Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture 48
Linear Dynamical Systems: Matrix Exponential via SVD

Hello, welcome to another module in this massive open online course. So, we are looking at the applications of linear algebra specifically in the context of LDS that is Linear Dynamical Systems and autonomous linear dynamic systems. Let us now look at what happens when one inserts the Eigen value decomposition or one essentially gets the Eigen value decomposition to the picture of the particular matrix. So, let us look at autonomous linear dynamical systems and what implications does the Eigen value have, Eigen value decomposition have in the context of LDS?

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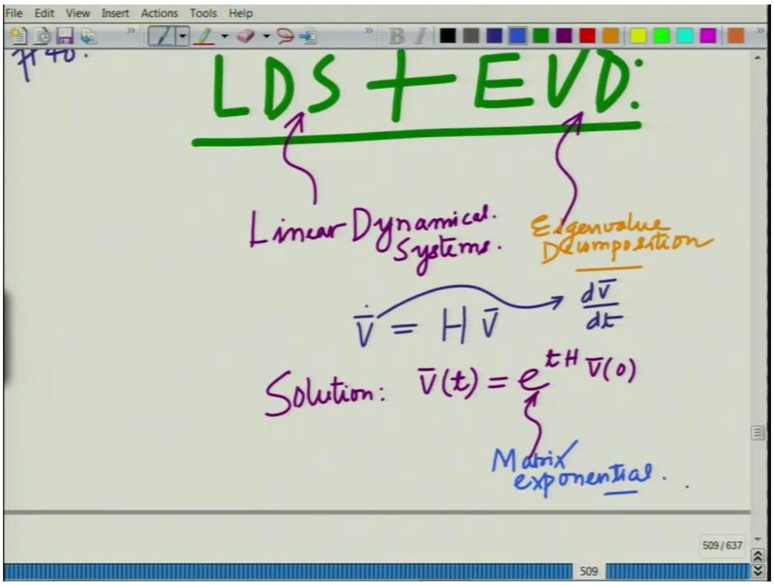

So, we are looking at LDS and what we are interested in looking at is; what are the implications of EVD that is the Eigen Value Decomposition for LDS? As you will know these are linear dynamical systems. So, this is a and this is the EVD which you must well be familiar by now. This is the Eigen Value Decomposition.

Now, let us look at what happens now, we remember the fundamental equation for EVD is you have your V bar dot equals H v bar, V bar dot is nothing but d v bar dot by dt and the solution of this is interestingly given as this is an autonomous linear dynamical system input is 0 solution is given as V bar t equals e raised to the power of you might remember this is e raised to the power of t times H into V bar 0 where this is a very interesting quantity.

This is what we have termed as the matrix exponential. This is what is termed as a matrix exponential. Now, let us look at what happens, how can you derive further insights into this solution of the autonomous linear dynamical system using the Eigen value decomposition? So, let us start with the Eigen value decomposition of this matrix H.

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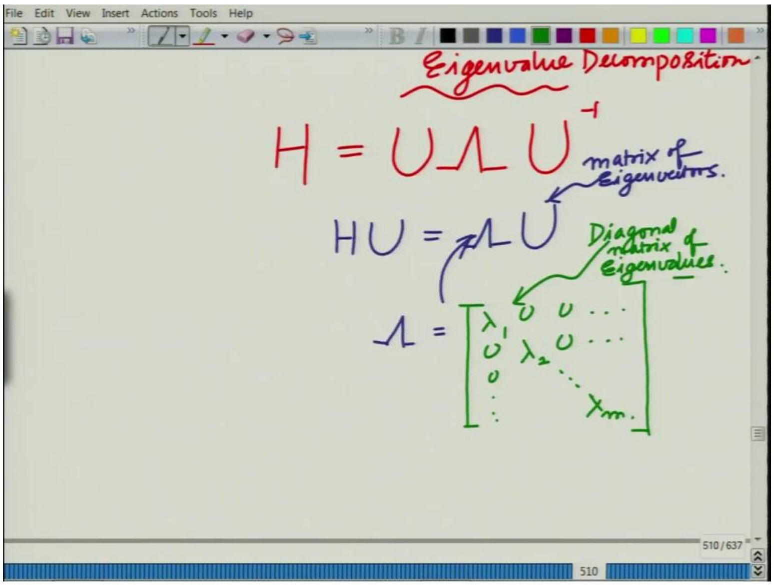

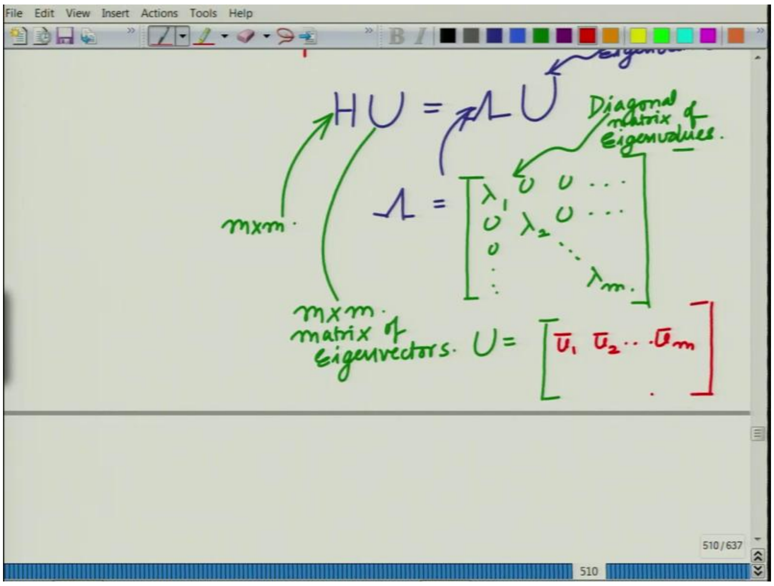

So, we have H equals U lambda, U inverse. This is, remember this is the Eigen value decomposition and remember it satisfies the property H U equal to lambda U where U is the matrix of Eigen vectors. Now, let us specifically look at and lambda remember this is the diagonal matrix of Eigen values; lambda equals you can write this as lambda 1 lambda 2 so on up to lambda m.

If this is an m cross m matrix, so these is the rest of the entries are of course, these are zeros. So, this is basically a diagonal matrix Eigen values and what happens here is that now we will form so, this is the m cross m matrix of this is let us say m cross m, U is the m cross m matrix of Eigen vectors. Let us say U equals u 1 bar, u 2 bar u m bar. These are the m Eigen vectors.

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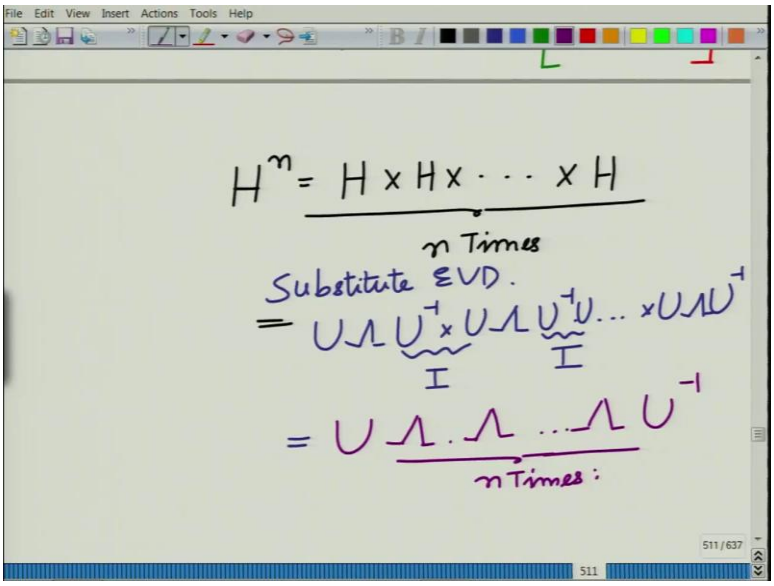

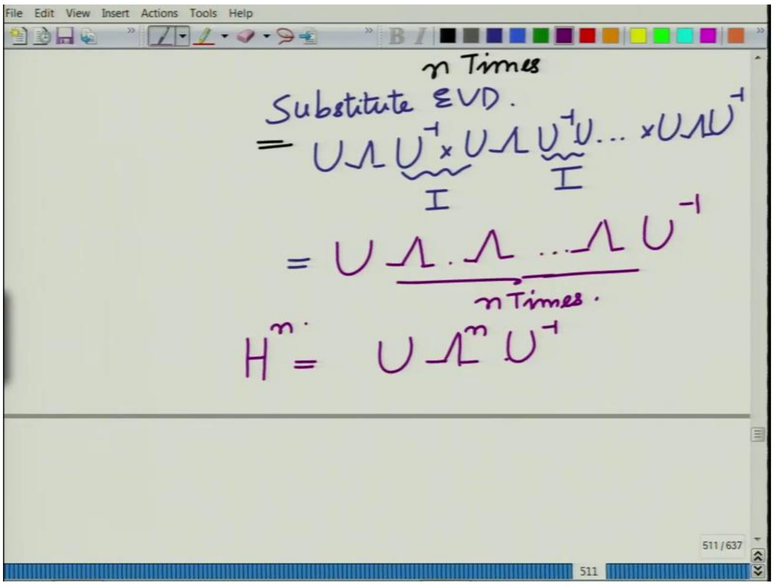

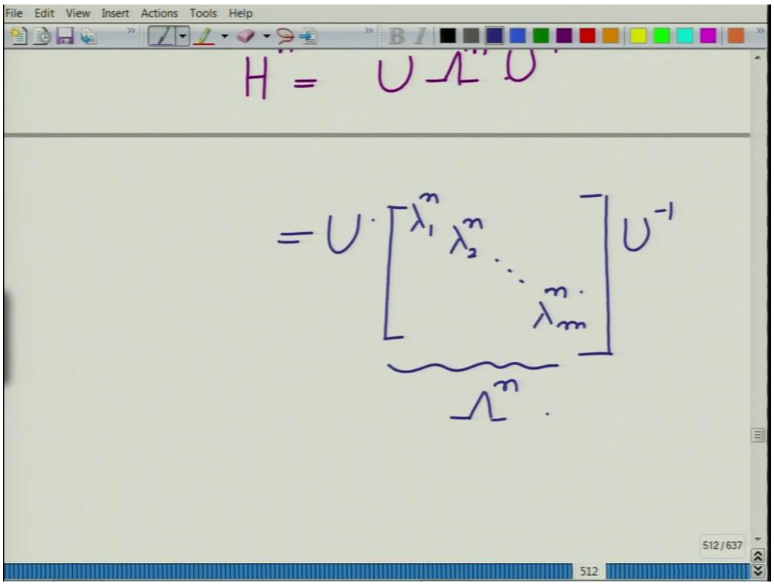

Now, what we are going to do now, observe an interesting probe. Let us see what is going to happen to H raised to the power of n. So, because H is a square matrix remember I can compute always compute H raised to the power of m that is multiply H with itself n times. So, let us look at H raised to the power of n, this simply means that the matrix H is multiplied with itself n times.

Now, substitute the EVD. So, we substitute the EVD, this becomes U lambda U inverse U, lambda U inverse so on and so forth U lambda U inverse. Now, you look at this U inverse into U this becomes identity and so on and so forth. So, you will have U inverse, you will have U over here and so on and so forth and U lambda U inverse and so on and so forth.

So, naturally all these U inverse U, this will become identity. All U will be left inside are these lambdas. So, this will be U. So, this will be u times lambda times lambda times U inverse. So, it will be this n times and this is ultimately equal to U lambda to the power of n U inverse. So, this is an interesting expansion. So, what you will see is H lambda, H raised to the power of n in terms of the Eigen value decomposition has a very simple expression.

It is simply the Eigen value decomposition is U lambda U inverse H raise to the power of n is U lambda is to the power of n U inverse. So, you simply take the matrix of a diagonal matrix of Eigen values lambda, raise it to the power of n which means that each Eigen values raise to the power of n lambda 1 to the power of n lambda 2 to the power of n and so on and multiplied by U post multiplied by U inverse.

So, and now you can see this is nothing but lambda raised to the power of n will simply be lambda raised to the power, lambda 1 raised to the power of n lambda 2 raised to the power of n, lambda m raised to the power of n and so on and this will be U inverse. So, this is basically your lambda raised to the power of n creating simply taking each Eigen value computing its nth power.

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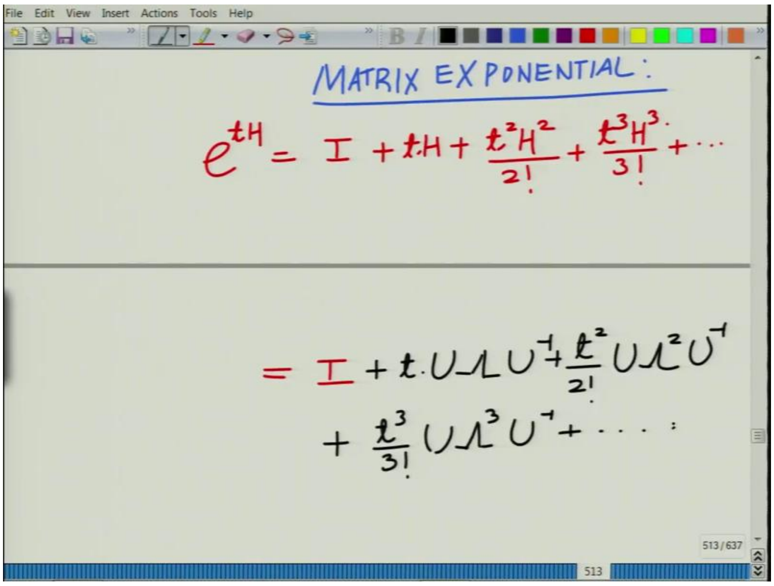

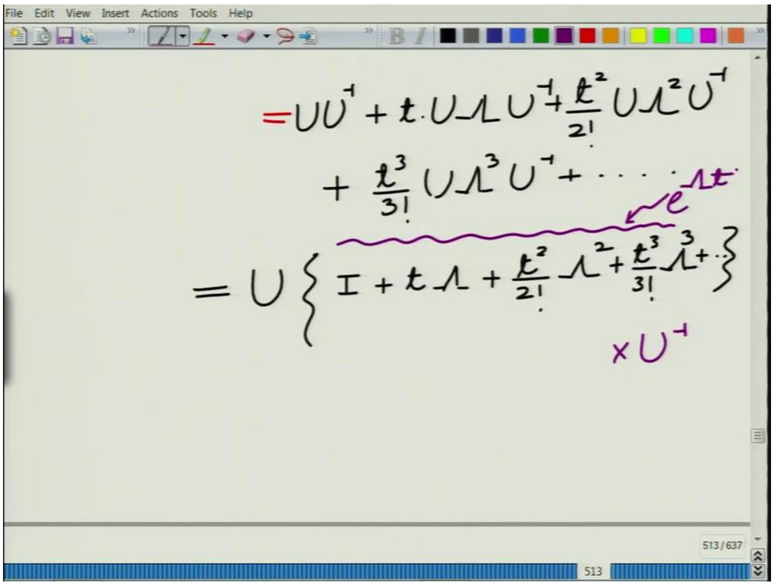

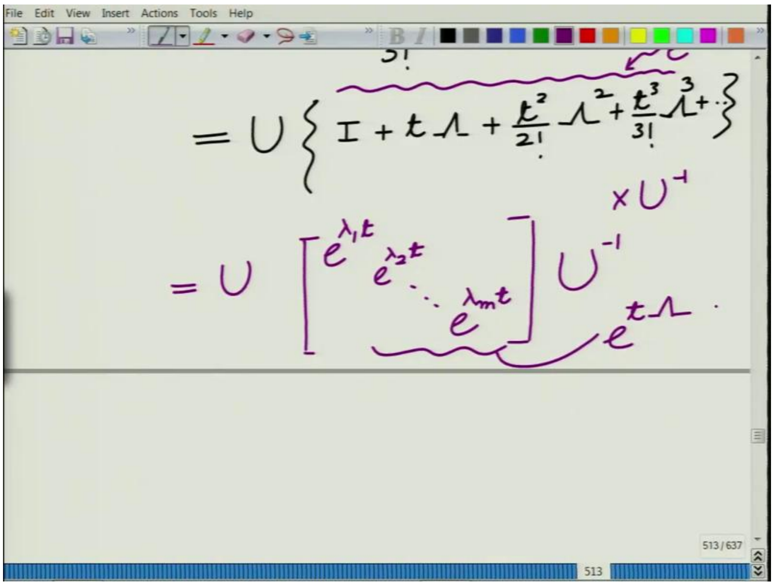

Let us now look at therefore, what happens to the matrix exponential now, let us ask the question what happens to the matrix exponential? Remember them recall the matrix exponential e raised to the power of t H that is equal to I plus t H plus t square H square by 2 factorial plus t cube H cube plus 3 factorial which is equal to I plus t times U lambda inverse plus t square H square by 2 factorial that is t square by 2 factorial U lambda square U inverse plus t cube by 3 factorial into H cube which is nothing but U lambda to the power of 3 U inverse plus so on which now, of course, I the identity matrix also I can write as U times U inverse.

So, net what is this going to be? This is going to be U times there is going to be I plus t lambda plus t square by 2 factorial lambda square plus t cube by 3 factorial lambda cube plus so on times U inverse and now, if you look at this you will notice something very interesting. This is nothing but e raised to the power of lambda t.

What do we mean by that, this is simply if you look at this this is simply U remember e raised to the power of lambda 1 t, e raised to the power of lambda 2 t so on and so forth e raised to the power of lambda m t times U inverse and this is nothing but e raised to the power of I can call it as t lambda, where lambda is the diagonal matrix of Eigen values.

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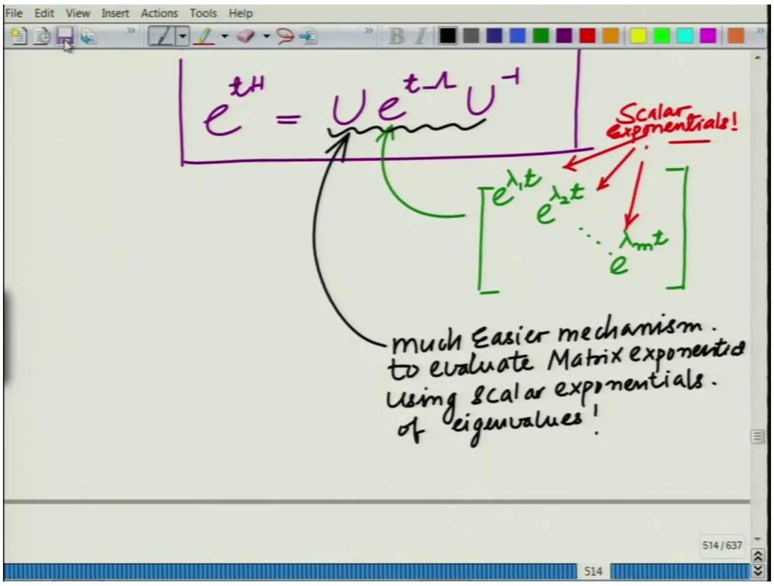

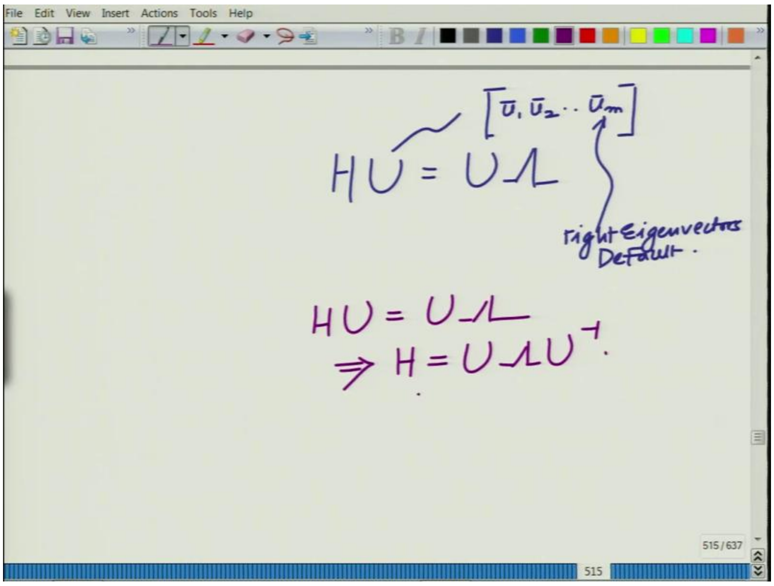

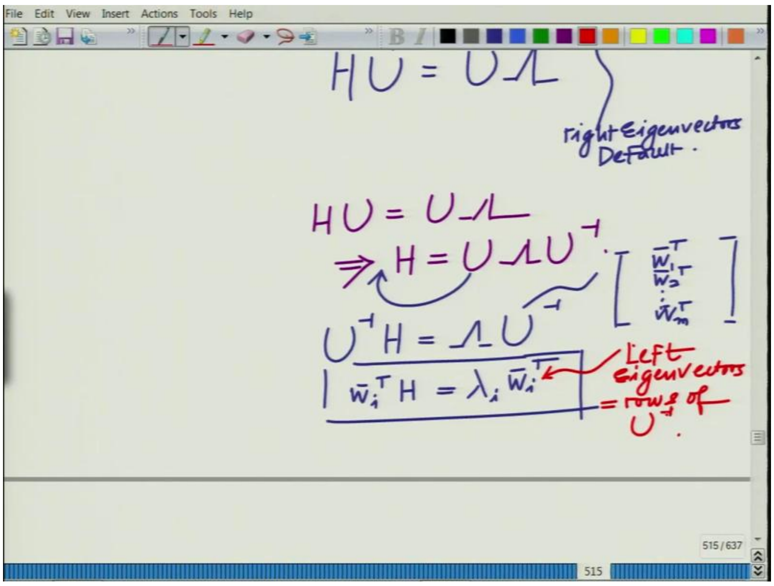

So, therefore, you can write e raised to the t H is U e raised to the t lambda times U inverse. So, this is a very interesting property because this tells you what is the matrix exponential in terms of the scalar exponentials the scalar exponentials. So, all this is you take the Eigen vector matrix U, U inverse and take each Eigen value raise it to the power of e raise to the power of t lambda 1, e raised to the power of t lambda 2.

So, these are scalar matrix. This are not, these are the scalar exponentials. So, you can see these are essentially; this is your e raise to the power of lambda 1 t, e raised to the order of lambda 2 t, e raised to the power of lambda m t. So, each of these is a scalar exponential. So, these are basically the scalar exponentials. So, it gives us a convenient, because we can easily evaluate the scalar exponentials.

We cannot easily readily evaluate a matrix exponential but the scalar exponentials we can easily evaluate. So, all you have to do is you take the scalar exponential of the Eigen values, construct the diagonal matrix, re-multiplied by U, post multiplied by U inverse that gives us the matrix exponential. So, this is essentially a very convenient way to evaluate the matrix exponential.

So, this gives us a much easier, rather than go through the much easier mechanism or you can say algorithm to evaluate matrix exponential using the scalar exponentials using the scalar exponentials or Eigen values. So, this is a very interesting expression. Now, let us look at this further. So, we have already seen H U equal to lambda times, U times lambda that is essentially what we have, where U is the matrix of Eigen vectors.

In fact to be more specific I can call this right Eigen vectors which is what we usually mean. So, this is what we mean by default. So, and therefore, this gives me a H of U equal to U lambda, this will bring U to the right. So, this gives H equal to U lambda U inverse. Now, there is also a notion in which you can define the left Eigen vector.

So, if you take U inverse on the left, you will have U inverse H equals lambda U inverse. Now, if this U inverse you can think of this as comprising of the rows w 1 bar transpose, w 2 bar transpose, w m bar transpose then what you can see clearly here is each w i bar transpose into H equal to lambda i w i bar transpose. So, these rows of U inverse are the left Eigen vectors.

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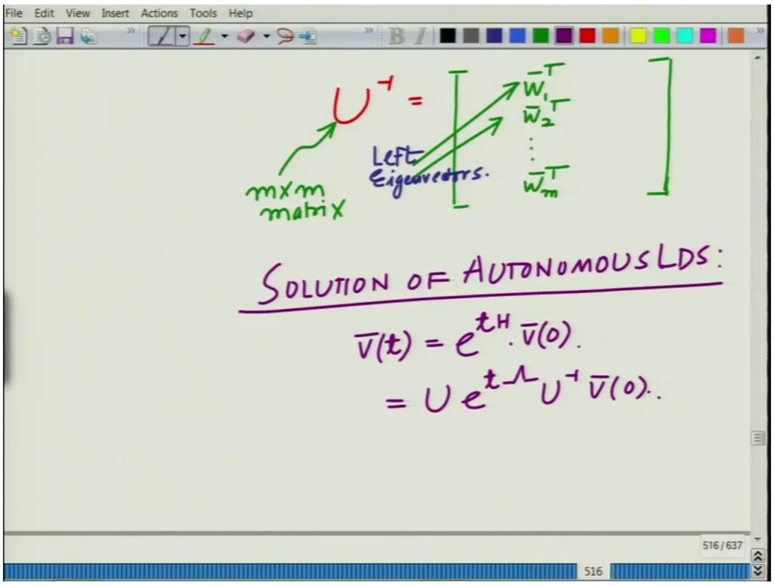

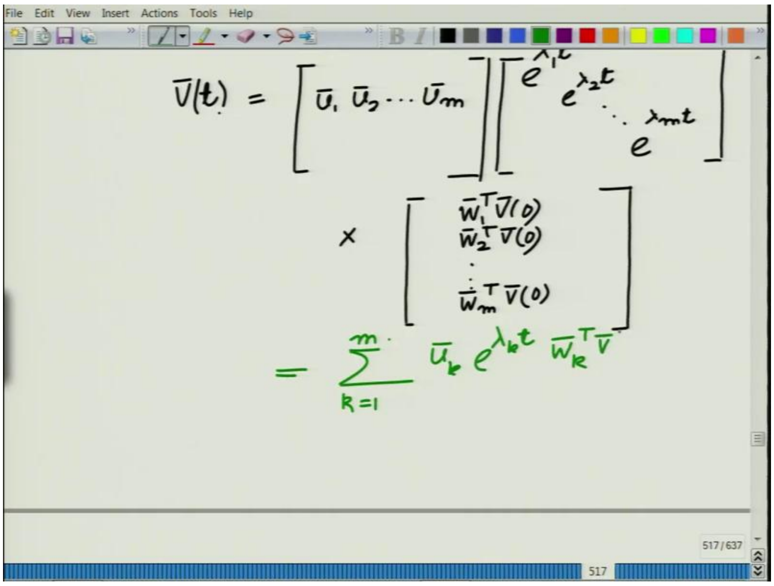

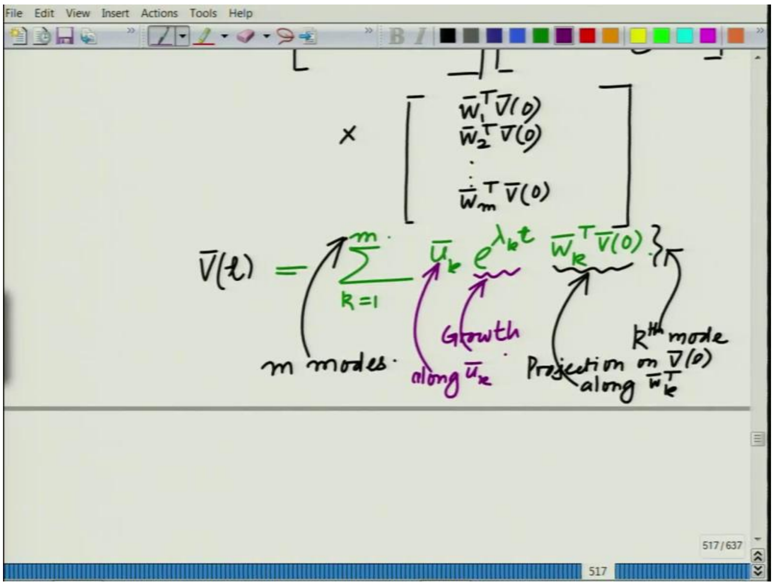

So, you can so, these are although we will not deal with these frequently these are the left Eigen vectors. These are rows of the matrix U inverse. So, by default we will always consider the right Eigen vectors but it is also useful to remember this notion of left Eigen vector; that is we take the matrix of right Eigen vectors U take the inverse U inverse rows of that will be the left Eigen vectors.

Eigen values are the same. Now, so we have U inverse just to be clear, you have U inverse will be w 1 bar transpose, w 2 bar transpose, w m bar transpose which as you can see will also be an m cross m matrix and these are essentially the left Eigen vectors. Now, let us look at the solution of the LDS. Therefore solution of the LDS becomes or specifically solution of autonomous LDS, for simplicity, we will only deal with autonomous LDS where the inputs are 0.

So, the solution of the autonomous LDS is remember V bar t is e raised to the power of t H v bar 0 and this is now we can write this as U is to the power of t lambda where lambda is the diagonal matrix of Eigen values U inverse times v bar 0 which now if I expand this, so I can write v bar t equals U which is the matrix of Eigen left, Eigen vectors or right Eigen vectors but we will simply call this as Eigen vectors wherever we need we will explicitly mention only the left eigenvector.

So, e raised to lambda 1 t, e raised to lambda 2 t so on, e raised to lambda m t times now, we have U inverse which is essentially your w 1 bar t multiplied by of course, whole thing is multiplied by v bar 0. So, this will be w 2 bar t multiplied by v bar 0. So on you will have w m bar t multiplied by v bar 0 which now essentially, you can easily see this can be written as K equal to 1 to m U k bar, e raised to the power of lambda k t w bar k transpose v bar 0.

So, you can think of this, each of these as a mode, you can think of each of these as essentially you have m modes. So, you can think of this so, this is your v bar t v bar t and so, you can think of this as m modes. So, the kth mode is essentially. So, this is essentially your kth mode, this is the projection of v bar 0 along w k bar transpose.

Remember which is the left Eigen vector and this grows, this is you can think of this as the growth factor, this is growing as grows as this growth. This is the growth e raised to the power of lambda k t and this is the final direction this is along. So, essentially the way you can think of this is essentially you are taking v bar 0.

So, kth mode, it has 3 components first take the projection along the left Eigen vector that is your, the right, left Eigen vector that is w k bar 0. Then grow it as e raised to grow it exponentially with the rate lambda k which is the kth Eigen value that is e raised to lambda k t and finally, which is the direction along with the kth mode is pointing that is U bar k.

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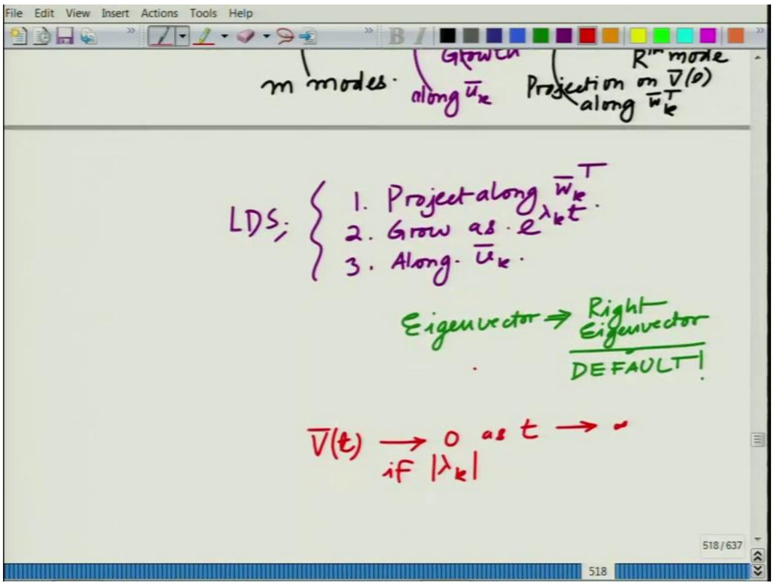

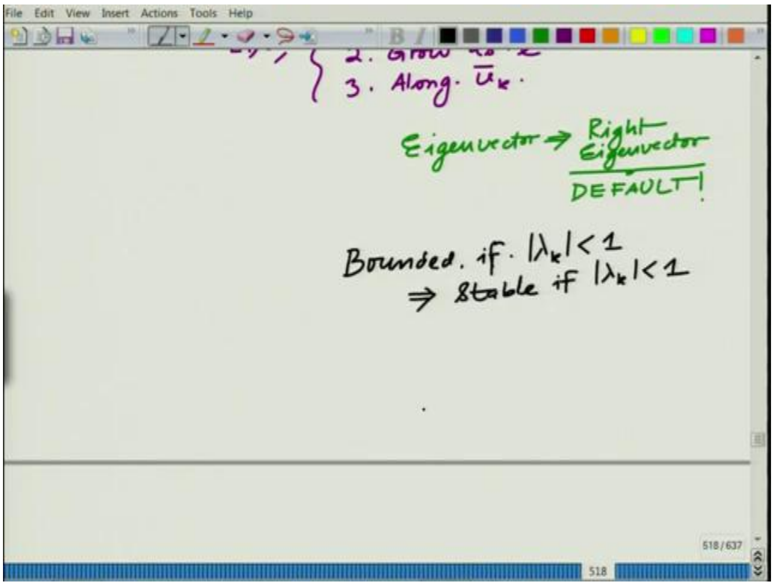

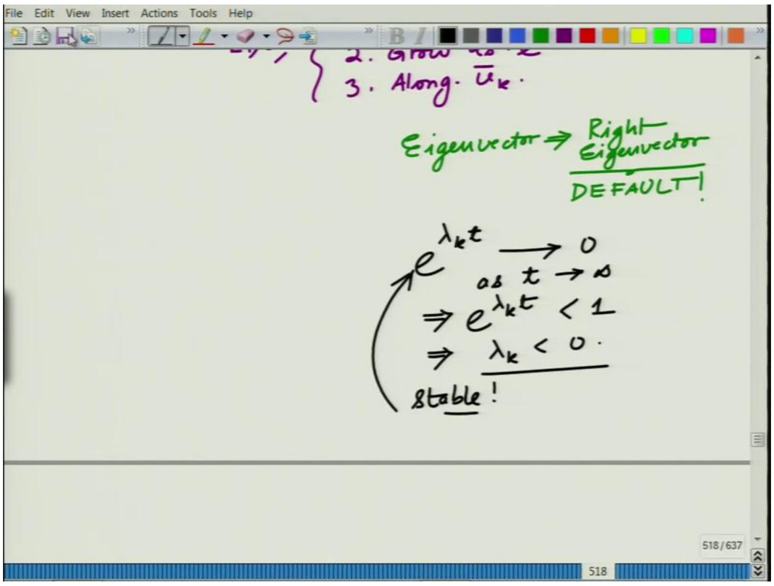

So, therefore, this is basically project along w bar k, so you can think of this as three steps essentially; project the trajectory, project along w bar k transpose, grow as e raised to lambda k t and finally, this is the whole thing is along U bar k. So, this is basically your, it essentially is your an interesting this thing an interesting way to think about you are autonomous LDS.

So, you project it along so, the projection along the kth left Eigen vector grows at the rate lambda k and pointing along U bar which is the kth right Eigen vector which will also call simply as the Eigen vector to avoid any confusion. Eigen vector always implies by default right Eigen vector. If it is a left Eigen vector, you will specifically mention it.

So let me also write that down Eigen vector implies right Eigen vector by default. So do not get confused. Now it is (essential) interesting to see when now one can ask the question, when is the system stable? When does this v bar t, when does this go to 0 as t tends to infinity. So you can see v bar t tends to 0 as t tends to infinity if magnitude lambda k is less than 1.

If magnitude lambda k or lambda k is lambda k is v bar k tends to 0 if or you can say this is bounded essentially. This is not, this is essentially bounded if magnitude lambda k less than 1 implies this is stable, if magnitude lambda k or e raised to or you can say this is stable if e raised to, therefore, e raised to lambda k t tends to 0 as t tends to infinity. This implies e raised to lambda k t less than 1. This implies lambda k less than 0.

So, if e raised to lambda k t, if this is essentially, if this quantity is essentially less than 1 or essentially the lambda k is if the Eigen value is less than 0 then we can say the system is stable. So, these are interesting interpretations that one can derive from the Eigen value decomposition. So, interesting things one can derive.

So, essentially what you can see is that the Eigen value decomposition gives you a very convenient mechanism to compute the matrix exponential in terms of the scalar exponentials of the Eigen values. So, you simply take the scalar exponential of the Eigen values and pre-multiply by U post multiply by U inverse that gives you the matrix exponential and one can use that in the solution of the autonomous LDS and the solution can be described very simply.

You take the projection along the left Eigen vector of the initial starting point that is v bar 0, grow it as e raised to lambda to the kth mode as e raised to lambda k t and finally, this point along the direction U bar k, that is essentially a very insightful and intuitive form of expressing the matrix explanation.

So, that I think presents a very interesting application and remember all this relies heavily on various principles of linear algebra. Of course, matrices, matrix multiplication, the Eigen value decomposition, projections and so on. So, again that shows how matrices can be used to efficiently compute the solutions of several complex systems and in this particular example the autonomous linear dynamical system which as we have shown has very interesting applications even in the context of circuits.

Remember, we built an example with a surface circuit a simple circuit with two capacitors and two resistors and showed how that can be modelled as a linear dynamical system. So, essentially this basically can be used as a technique to solve complex systems. So, let us stop here and let us continue this discussion in the subsequent modules. Thank you very much.