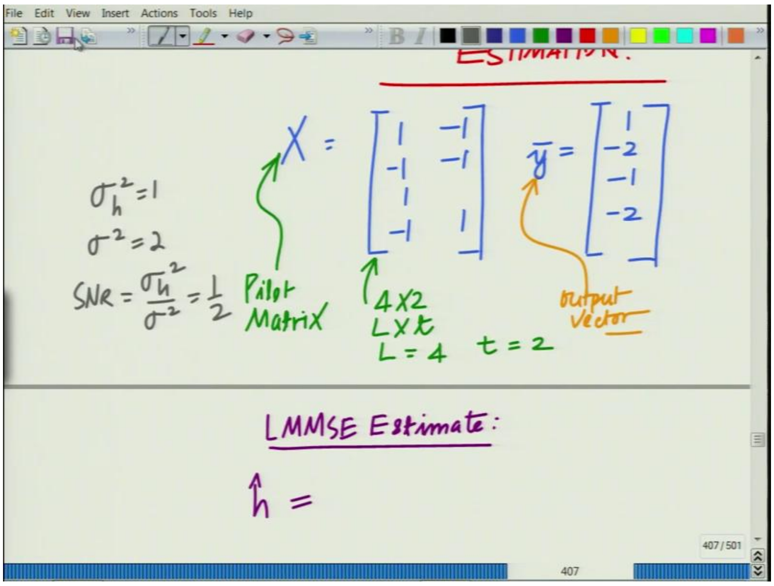


And therefore, the estimate is given as h hat. Now, the question we want to ask obviously is, what is the LMMSE estimate h hat equals, of course, what is LMMSE estimate and we also need the prior variance. So, let us assume sigma square equal to 1, noise variance sigma square equal to 2. So SNR equal to sigma h square divided by sigma square which is equal to half.

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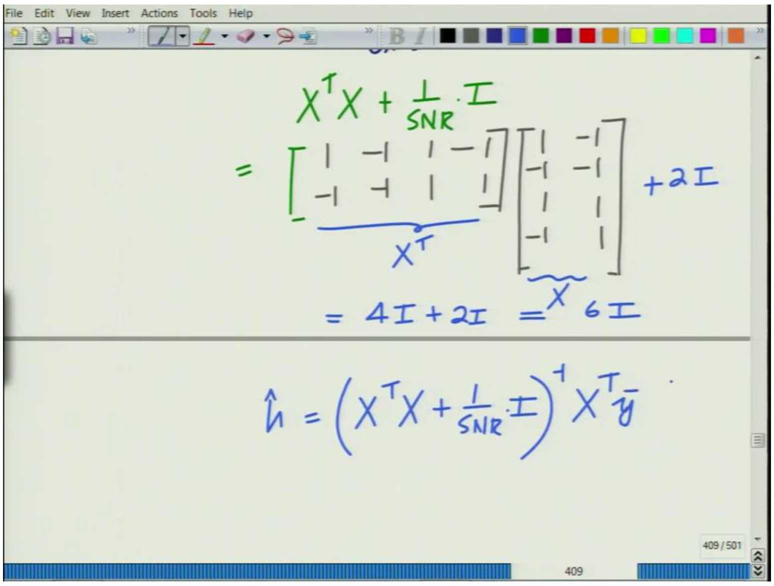

Now, let us ask the question what is LMMSE estimate, LMMSE estimate for this case is h hat equal to x transpose x plus 1 over SNR identity inverse x transpose y bar. Remember this identity is of size remember X transpose X, this is t cross t so this identity also has to be of size t cross t.

Let us first evaluate this quantity x transpose x plus 1 over SNR times identity, this is equal to well, you have your x transpose x 1, minus 1, 1, minus 1, minus 1, minus 1, 1, 1, 1 minus 1, 1, minus 1, minus 1, 1, 1, 1 which is essentially equal to 4 times identity, you can check this and 4 times identity.

So this quantity this is your x transpose and this is your x plus 1 over SNR is 2, 2 times identity so this is essentially X transpose X is 4 times identity plus 2 times identity, this is equal to 6 times identity. And therefore, your LMMSE estimate h hat equals x transpose x plus 1 over SNR times identity inverse x transpose y bar.

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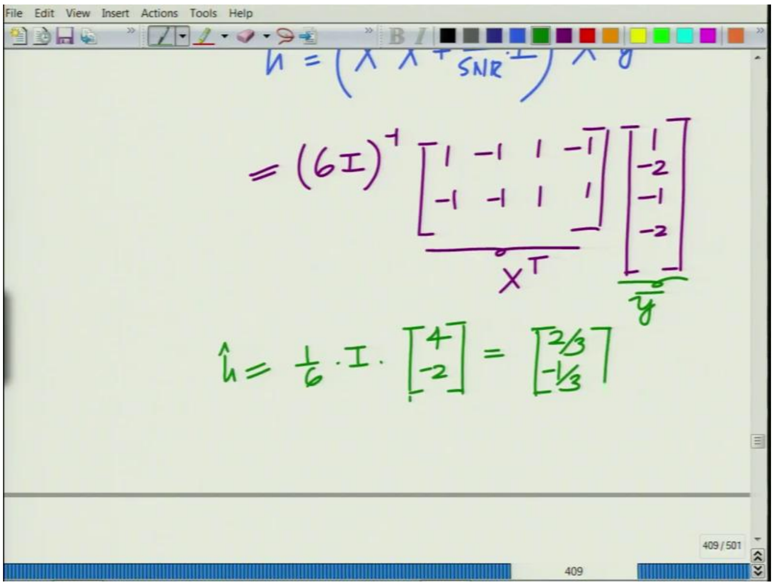


Which is essentially this is your 6 identity inverse times your x transpose which is 1, minus 1, 1, minus 1, minus 1, minus 1, 1, 1, which is your x transpose and what is your y bar, 1, minus 2, minus 1, minus 2, this is your x transpose and this is your y bar. And therefore, if you look at this, this is 1 over 6 times identity into x transpose y bar which is essentially you can work this out. This is 4 comma minus 2, which on simplification yields 4 by 6 which is 2 by 3 minus 2 by 6 which is minus 1 by 3 and that is your LMMSE estimate.

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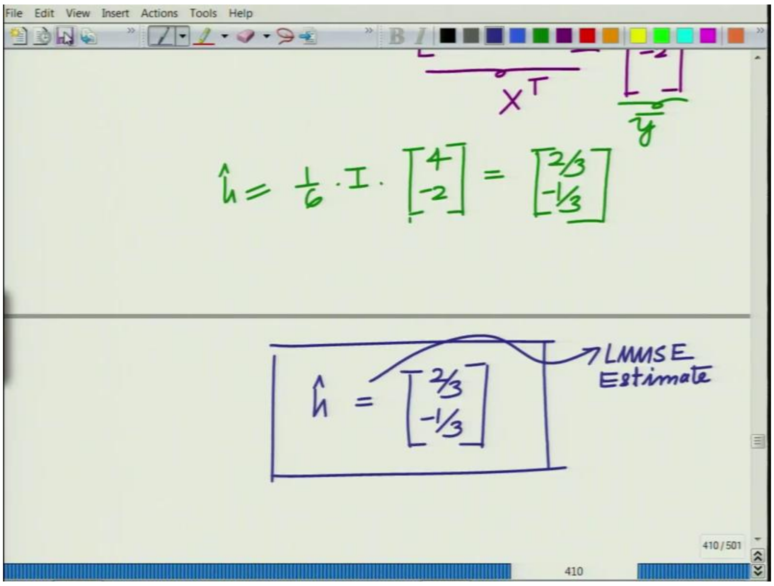


So, h hat equals 2 by 3 minus 1 by 3 and this is your and this is your LMMSE that is the linear minimum mean squared error. Let us again ask the other question, what is the error covariance of this estimation.

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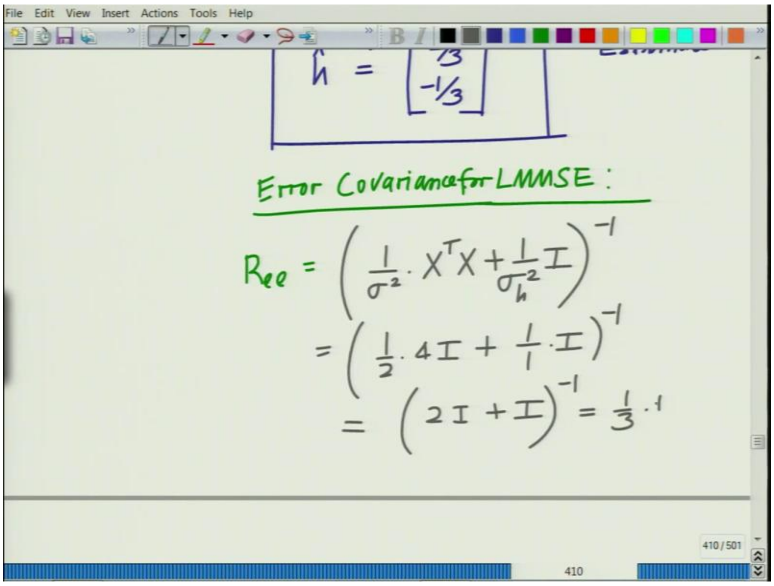


So, error covariance for the LMMSE estimate, this is equal to R e e, remember this is 1 over sigma square x transpose x plus 1 over sigma h square identity inverse which is essentially 1 over sigma square is half x transpose x is 4 times identity plus 1 over 1 times identity. So this is 2 identity plus identity inverse, this is equal to 3 identity inverse which is 1 over 3 times identity.

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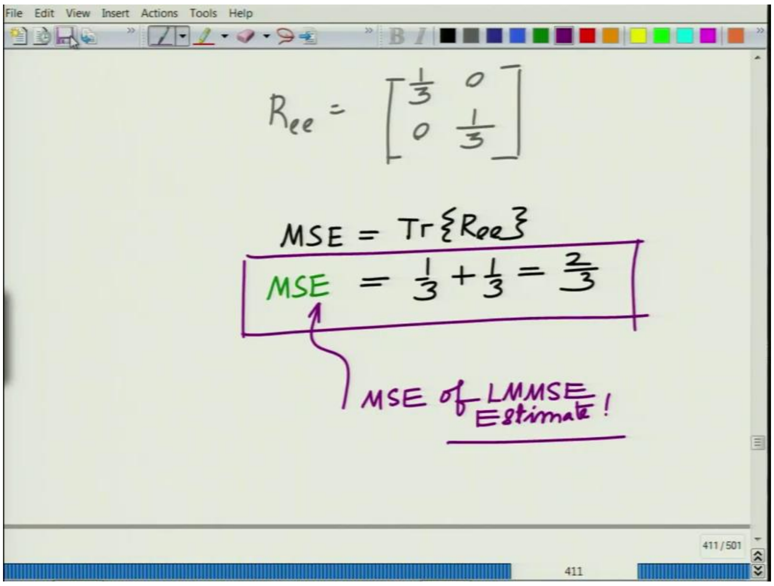


So, this is equal to 1 over 3, 0, 0 so this will be if you write this carefully, and the MSE finally is trace of this sum of the diagonal elements which is 1 over 3 plus 1 over 3 which is equal to 2 over 3, this is your MSE for, this is the MSE of the MSE of the mean squared error of the LMMSE estimate.

So, we have looked at a beautiful application of this LMMSE principle in a very practical example that is channel estimation which is of course, carried out very frequently in practical wireless systems that is we have to estimate, one has to estimate the channel, the channel is estimated as I have already told you either using the LS least squares or LMMSE technique, I mean these are all very popular techniques, and again there are other techniques such as, for instance you might have advanced techniques such as the Kalman filter or slightly more efficient techniques such as adaptive channel estimation, and so on and so forth.

But all of these are essentially what you have to realize are based on beautiful principles, the theory and analysis based on beautiful principles borrowed from, and the framework and the models borrowed from linear algebra. And as you can see, all the manipulations involve the system that is essentially designed and analyzed using principles from linear algebra.

And I encourage you to look at this application and also calculate what the LS estimate would be, what is the least squares estimator and what would be the corresponding error covariance for the least squares estimator transpose X inverse X transpose y bar that will be the estimate and error covariance will be sigma square X transpose X inverse, so I encourage you to look into this and learn more, observe these concepts and appreciate these things, so we will continue this discussion and look at other applications of this principle in the subsequent modules. Thank you very much.