Applied Linear Algebra for Signal Processing, Data Analytics and Machine Learning
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Lecture 66
Woodbury Matrix Identity – Proof

Hello, welcome to another module in this massive open online course. So, we are looking at the Woodbury matrix identity or also which is known as the matrix inversion lemma or the matrix inversion identity, we have looked at that in the previous module we understood its proof and now, let us look at a simple application of that.

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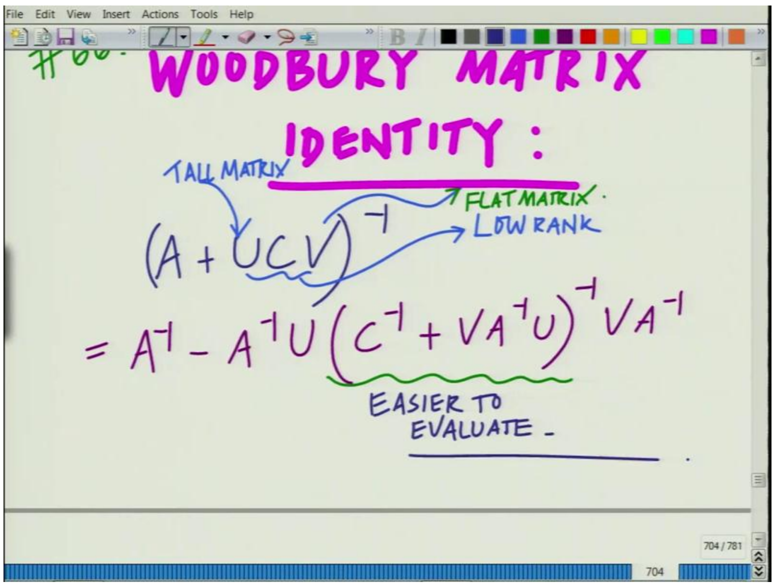

So, we are looking at the Woodbury matrix identity or what is also known as the matrix inversion lemma, which states the following thing which is essentially if you have matrices A plus UCV inverse this can be written conveniently as if A inverse is already known or if A is easily invertible, A inverse minus A inverse U times C inverse plus V A inverse U inverse times VA inverse and we said this especially useful if UCV is a low rank matrix.

This is especially useful if UCV, if this is a low rank matrix such as when U is a tall matrix and V is a what we call as a flat matrix that is more columns than rows and in which case this C inverse plus VA inverse U inverse that is of a much smaller size and therefore, it is much easier to evaluate in comparison to what we have in the left that is A plus UCV inverse, we also looked at a simple example in the previous module. So, this is much easier to evaluate. Fantastic.

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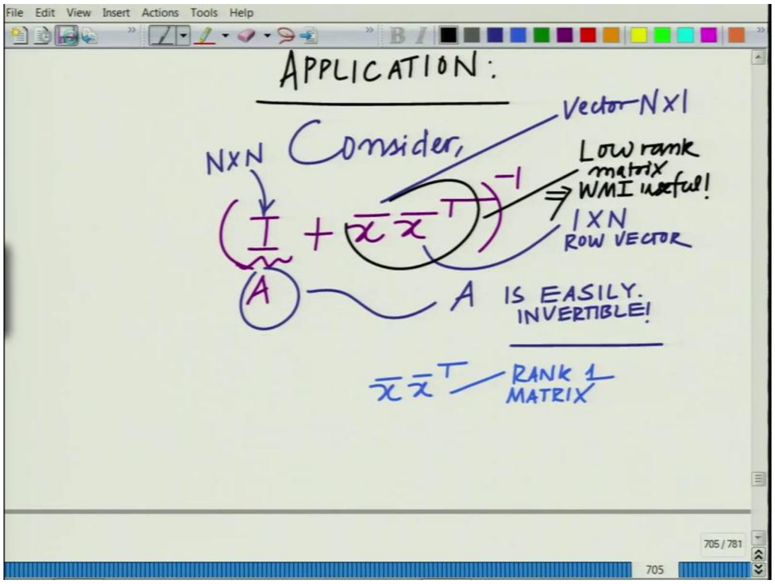

Let us now look at a simple application of this, consider the following example you have I plus x bar x bar transpose is a very simple example inverse, we want to compute the inverse of this matrix, where I you can see this is our quantity A and you can now easily see A is easily invertible, because A inverse is also equal to identity this is an exactly a scenario where your matrix inversion lemma or your Woodbury matrix identity is very helpful. And more importantly look at this, this is an extreme example x bar is a vector, this is a vector.

So, let us say this is your N cross N identity matrix. So, this is an N cross 1 vector x bar transpose this is 1 cross n vector row vector and therefore, this x bar x bar transpose, if you look at x bar x bar transpose this a rank 1 matrix. Because, see, this is a column vector times a row vector. So, every column of the resultant matrix will be a linear combination of the vector x bar. So, this is a rank 1 matrix.

So, this essentially if you look at this, this is what is known as a rank 1 matrix or a matrix that is formed from an outer product and therefore, this has rank 1. And this is especially this kind of scenario where the matrix inversion lemma or the Woodbury matrix identity the WMI is very helpful because this is a low rank matrix. So, that is the whole point. So, this is if you look at this quantity over here, this is a low rank matrix, this implies your Woodbury matrix identity is very useful for the inversion of this matrix this is precisely the kind of scenario where your matrix inversion identity is very helpful.

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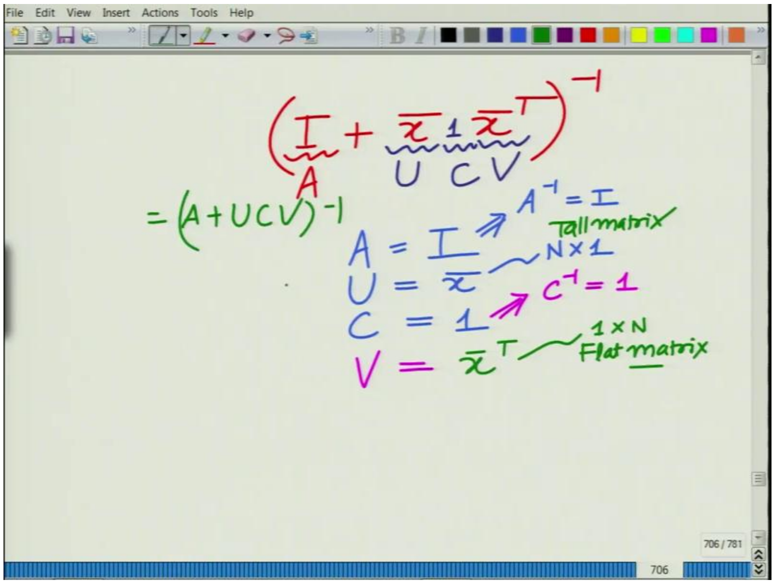

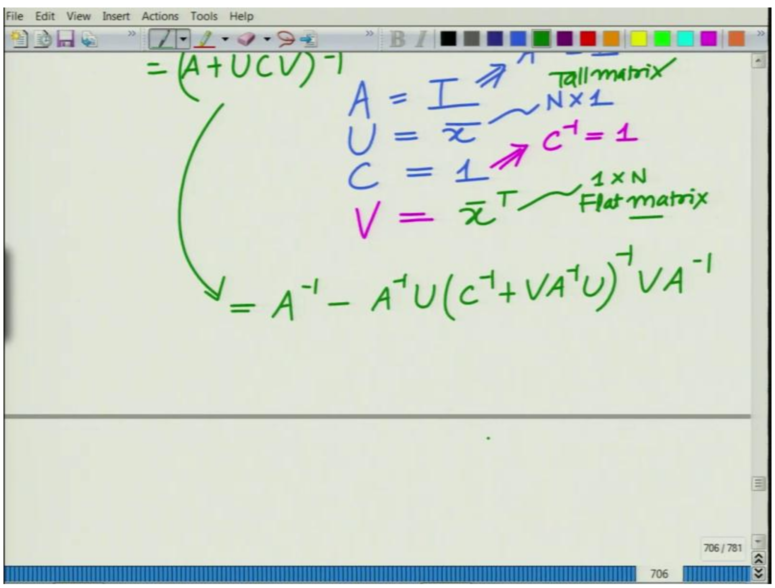

So, let us look at this counter quantity. So, this I plus x bar x bar transpose inverse this is basically your quantity A this is basically your matrix A this is basically your U this is basically your V. Now, what is C? Now C is not explicit we can right multiply this by 1 in between and this becomes your quantity C. So, let us note what these quantities are you have A equals the identity, U equals the vector x bar remember U is a tall matrix in this case it is simply a vector extreme example of a tall matrix.

So, this is your N cross 1 vector C is in fact a scalar quantity, so, C is 1 cross 1 can be readily invertible in fact A inverse equals identity, what about C inverse trivial C inverse equal to 1, C is

1 C is a constant 1, C inverse is also equal to 1 so, this is trivial. What about this V? V equals x bar transpose this is 1 cross N this is your tall matrix.

In fact, this is just a column vector this is a flat matrix, tall matrix flat matrix or tall matrix wide matrix. So, this is your, we have written this in terms of your A plus UCV inverse. So, that is essentially what this is, this is your A plus UCV inverse, now, make this into your matrix inversion lemma, so that becomes your A inverse minus inverse U into C inverse plus V inverse A inverse U inverse into VA inverse.

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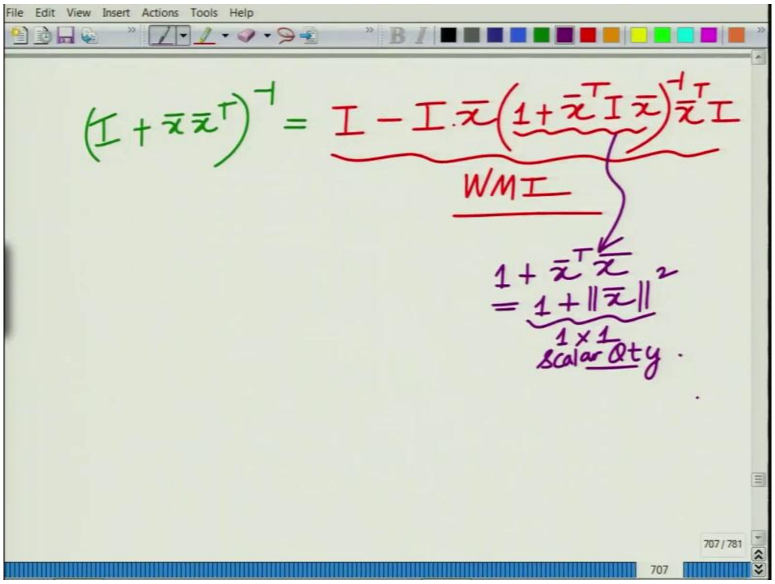

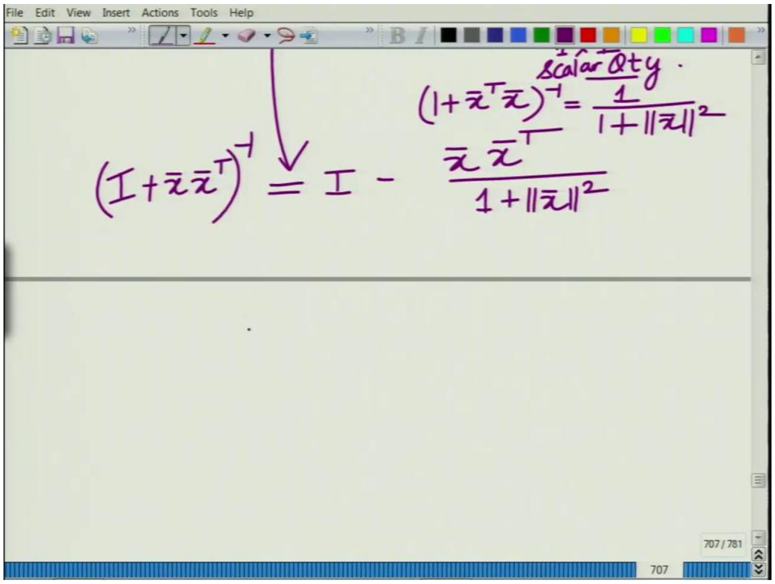

And therefore, now I can write the inverse as I plus x bar x bar transpose, this inverse is going to be A inverse. Let us write this out for term by term I hope you can see the expression above. So, this is going to be a inverse A inverse is identity minus A inverse identity into U which is your x bar times C inverse which is 1 plus V which is x bar transpose A inverse identity into x bar inverse V x bar transpose A inverse identity done. This is what we obtain from the Woodbury matrix identity please check that this is exactly what we obtain by applying the Woodbury matrix identity here to this expression that is identity matrix plus a rank 1 matrix which is expressed as x bar x bar transpose.

Now, let us simplify this. Let us start with this quantity 1 plus x bar transpose identity into x bar. You can say this is very simple, this is 1 plus x bar transpose x bar which is equal to 1 plus the norm x bar square which is a scalar quantity. This is a scalar quantity remember this is a 1 cross 1. This is a scalar quantity, so this becomes so the inverse of this is nothing but so 1 plus x bar transpose x bar is a scalar quantity.

So, this inverse is nothing but the reciprocal. So, this becomes 1 over 1 plus norm of x bar square 1 over 1 plus the norm of x bar square that is essentially what this quantity becomes and now you can apply this, so simplify this, this becomes equal to the identity minus identity into x bar divided by that is 1 plus x transpose x inverse that is divided by 1 plus norm x bar square times x bar transpose into identity, so that is it.

So, this is your very simple way to calculate the inverse. So, this is x bar into x bar transpose inverse and that is a very efficient. Now, look at this you do not need to do any matrix inversion simply I minus x bar x bar transpose compute the matrix product divided by 1 plus norm x bar square. So, this is a very low complexity. So, this right hand side can be computed very efficiently much more efficiently than the left hand side.

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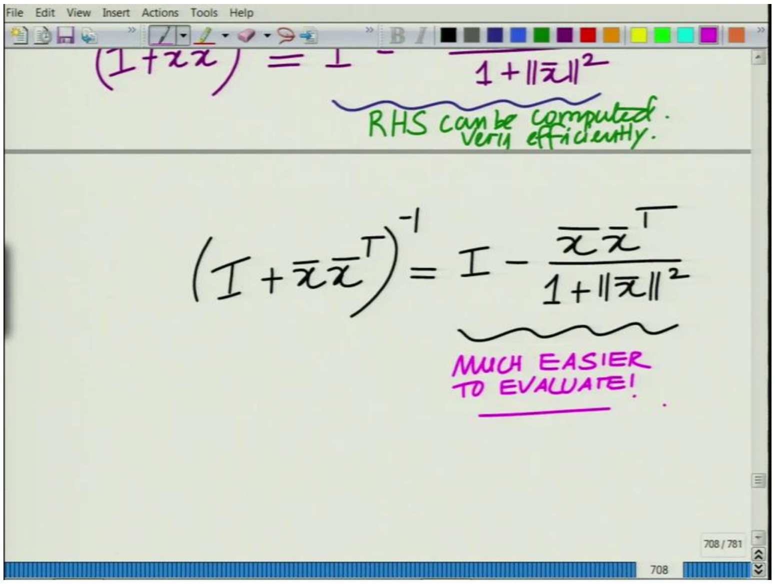

So, this LHS, left hand side, LHS can be computed very, I am sorry, RHS can be computed very efficiently the right hand side that is what I meant to say, the RHS can be computed very efficiently and therefore, just rewriting this I plus x bar x bar transpose inverse equals I minus x bar x bar transpose divided by 1 plus norm of x bar square and this is much easier to evaluate, evaluated very, very, this can be evaluate in a very, very simple fashion. Let us look at a simple example to understand this better.

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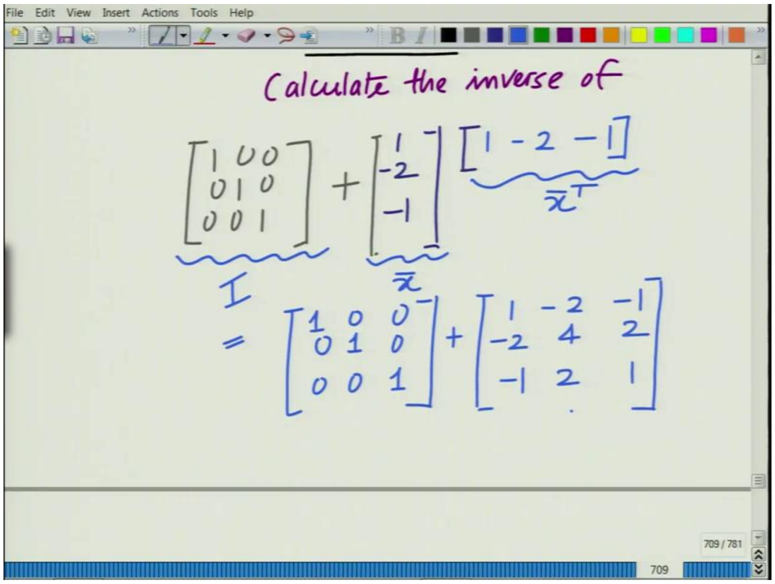

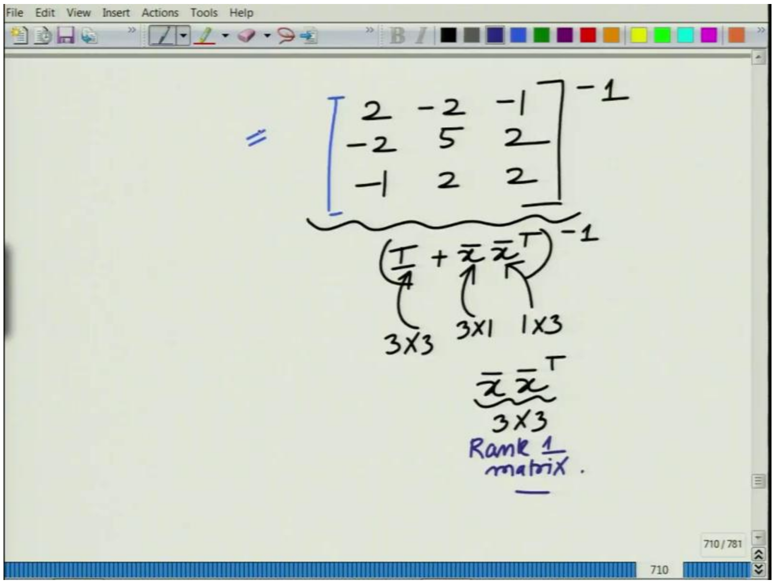

So, let us look at an example. Calculate the inverse of this matrix plus this one which is 1 minus 2 minus 1 times 1 minus 2 minus 1. This is the matrix that we want to calculate the inverse of, this is your matrix identity plus this is your vector x bar this is your vector x bar transpose. So, you can think of this as identity plus the matrix which is the following thing, plus the matrix which is 1 minus 2 minus 1 minus 2 4 2 minus 1 2 1.

So, sum of these two matrices. So, which is if you look at this, this will essentially be this matrix, which is 1 plus 1 this is 2 minus 2 minus 1 minus 2 5 2 and minus 1 2 2. So, we want to calculate the inverse of this matrix I believe that is correct. So, this is minus 2 4 2 minus 1 2 1. So, this is 2 minus 2 minus 1 minus 2 5 2 and minus 1 2 2. So, this is the matrix we have to calculate the inverse.

This is our I plus x bar x bar transpose inverse and this is remember, this is a 3 cross 3 matrix this is 3 cross 1 and this is 1 cross 3. So, now remember although if you look at x bar x bar transpose and we know this, this is a 3 cross 3 matrix this is an outer product, but you can see because the vector, outer product of a vector with itself that is x bar into x bar transpose this will be a rank 1 matrix. So, that should not be very difficult to follow. This is a rank 1 matrix. So, this is not very difficult to see.

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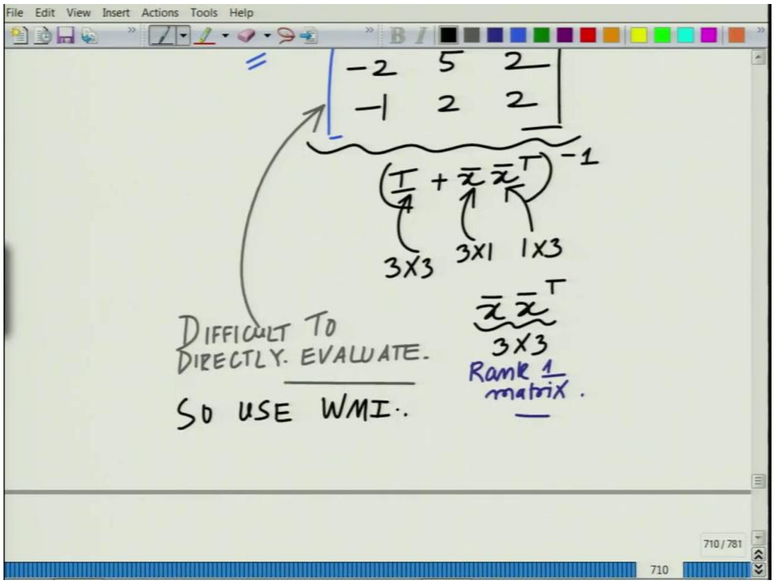

Now, of course, this matrix inverse it is difficult to evaluate this directly. So, difficult to directly evaluate this, so we use the Woodbury matrix identity. So, use the Woodbury matrix identity, which is essentially.

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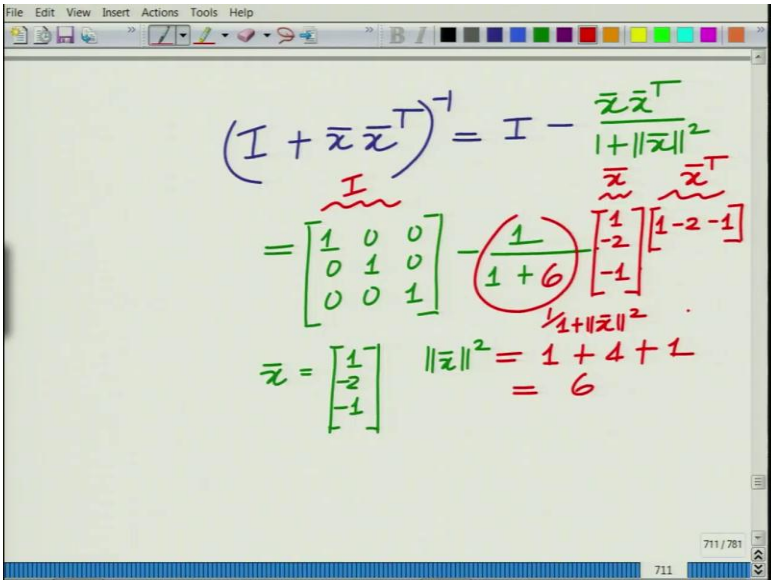

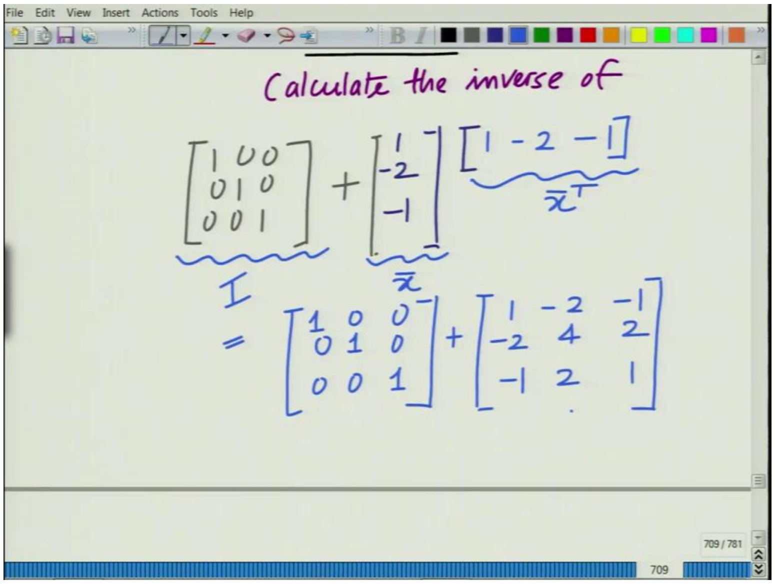

Now if we use the previous one result, we know that I plus x bar x bar transpose the inverse can be very efficiently and simply evaluated as this is I minus x bar x bar transpose divided by 1 plus norm x bar square. And in this case, this is going to be the I which is the 3 cross 3 matrix 1 0 0, 0 1 0, 0 0 1 minus 1 over 1 plus, now let us look at norm x bar square, now you see x bar equals this vector 1 minus 2 minus 1. So your x bar equals 1 minus 2 minus 1. So, norm x bar square if you think about it is not very difficult to see that is 1 plus 4 plus 1, so that is equal to 6. So, this will be 1 by 1 plus norm x bar squared, which is 6 into x bar into x bar transpose.

So, 1 minus 2 minus 1, So, that will be 1 minus 2 minus 1 times 1 minus 2 minus 1, so this is your x bar, this is your x bar transpose and this is of course, this is your in case you are wondering what is the mapping, this is your identity and this is your 1 over this whole quantity. If you look at this whole quantity, this is your 1 over 1 plus norm x bar square that is what this quantity is.

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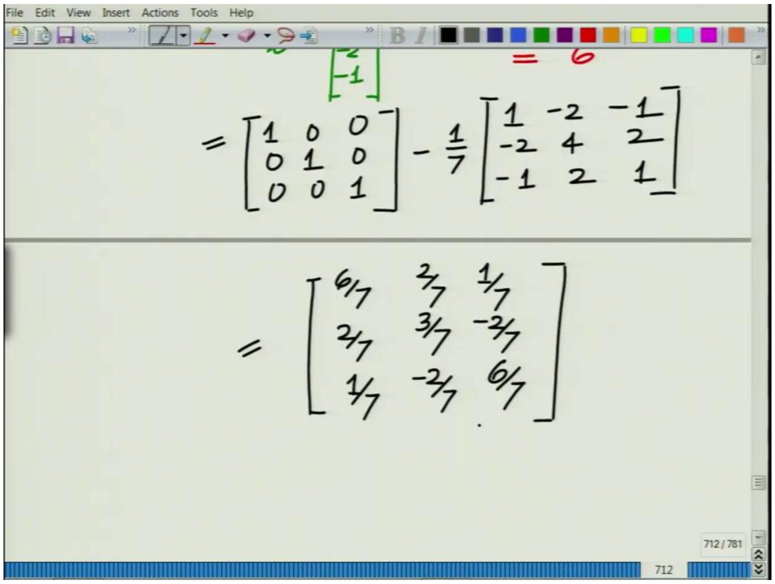

So, this becomes needless to say, this becomes if you simplify this, just keep this in view and you simplify this, this becomes the identity 1 0 0, 0 1 0, 0 0 1 minus 1 over 7 times x bar x bar transpose which is what we already written 1 minus 2 minus 1 minus 2 4 2 and minus 1 2 1. We have already written this, so this should be clear and now you do this subtraction.

So, you have 1 minus 1 by 7. So this is equal to 1 minus 1 by 7, so, this is 6 by 7, 2 by 7, 1 by 7, and you have 1 by 7, 1 minus 4 by 7, so this will be 3 by 7, and then you have 0 minus 2 by 7, so this is minus 2 by 7, and then you have 1 by 7, you have minus 2 by 7 and then you have the 1 minus 1 by 7, which is needless to say your 6 by 7.

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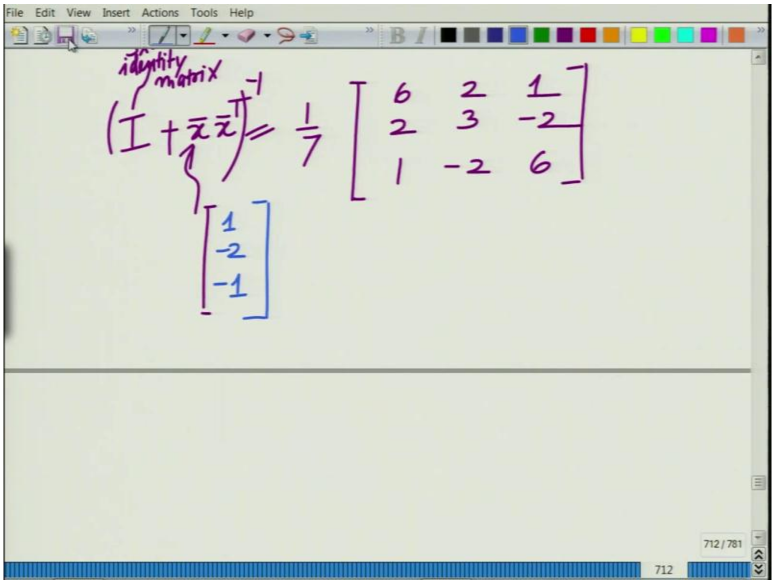

And therefore now you take out the 1 over 7 common. And then you can write this as your matrix 6 2 1, 2 3 minus 2, 1, minus 2 6, 6 2 1 2 3 minus 2, 1 minus 2, 6. So this is your I plus x bar x bar transpose inverse or this is in fact your whatever we had, where you had, this is a 3 cross 3 identity matrix and your x bar is the vector. Remember, what is this vector x bar? x bar is the vector, what is this vector? This is your vector 1 minus, sorry, 1 minus 2 minus 1.

This is your vector 1 minus 2 minus 1, 1 minus 2 minus 1. This is your vector. Fantastic, all right. So essentially, what we have done is we will illustrate an application of this matrix inversion lemma to the interesting case where you have an identity matrix, it is going to be readily evaluated plus a rank 1 matrix and this can be readily evaluated very efficiently evaluated in fact, without doing any inverse in the manner that we have shown, and we have also consider an example to illustrate this. So, let us stop here and we will continue this discussion and explore other such concepts in the subsequent modules. Thank you very much.