Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture No. 56
Discrete Time Markov Chains and Transition Probability Matrix

Hello, welcome to another module in this massive open online course. So, in this, in the past module, we have started our discussion regarding stochastic processes.

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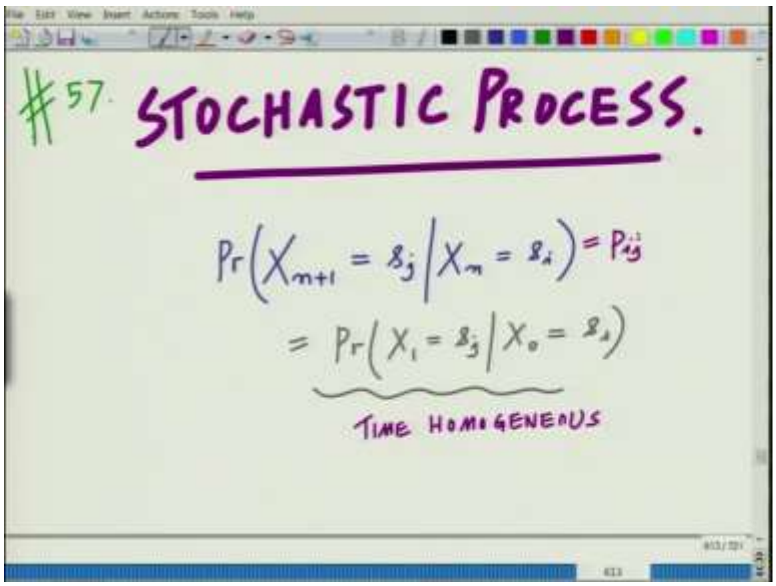

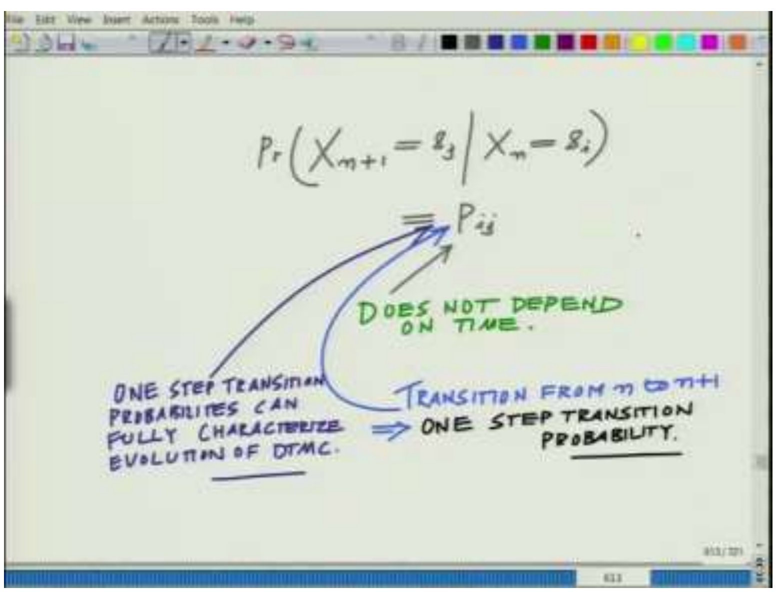

So, let us continue our discussions regarding stochastic process. So, we were talking about the one step transition probability that is the probability that Xn plus 1 equal to sj given Xn equal to si this is equal to the probability that X1 equals s j given X naught equals si and this is essentially does not depend on n. It is time homogeneous. We call this essentially as something that is time homogeneous and this quantity is denoted by P ij.

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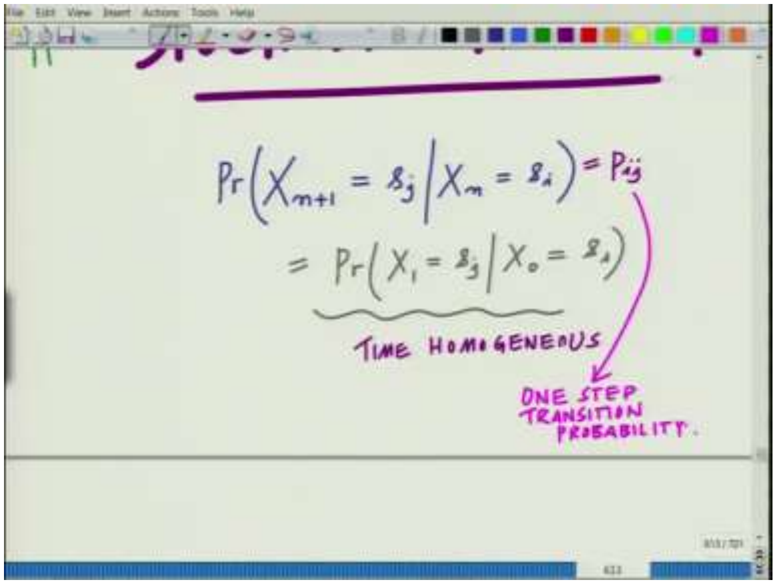

We are denoting this transition probability by P ij and this is essentially what we term as the one step transition probability, remember this what we are calling as the one step transition probability. And now out of these one step transition probabilities one can create this transition, what is known as transition probability matrix.

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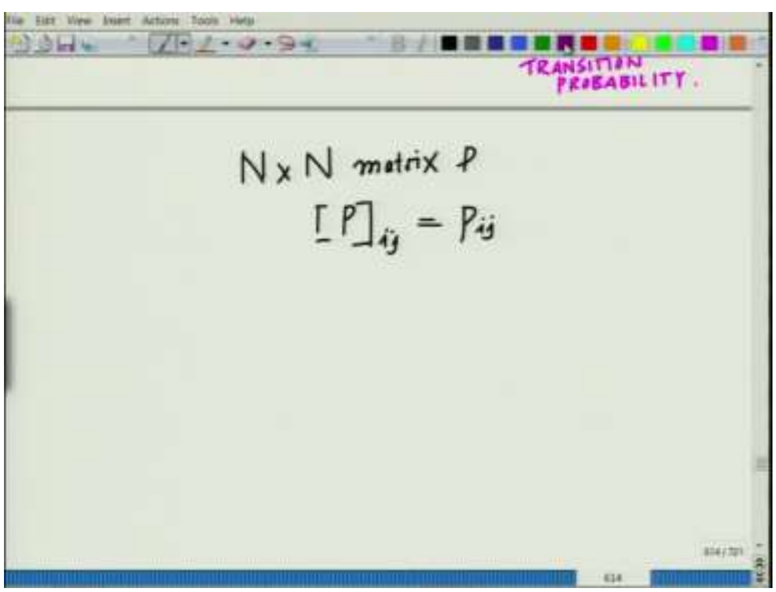

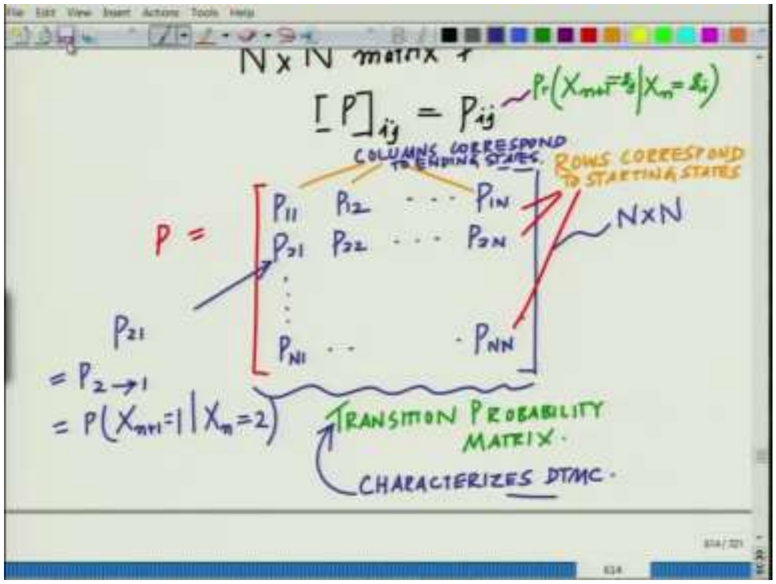

So, I am going to have N cross N matrix P such that the i th, ij th element of P is basically P ij. And I can formulate therefore which is again remember the probability Xn plus 1 equal sj given Xn is si and therefore I can form this matrix P as, I can form this matrix P with the elements P 11, P 12 so on P 1N, P 21, P 22, P 2N, P N1 so on up to P NN. So, this is naturally the N cross N matrix, corresponds to the one step transition probability.

So, this is simply known as the transition probability matrix, this is simply termed as the transition probability matrix, which essentially fully characterize to your discrete time Markov chain that is the importance. So, we have the one step transition probabilities and from Markov property that is essential to characterize the evolution of the stochastic process. So, this essentially characterizes, you can clearly see this characterizes this DTMC, its evolution and behavior and this can tell us many properties of DTMC that are going to keep in front.

Now, and now here you can see for instance, now you can see P 2, let us look at some of these elements, for instance let us say, let us ask this question what is P 21? P 21 equals the probability of transition from state 1 to state 2 that is, probability Xn plus 1 equal to 2 given Xn equals to 1, probability Xn plus 1 equals 1, there is a probability transitions from state 2, state 1 that is P ij, this is probability Xn plus 1 equal to 1, given Xn equals 2.

So, therefore, if you look at the rows, the rows correspond to the starting states that is the Xn. Rows correspond to the starting states and if you look at the columns, columns corresponding to the ending states in the transition. Columns correspond to the ending state, so rows correspond to the starting states, columns corresponds to the ending states in the transition. And this is essentially your N cross N transition probability matrix, which P ij denotes the probability of transition from state i to state j. There is a probability that Xn plus 1 equals sj given Xn equals state i and note that also, this is important and this is straight forward to see that, not difficult to see.

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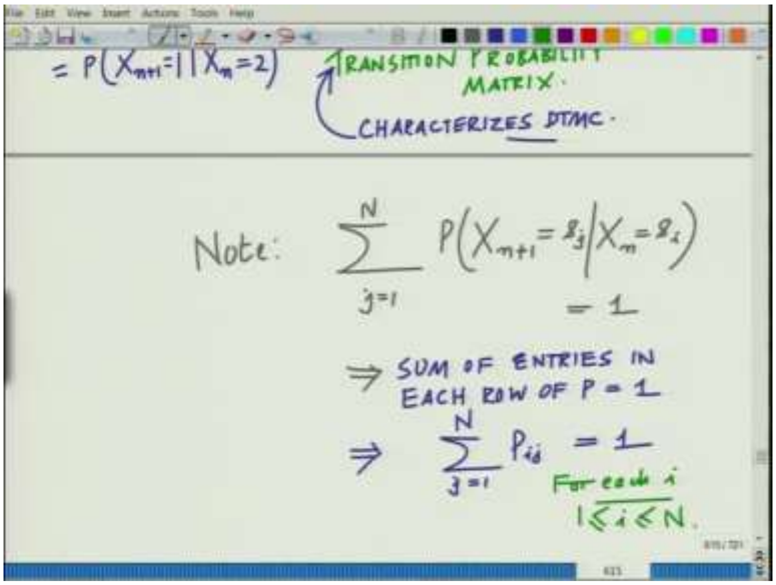

Note that if you look at the summation overall j, that is j equal to 1 and N, if you look at all these conditional probabilities Xn plus 1 is equal to sj given Xn equals si this must be equal to 1 that is starting in state si, if you look at all the conditional probabilities, some of the probabilities of transitions to all the other states that must be equal to 1, and therefore this implies that some of the probabilities in each row must be equal to 1.

This is essentially implies some of all entries in each row of P. This implies that if you look at summation j equal to 1 to N P ij this must be equal to 1 for all i for each i, that is 1 less than equal to i less than or equal to N, that is when I take any particular i and take the some of P ij or all j that must be equal to 1, that is the some of the transition probabilities, all the transition probabilities starting from the state i.

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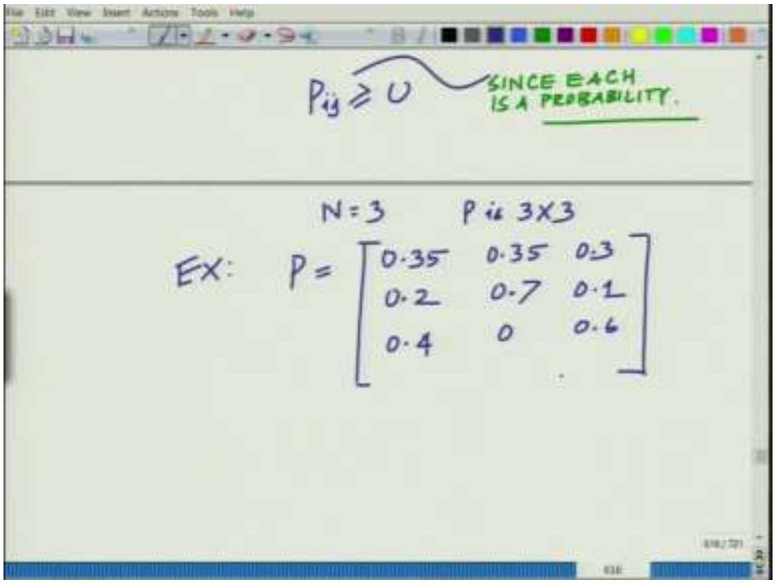

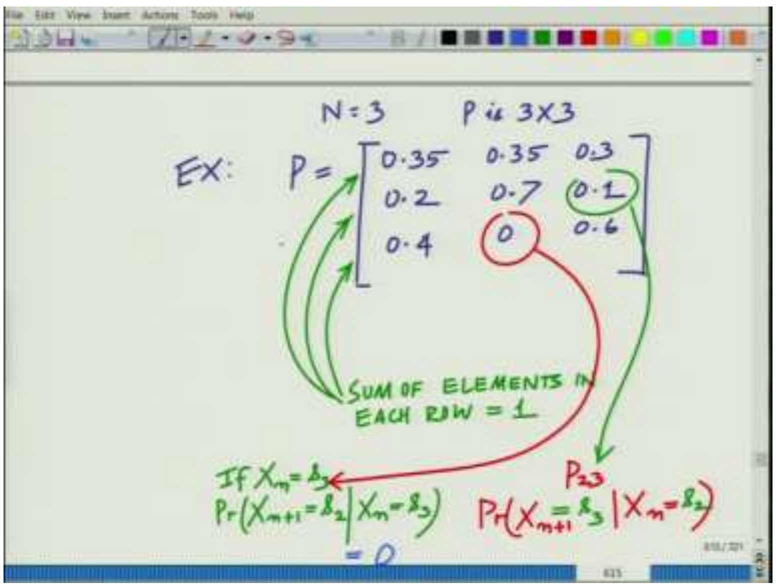

And further it goes without saying because each of these is a probability it naturally follows that P ij has to be greater than equal to 0, cannot be negative, it can be 0, probability can be 0, but cannot be negative, since each is a probability. Let us take a simple example, probability matrix P equal to 0.35 let us take N equal to 3 case, discrete time Markov chain that is your P is of size 3 cross 3. We can have a probability transition matrix that looks like this 0.35, 0.35, 0.3, 0.2, 0.7, 0.1, 0.4, 0, 0.6.

You can see sum of the elements in each row equals 1. For instance let us take it, look at this element P 23, what is this? This is your P 23. This is the probability Xn plus 1 equals 3 given Xn equals 2, that is starting from the state 2, what is the probability that the next state is 3 there is a point 1 probability that the next state is 3.

Now this 0 shows something interesting. 0 shows that starting from state, if the state Xn is equal to s3, probability Xn plus 1 equals 2 given or Xn plus 1 equals, I have to say Xn plus 1 equals s3, s2, Xn plus 1 equals s3 or Xn plus 1 equals s2, given Xn equal to s3, this probability is equal to 0 and similarly let me just trying to see if I have, so this is the probability of transition of state 2 to state 1, so I have to write Xn plus 1 equal to s1 given Xn equal to s2, so on.

And therefore, now we can write the state transition diagram, now we can also represent this transition probability matrix as a state transition diagram. We can draw each state and represent these transition probabilities on arrows so that is your, that is another way to represent a DTMC, although a slightly more elaborate, it is more visual, the matrix representation is more compact and suitable for mathematical manipulations. The pictorial representation, the figure we going to draw is better, it is easier to visualize. So, it is a visualization tool.

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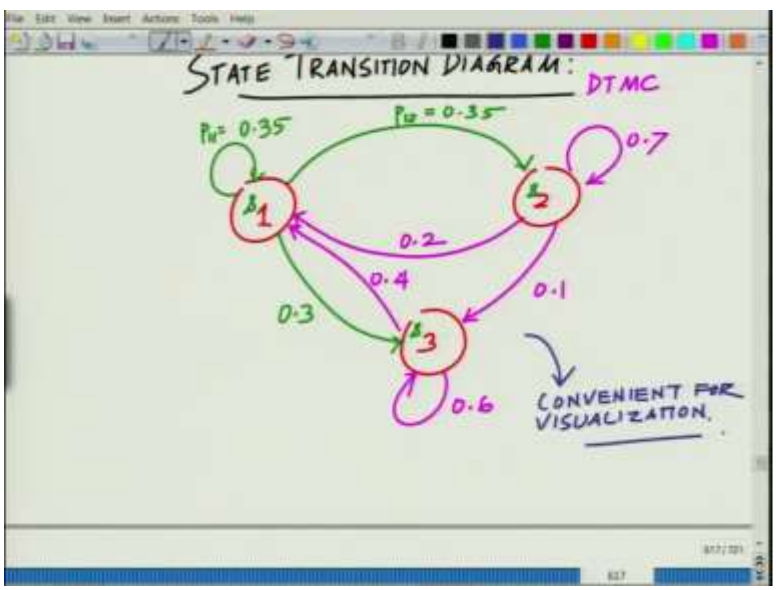

So, one can draw the state transition diagram for this DTMC. So, the state transition diagram, so you have 0.35, the probability, this is basically your P 11 transitions from 1 to 1, state 1 to 1, you can call this as s1, s2, s3. s1 to s2 this is P 12 which is equal to 0.35. s1 to s3 this is P 13 this is equal to 0.3. Now s2 to s1 this is 0.2. This is P 21, I believe this is 0.2. s2 to s2 that P 22, this is 0.7. And finally s2 to s3 this is P 23, this is 0.1. s3 to s1 this is essentially, this is 0.4. s3 to 2, remember the probably 0 so there is no branch which shows that it cannot transition from state 3 to state 2 because of probability 0. The only remaining one is transition 3 to 3 and that probability is point 6.

And this is essentially you state transition diagram. So, this is the state transition diagram for the DTMC. This is something that is very convenient. I would like to say, this is add, that this is convenient for visualizing. Something that is very convenient for visualization. All right, so this basically is the state transition diagram.

Let us try to look at a simple example, application of this discrete time Markov chain. For instance let us consider an example in the context of the industrial reliability. Let us assume that there is a machine. The machine can be in one of two states that is it can be in an operating condition that is it can be operational or it can be broken, it is faulty and it can transition between these states. Now let us try to build a model for this.

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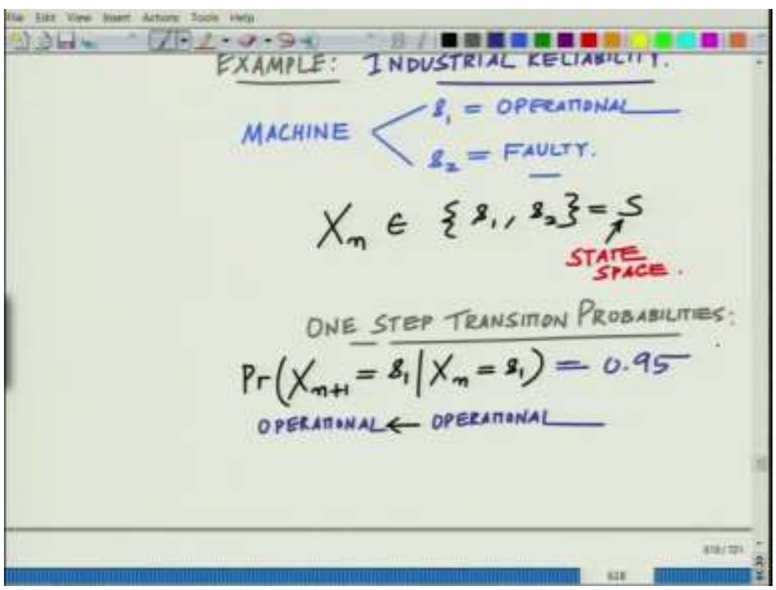

Let us try to look more I would say a practical example. So, we have industrial reliability, let us talk about, now what happens with industrial reliability? So, let us say we have a machine and this can be in two states. s0 equals or we can say s1 equals operational, s2 equal to faulty that it is not operational. Now let us try to build a model and it can transition between these two states. From operational it can become faulty, get broken and then from faulty it can be repaired and become operational ones again and then we can, one can build a model for this.

Now of course, practically one would hope that the machine does not break down too often. And the probabilities will reflect that. All right so Xn, we know now there are two states, so each time instant s n can belong to one of these two states, that is s1 or s2. Now we need the transition probability. So, this is essentially your state space, you might well remember, this is your state space and the probability.

Now let us look at the transition, the one step transition probabilities. Now to characterize this we need to give the one step transition probabilities. Probability Xn plus 1 equal to s1 given Xn equal to s1. That is it goes from a working state, an operational state to operational. That is it is originally operational s1 goes into the operational state in the next hour.

Let us say this n each time instant is an hour, first hour, second hour and so on. Hourly, that its plant is, or the industry is running, the plant is running continuously and we are monitoring the status of this machine. So, it goes from operational to operational, operations in one hour to operations in the next hour. Probability of that is 0.95 which shows that it is a fairly reliable machine, might not be extremely reliable but fairly reliable.

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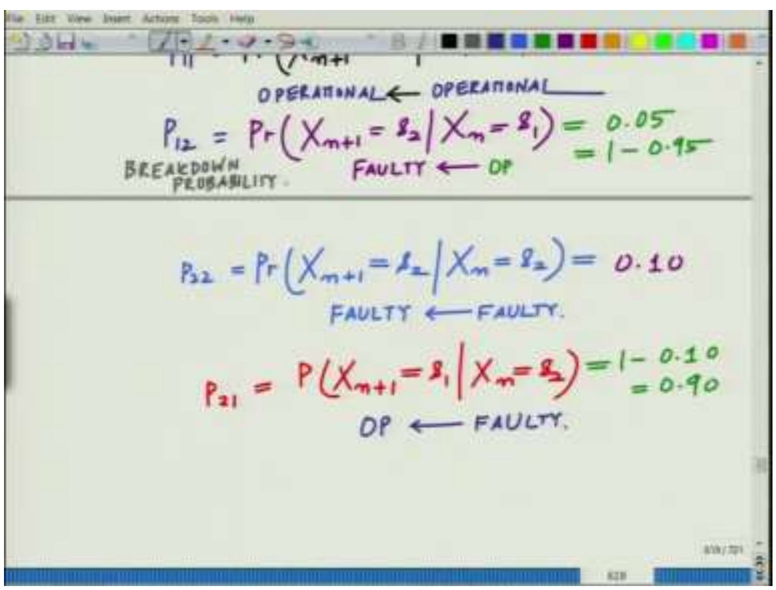

Similarly, now let us characterize the probability of breakdown that is probability of Xn plus 1 equal to s2 given Xn equals s1 that is operational. This is, it is going to faulty state from 1 that is originally operational. Now remember, we take all the probabilities corresponding to each state s i, s1 in initial state then they should add up to 1. So, naturally probability of Xn equals to s1 given Xn plus 1 equal to s 1 that is P 11 equals 0.95. So, that automatically means that P 12 it goes from operational to faulty that must be equal to 1 minus 0.95 that is 0.05.

So, this must be equal to point 05. That is 1 minus 0.95 and further this is your basically P 11 and this is your P 12 and this essentially is what we are calling as the break down probability. Working to, operational to faulty. This is essentially your break down probability. There is a probability of break down.

Now let us look at the probability Xn plus 1 equals s2 given Xn equals to s2 that is P 22. That is, it goes from faulty. So, it is faulty at the beginning of the hour and it continues to be faulty that there is no repair. There is certain probability with which it cannot be repaired and in that hour, and probability is basically point, let us say that the probability is point 10.

And therefore, P 21, the probability of it being repaired in that hour after, in an hour after break down is probability Xn plus 1 equals s1 given Xn equal to s2 that is equal to 1 minus 0.10 equal to 0.90 and this is the probability of repair. So, it goes from faulty to operational.

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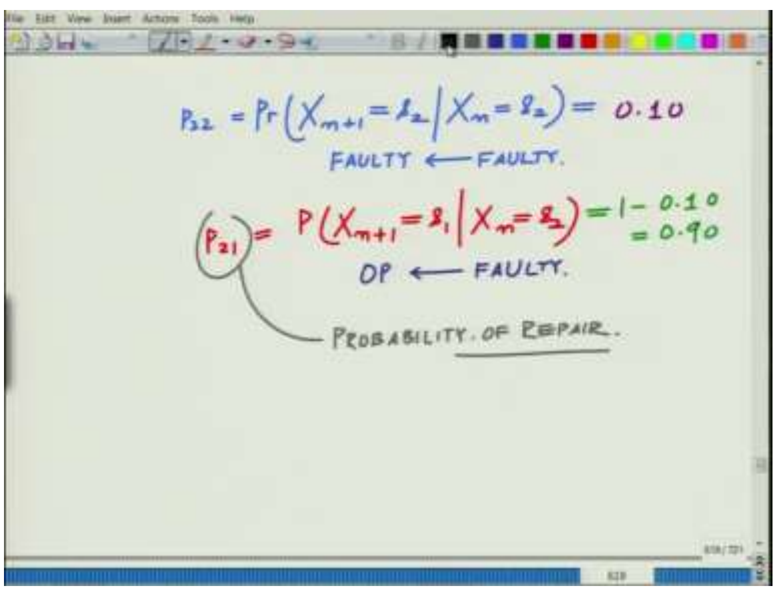

And this is essentially the probability of repair and now we have all these quantities that is each P ii. We have the P ij, so each P ij is greater than equal to 0. Summation of P ij or all j is equal to 1. Now we are at the one step transition probability matrix or simply the transition probability matrix.

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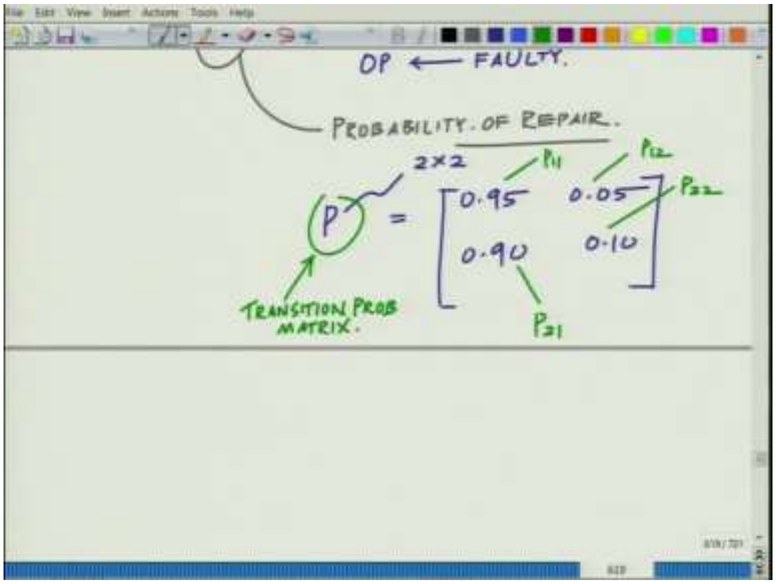

And therefore, now the P which is the transition probability matrix. Remember there is two states, so this is going to be a 2 cross 2 matrix this is going to be 0.95, 0.05, 0.90, 0.10. This is your transition probability matrix. That is basically built for, and now you can see the rows correspond to initial state, columns corresponds to the final state and sum of each row is equal to 1. Sum of the transitional probabilities, for instance this is your P 11, this is your P 12, this is your P 22 and this is your P 21 and you can see the sum over each row equals unity. So, this is the transitional probability matrix for this problem.

So, with this, let us stop this module here and let us, we will consider other similar examples and continue to build on these concepts, especially pertaining to the analysis of the discrete time Markov chain in the subsequent modules. Thank you very much.