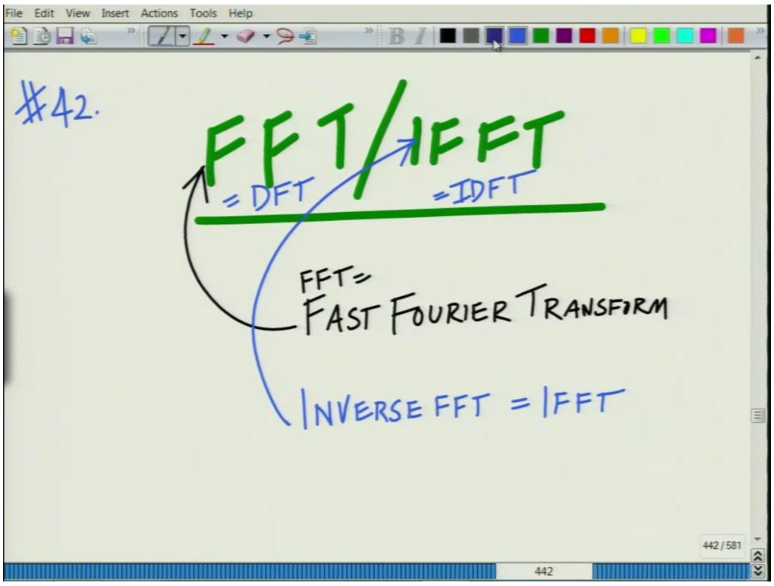
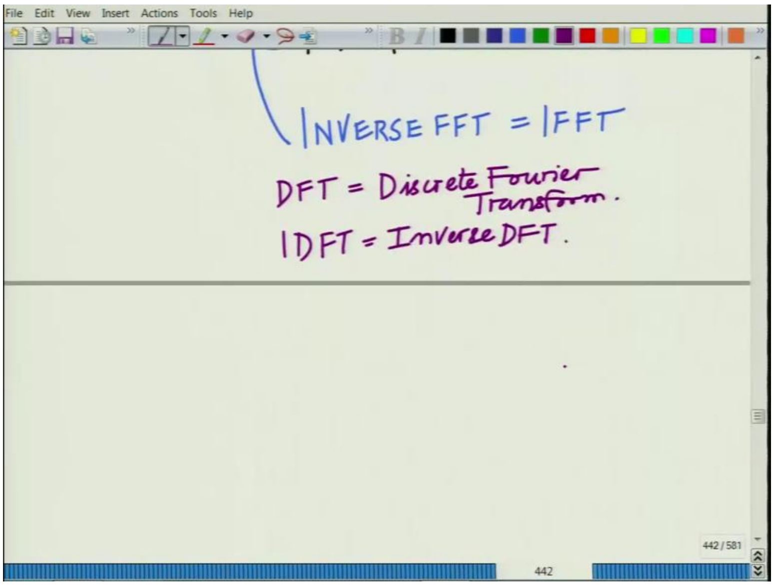
Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
Professor Aditya K.Jagannathan
Department of Electrical Engineering
Indian Institute of Technology, Kanpur
Lecture 42
Fast Fourier transform (FFT) and Inverse fast Fourier transform (IFFT)

Hello, welcome to another module in this massive open online course. So, let us continue our discussion on applications of various applications of linear algebra and let us look at yet another interesting area where, linear algebra has several implications and that is in Fourier analysis of signals systems and so on, we are what we know as the fast Fourier transform.

(Refer Slide Time: 00:42)



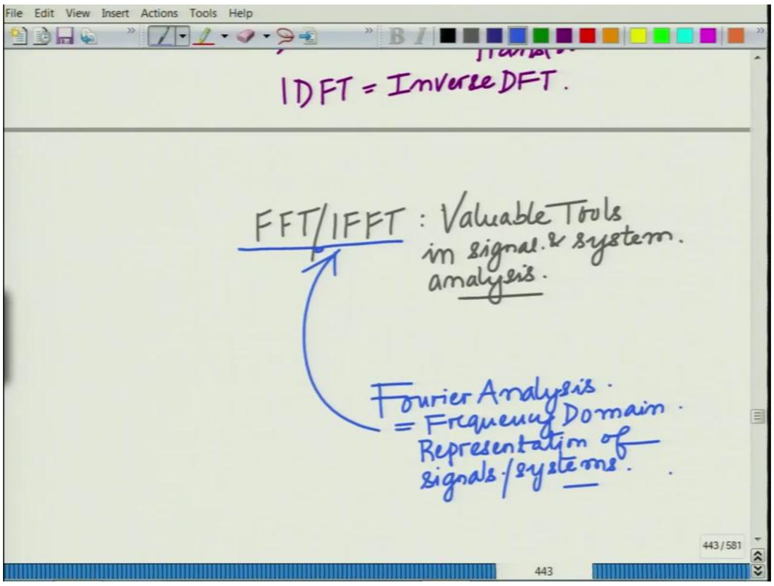


So, let us talk about the application of linear algebra and the fast Fourier transform that is the FFT and inverse of the fast Fourier transform so, this is one of the fundamental operations or we could say one of the fundamental linear operations in system analysis and signal analysis.

So, FFT stands for fast Fourier transform that is your FFT and I have IFFT many of you might be familiar that is the inverse of the FFT that is the inverse fast Fourier transform so, this is your IFFT and these are essentially basically if you look at it these are nothing but, fast algorithm so, FFT is essentially nothing but, a fast algorithm for DFT and IFFT is a fast algorithm for IDFT and what are DFT?

So, DFT equals again this is your this discrete Fourier transform and IDFT naturally this is inverse DFT and these are actually as you see this is basically fundamentally related to the Fourier analysis of signals and systems that is to convert this time domain representation and a certain domain time or in fact if you look at images and video spatial domain into the frequency domain and draw variable in, valuable insights such as what is the high frequency content of the signal, what is the low frequency content of the signal and so on.

(Refer Slide Time: 03:03)

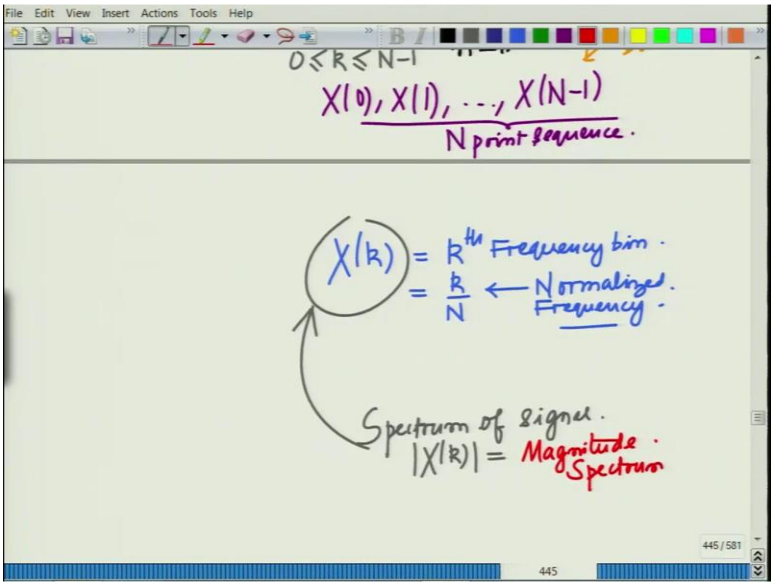


So, FFT and IFFT essentially so, your FFT and IFFT and we look at applications of this so, FFT and IFFT these are valuable tools in the analysis of signals and systems that is signals and system analysis and this is basically, what is known as the Fourier analysis that is Fourier or the harmonic analysis this gives rise to the frequency domain representation of signals and systems.

Now, what is the FFT or the DFT so, the FFT or DFT which is essentially the same thing as the DFT discrete fourier transform this is defined for a sequence, so FFT or DFT is defined for a sequence and we have n n points sequence and so this is a finite N point sequence and the FFT is given as that is if you look at X of k that is the kth FFT point or the kth frequency, the kth frequency bin this is given as summation N equal to 0 to N minus 1 xn e raised to minuj j 2Pi k over N so this is essentially the k^{th} FFT point Xk and we have FFT points X0 and this is defined for k equal to 0 1 upto N minus 1 so we have N points so, for an N point sequence capital N point sequence you have capital N FFT bins and the corresponding capital N FFT values.

So this defined for so its one to one transformation as you are going to see this is defined for 0 less than equal to k less than equal N minus 1 and therefore, you will have X of 0 X of 1 you will have another output N point sequence but, this is in the frequency domain.

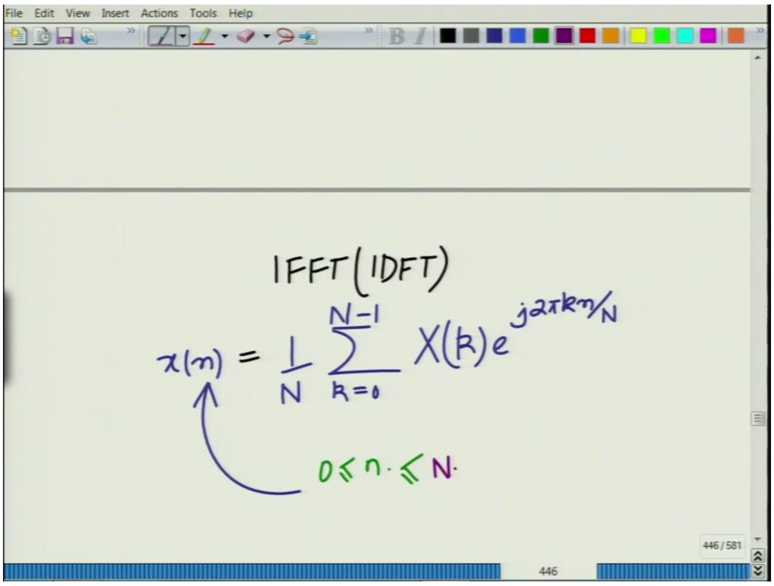
(Refer Slide Time: 07:51)

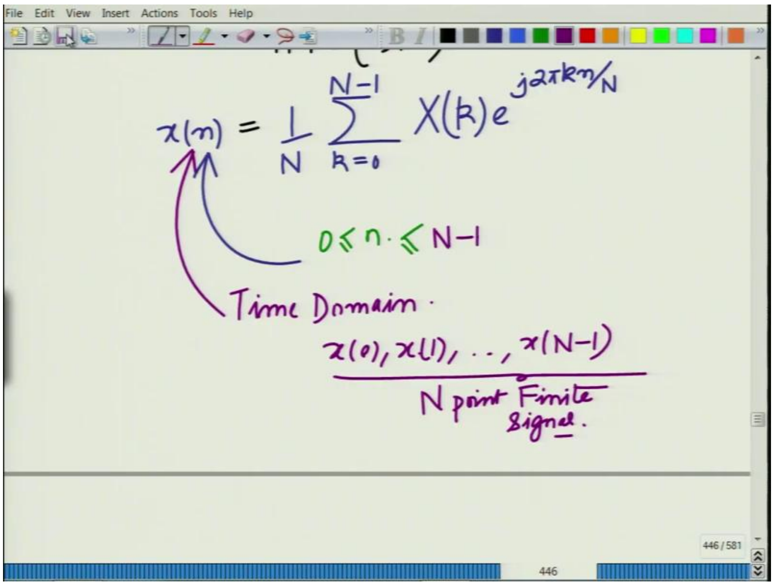


And Xk corresponds to as I have said k^{th} FFT point this corresponds to the k^{th} frequency bin with normalised frequency that is k by N this is the normalised frequency so the k^{th} FFT bin so this is essentially the normalise frequency you can think of frequency bin 0 1 2 0 1 over N, 2 over N, k over N up to N minus 1 over N so these are so, you will essentially have N frequency bins and this gives essentially spectrum.

So, if you look at this so, Xk is nothing but the spectrum of the signal the frequency spectrum or simply called as the spectrum of the signal and you will call and this is used in all domains for instance you will see the notation spectral analysis. So, this is the spectrum of signal and if you being more particular you will have magnitude Xk this is what is known as the magnitude spectrum and typically, what is plotted in fact this is what is known as the magnitude spectrum.

(Refer Slide Time: 09:30)



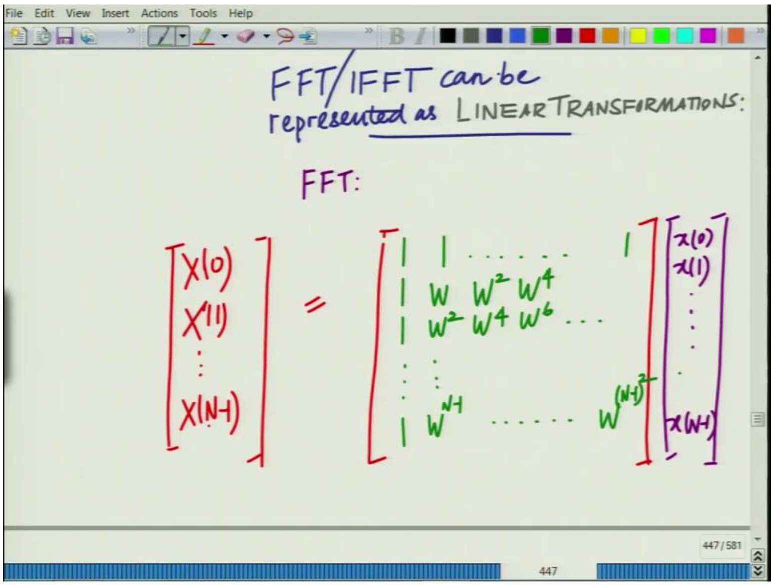


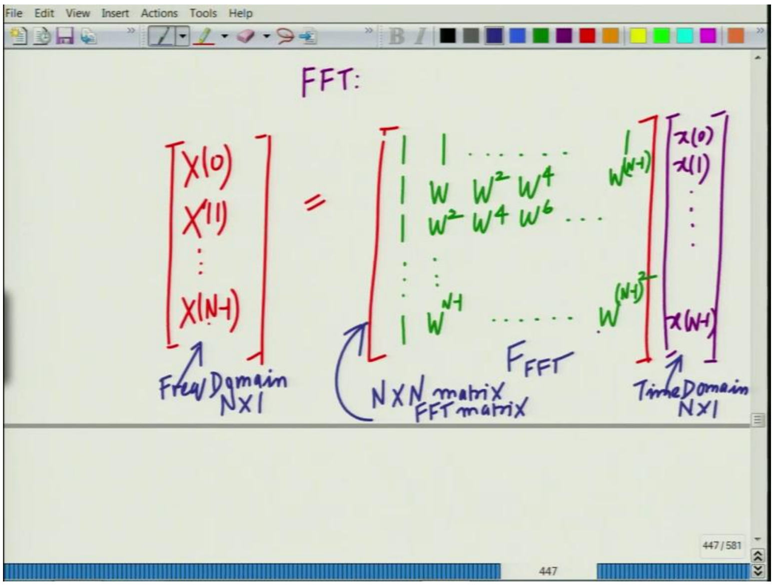
And now, if you look at the IFFT or the IDFT so, the IFFT or IDFT this can be defined as the IFFT or IDFT which is the same as IDFT this can be defined as naturally this performs the inverse operation so, this will give you xn in terms of the frequency bins or the fourier values.

So, this will be k equal to 0 to N minus 1 1 over n Xk e raised to j 2 Pi kn over N and once again this is defined for 0 less than equal to n again this is easy to see because, you will get back the n signal values that is 0 less than equal to N minus 1 and this is essentially the time domain again this is the time domain signal and therefore, you will once again you will have the N points x0 x1 up to so you have the N point discrete and in fact finite signal, finite you will have the N point finite signal.

So, that is essentially the summary so, you have FFT takes the signal x0 small x0 small x1 small xn FFT gives you the Fourier or the spectrum of that capital X0 capital X1 capital XN minus 1 you take the IFFT that transforms back from the frequency that can gives you transforms from the frequency domain and gives back the signal the sequence in the time domain that is a small x0 small x1 small xn small x of capital N minus 1.

(Refer Slide Time: 11:55)

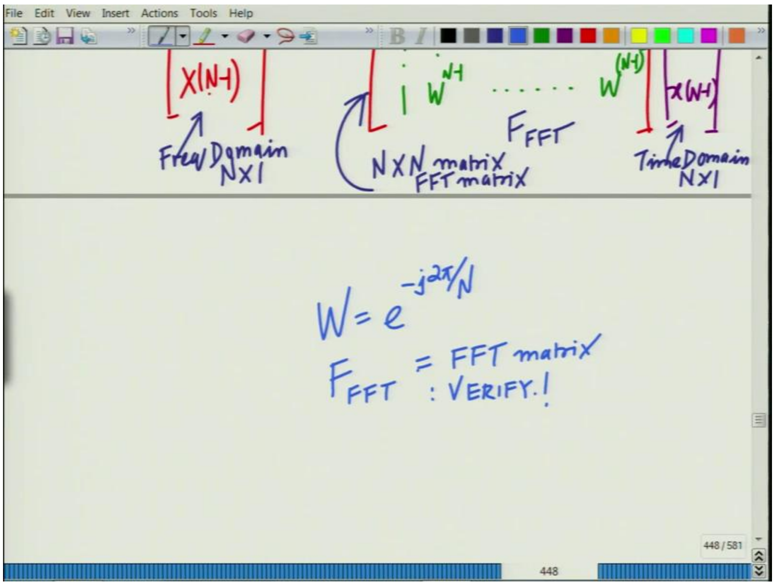




And now, the linear algebra part now FFT and IFFT can be represented FFT operations and IFFT operations can be represented as matrices or rather you can say matrix transformations or linear transformations so, these are nothing but, linear transformations, what are these linear transformations?

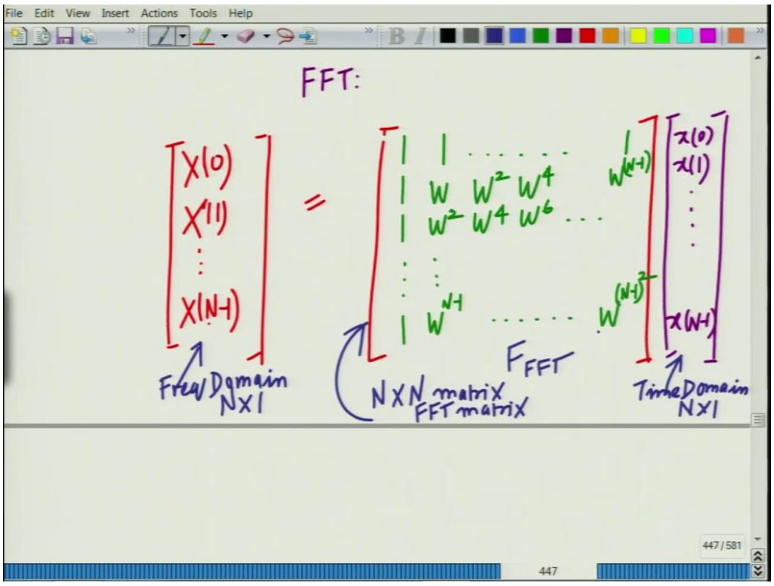
That is if you look at the FFT for instance, you have the FFT the output is the N dimensional vector capital X0 capital X1 open represented as a matrix in fact, n cross in matrix and these entries will be first all rows will be 1 first column all entries are 1 and this can be W W square W four W square W four W six and so on WN minus 1 and the last entry will be WN minus 1 times or you can say WN minus 1 square W raised to N minus 1 square and this is essentially your for instance, the entry here will be W raised to the power of N minus 1 and so on and you can fill the rest of the entries and this is an N cross N matrix corresponding to the N dimensional FFT so this is the frequency domain vector this is capital N cross 1 and this is the time domain vector and this is the N cross N this is what you call as the FFT matrix this I can denote by F FFT.

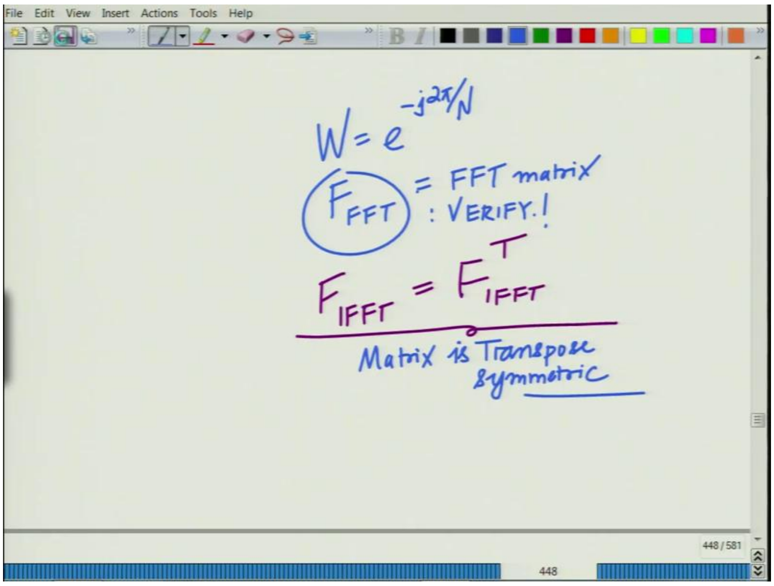
(Refer Slide Time: 15:09)



Where, what is this quantity W this quantitative W is nothing but the root of Nth root of unity this is given as e raise to minus j 2 Pi over N this is the quantity W and you can easily verify and this verify you can see FFT this is the FFT matrix and you can easily verify this if you take the Fourier transform expression for the Fourier transform that you have just seen above write it in the form of a matrix that is you have the N input values x small x0 small x1 small xn minus 1 and output values capital X0 capital X1 capital XN minus 1 if we look at this as an N to N point to N point transform write down the matrix can easily convince yourself that this will be given by this matrix FFT and the important thing you can see here one of the important things you can see here is that, this is a transpose symmetric matrix that is.

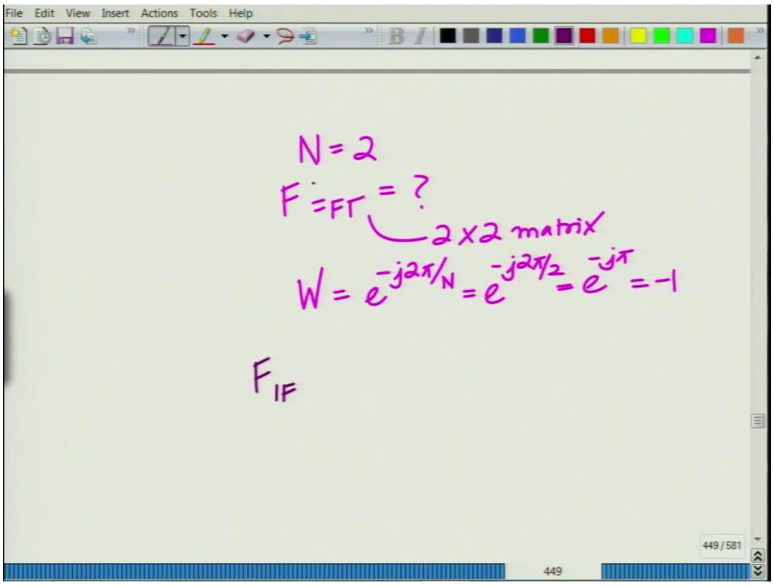
(Refer Slide Time: 16:23)





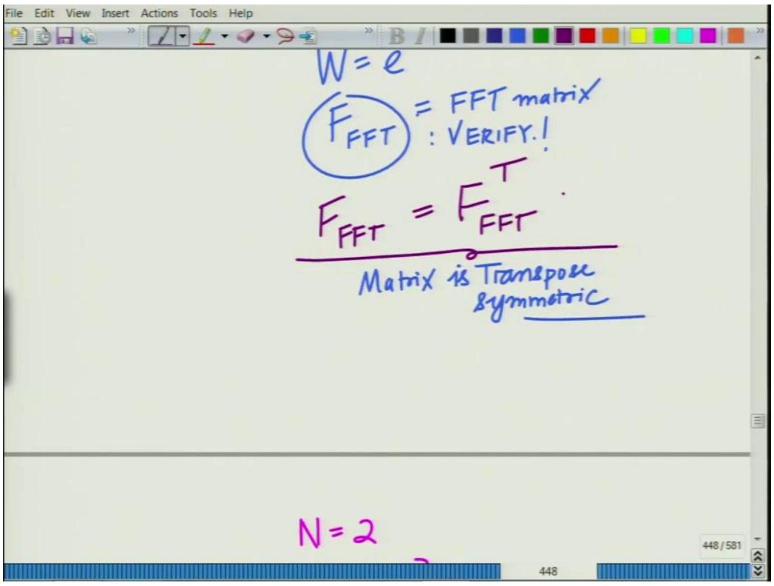
If you look at any two element that is the there is a transpose this matrix is transpose symmetric that is F IFFT of ij equal to F IFFT of ji, so this is the transpose symmetry matrix not Hermitian symmetric, transpose symmetric so, this matrix is do not get confused all the it is a complex matrix exhibits only Hermitian transpose symmetric so, do not there is a tendency to get confused because it is a complex matrix what is exhibit only transpose symmetric.

(Refer Slide Time: 17:13)



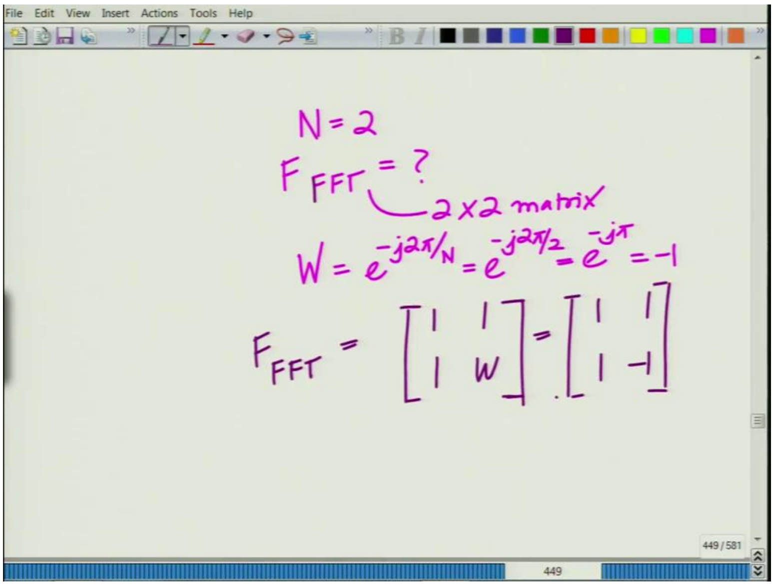
So, let us look at a simple example to understand this for instance, let us take the case of N equal to 2 ask our self, what is F IFFT F IFFT equal to what naturally this will be an N cross N this will be 2 cross 2 matrix the W equals e raise to minus j 2 Pi over n equals e raise to minus j 2 Pi over 2 which is e raise to minus j Pi which is minus 1 and F IFFT this is equal to this matrix.

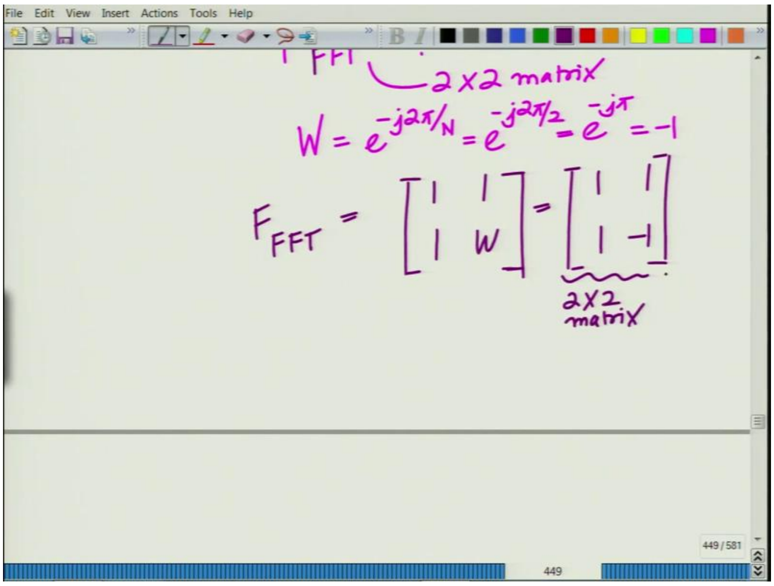
(Refer Slide Time: 17:54)



F FFT everywhere here this has to be F FFT this is F FFT.

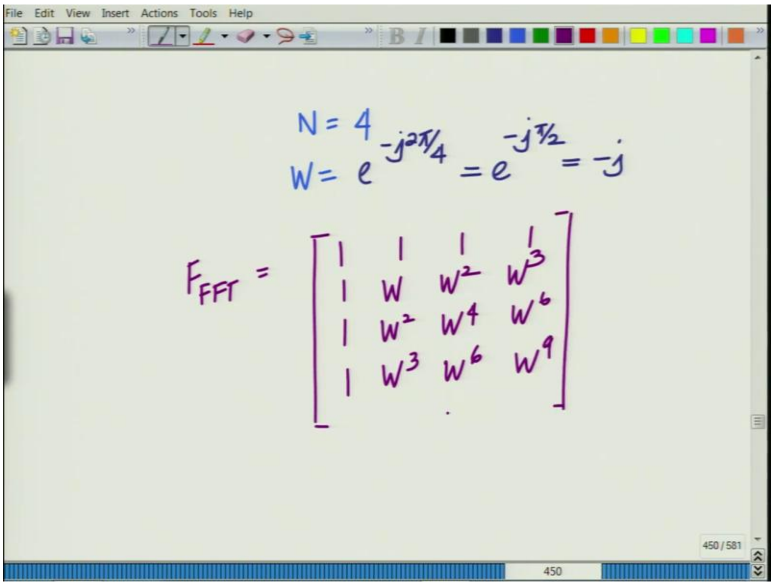
(Refer Slide Time: 18:12)

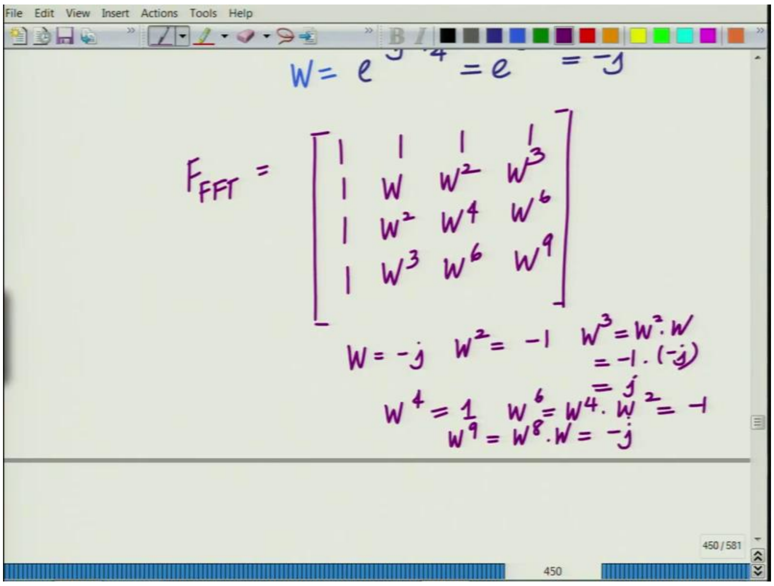




And F FFT equals 1 1 1 W which you can see this case it is very simple this is 1 1 1 minus 1 so this is a 2 cross 2 matrix, this is the F FFT which is a 2 cross 2 matrix.

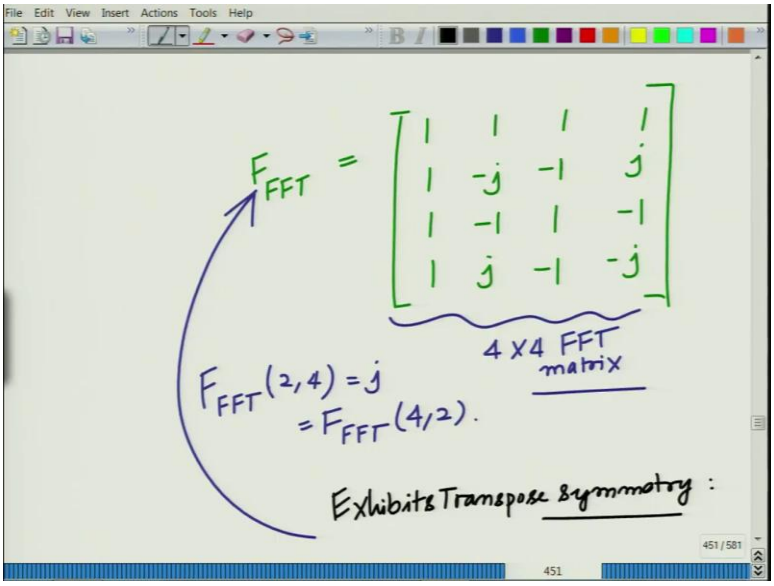
(Refer Slide Time: 18:46)





Let us look at this for N equal to 4 it becomes a little bit more complex N equal to 4 W equals e raise to minus j 2 Pi divided by 4 which is e raise 2 minus j Pi divided by 2 which is minus j and therefore, F FFT the Fourier transform matrix this will be 1 1 1 1 1 1 1 this will be W W square W cube W square W four W six W raise to three W raise to six W raise to nine N minus 1 square and if you substitute this and you can simplify these various expressions for instance you can see W equals minus j W square equals minus 1 W cube equals W square into W which is minus 1 times minus j equals j W four this is a fourth root of unity so W four is naturally equal to 1 W six equal to W four into W square which is equal to minus 1 W cube we have already seen W nine equals w eight into W eight it is 1, so W is basically minus j you use all these things and you substitute this matrix.

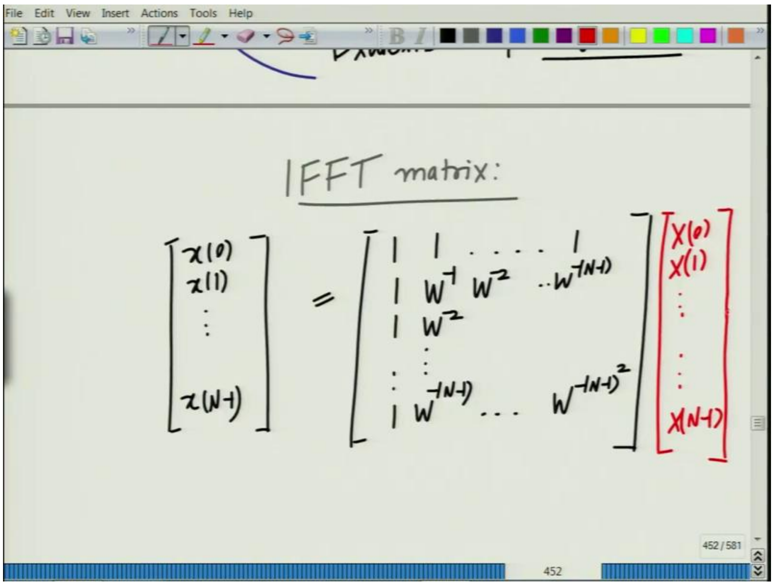
(Refer Slide Time: 20:39)

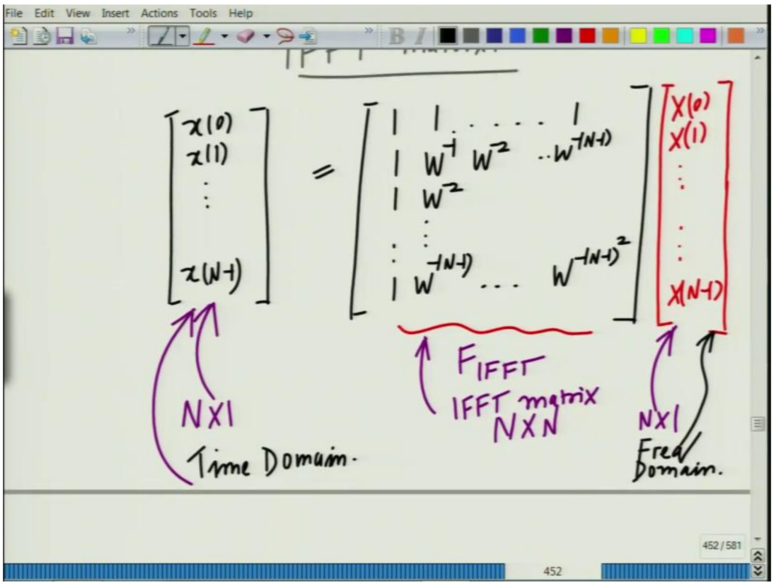


So, you will get F FFT this is the 4 cross 4 matrix this will be equal to 1 1 1 1 1 1 1 this will be W which is minus j W square which is minus 1 W cube which is j W square which is minus 1 W four which is 1 W raised to six which is minus 1 this will be W raised to W three which is j this will be W six which is minus 1 and finally we have W nine which is minus j and this is the Fourier matrix this is a 4 cross 4 Fourier matrix.

And you can clearly see it exhibits transpose symmetry that is if you look at F FFT of let us see F FFT of 2 comma 4 equal to j which is the same thing as F FFT of 4 comma 2 exhibits transpose symmetry and it is not Hermitian symmetry because if it was Hermitian symmetry then F FFT of 4 comma 2 would be equal to the conjugate of FFT of 2 comma 4 that would it would be minus j so it is not Hermitian symmetry, but it is rather transpose symmetry, so this is something that is important to keep in mind not to get confused.

(Refer Slide Time: 22:47)





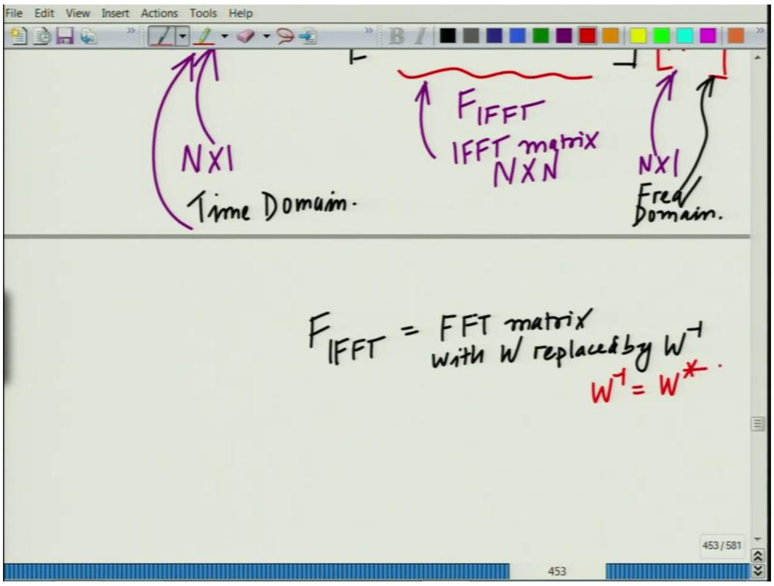
Now, let us now come to the IFFT matrix and IFFT matrix can be written similarly you can see let us now come to the IFFT matrix and the IFFT matrix is given similarly this is an N cross N matrix if you look at this again this is nothing but you will get back the time domain representation what does the IFFT do?

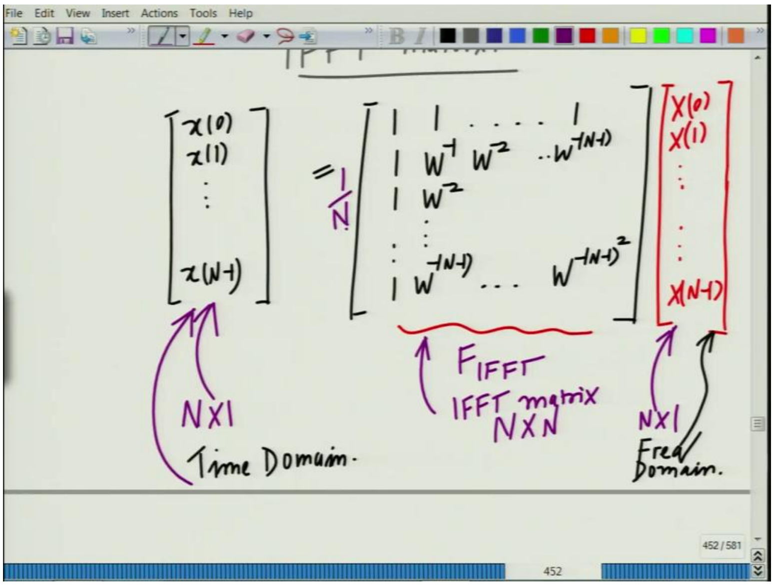
It gives back that time domain representation from the frequency domain quantity so this will be 1 1 1 1 once again 1 1 1 W raise to minus 1 W raise to minus 2 minus N minus 1 W raise to minus 2 W raise 2 minus N minus 1 W raised to minus N minus 1 square and you will have here capital X0 so from the frequency domain get back the time domain and therefore, you can see this is again this is essentially your F IFFT this is the IFFT matrix this is of size

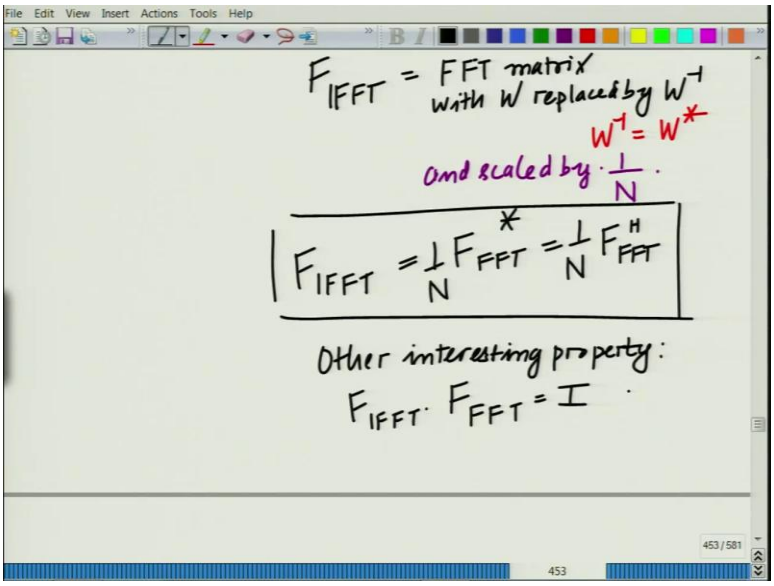
N cross N this is N cross 1 this is of size N cross 1 capital N cross 1 this is your small x's remember, we are using the small x's to denote the time the capital X's to denote the frequency.

So, this is the time domain this is the frequency domain once again you have the intro and if you look at this here all the only difference here is this is the FFT matrix if we are shrewd you will observe that this is the FFT matrix with W replaced by w inverse and what is W.

(Refer Slide Time: 25:21)





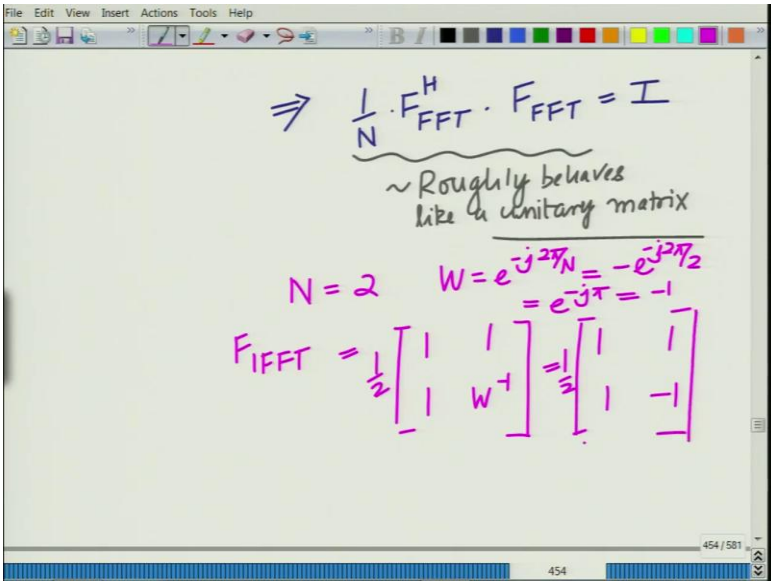


So, this is nothing but F IFFT equals FFT matrix with W replaced by W inverse and which is nothing but, also W conjugate and of course, plus there is a scaling by 1 over N not to forget that, there is a scaling not to forget in front of this there is a little scaling factor 1 over N it, so replaced by W inverse or W conjugate and scaled by 1 over N.

And therefore, now since it is replaced by w conjugate scaled over 1 over N you can see it is essentially the conjugate of IFFT matrix and plus you will in addition you will have the scaling factor of 1 over N so, you can see FFT F IFFT is an important relation F IFFT equals F you can say F FFT conjugate or since it exploits the transpose symmetry conjugate will be the same as Hermitian, so because Hermitian is nothing but conjugate and transpose and this is nothing but F FFT you can say Hermitian.

Now, the other interesting property that you will see here is that you have F IFFT into F FFT equal to identity because if you take the I FFT and then followed by the IFFT you will get back to the original signal but remember F IFFT is nothing but the Hermitian of F FFT of course, divided by 1 over N.

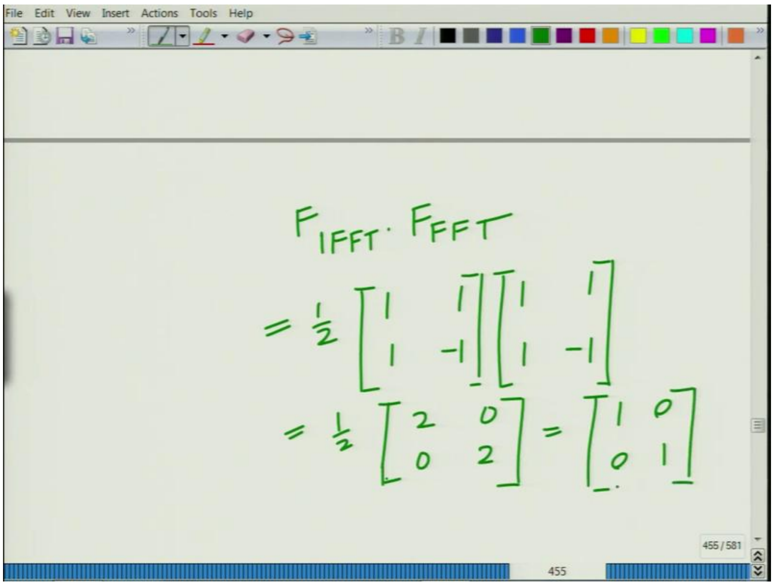
(Refer Slide Time: 27:55)

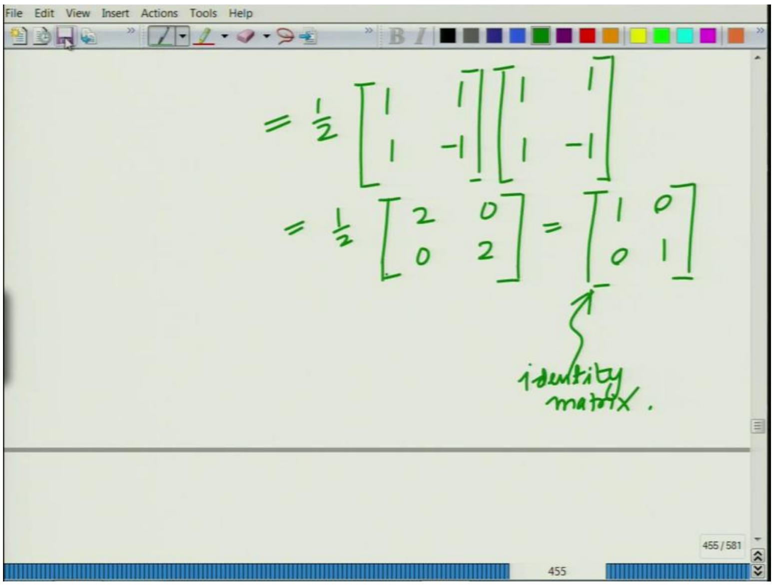


So, this implies that 1 over N F FFT Hermitian into F FFT equals identity so you can say this is roughly behaves like a unitary matrix because unitary matrix is when you have a matrix U says that you Hermitian U U U Hermitian equal to identity but here you do not have exactly U Hermitian but, you have 1 over N times U Hermitian.

So, it roughly behaves like an unitary matrix so something so, roughly behaves like unitary matrix that is an interesting property let us look at the IFFT matrix so N equal to 2 let us again write this F IFFT of course we know W equals e raised to minus j 2 Pi over N which is equal to e raised to minus j 2 Pi over 2 which is e raise to minus j Pi which is minus 1 so F IFFT equal to 1 1 1 W minus 1 or W raise to minus 1 W inverse which is nothing but again and of course, there will be a factor of 1 or 2 not to forget that and that is essentially that is 1 over 2 1 1 1 minus 1.

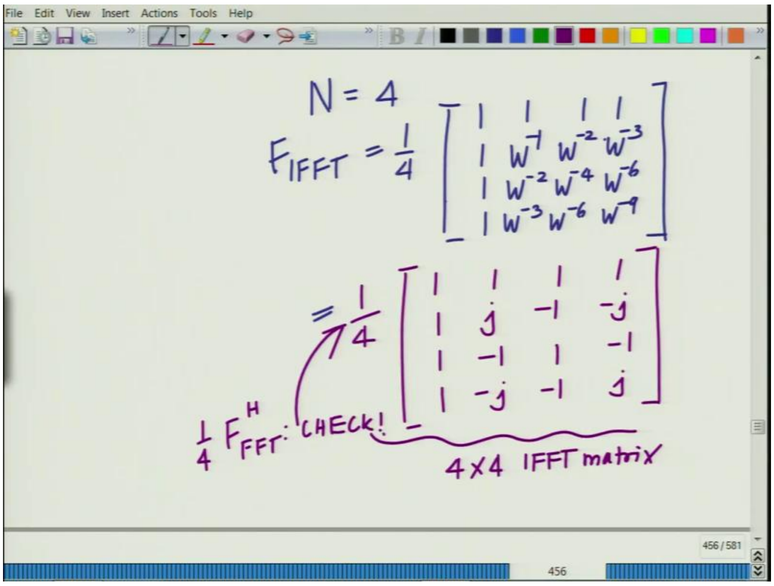
(Refer Slide Time: 29:53)

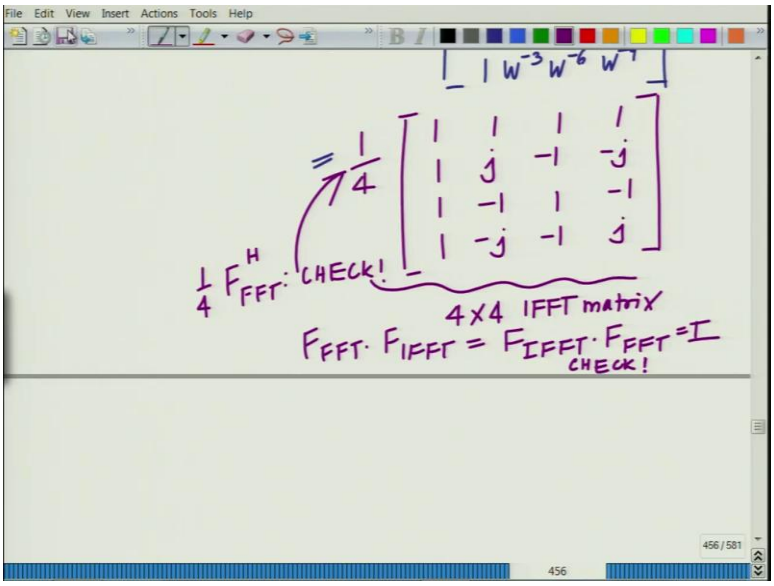




And if you now look at IFFT into FFT this is equal to half 1 1 1 minus 1 times F FFT which is again the same matrix 1 1 1 minus 1 which is equal to half now you can easily see 1 comma 1 entry 1 into 1 plus 1 into 1 2 1 comma 2 entries 1 into 1 1 into minus plus 1 2 minus 1 that is 0 2 comma 1 is 1 into 1 plus 1 minus 1 is 1 that is 0 and 2 comma 2 is once again 2 and this is nothing but essentially your 1 0 0 1 that is essentially the identity matrix this is the identity matrix 2 cross 2 the identity matrix.

(Refer Slide Time: 31:04)





Quickly look at N equal to 4 what happens to the IFFT matrix N equal to 4 so F IFFT becomes 1 over 4 this is 1 1 1 1 1 1 1 and everywhere I have W I replace by W minus raise to minus 1 so W minus 2 W minus 4, W minus 1 W minus 2 W minus 3 you can also obtain by as I told you taking the Hermitian of the FFT matrix and dividing it by N.

So, this will be equal to 1 over 4 1 1 1 1 1 1 1 this will be j minus 1 minus j minus 1 1 minus 1 minus j minus 1 j and you can once again so this is the 4 cross 4 IFFT matrix this is given as F FFT Hermitian divided by 1 by 4. So, this you can check and also check that F FFT into F IFFT equals F IFFT into F FFT equals identity again, you can check this and check these two propositions which should be fairly simple for you to check so these are the FFT matrices IFFT of size 2 cross 2 and 4 cross 1.

Similarly, you can construct other both FFT as well as IFFT matrices and it is very simple to see the k comma L^{th} element in the FFT matrix is essentially W raised to the power kL or you can say if your indexing starting from 1 comma 1 you can say W raise to K minus 1 L minus 1 and the k comma L^{th} element in that IFFT matrix will be 1 over N where N is the FFT or IFFT size times W raised to minus k minus 1 into L minus 1.

So, that is essentially the relation and we have seen both these matrices exhibit transpose symmetry and you can say the FFT and IFFT matrix are unitary except you have this factor this annoying little factor 1 over N sticking around so you have to take the Hermitian to get the inverse of the FFT matrix, you have to take the Hermitian and divide by 1 over N and to take the inverse of the IFFT matrix, you have to take the Hermitian and multiply by N depending on which matrix you are looking.

And as I told you Fourier analysis is fundamental to the analysis of linear system and signals, I mean, these matrices have several interesting properties and these matrices arise extremely frequently in the whole of system analysis and the whole of signal analysis wherever you have images, wherever your image processing wherever your image, for instance compression, we do something called your DST discrete cosine transform, which is roughly similar to this thing not exactly is this thing, your FFT, but it is very similar to the concept of re-analysis so image processing, image compression, image denoising and in everywhere you can think of image reconstruction this arises.

We are going to see a very interesting application of this in the context of OFDM that is orthogonal frequency division multiplexing that is, how do you analyse the OFDM and formulated in terms of a linear system and analyse it using FFT FFT matrices and describe what the properties are going to be what the resulting system model is, so that is something we are going to do in the next module, so let us stop here and I urge you to again go through this once again to understand the concept of Fourier analysis thoroughly. Thank you very much.