Evolution of Air Interface towards 5G
Prof. Suvra Sekhar Das
G. S. Sanyal School of Telecommunications
Indian Institute of Technology, Kharagpur

Lecture – 36
Mimo Signal Processing (Receive Diversity)

Welcome to the lectures on Evolution of Air Interface Towards 5G. So, till now we have seen various waveforms, and then we have also characterized the communication channel how it is modeled. So, now, it is time that we look into the different multi antenna transmission schemes which are helpful in providing higher spectral efficiency in meeting the new requirements of data rates and spectral efficiency.

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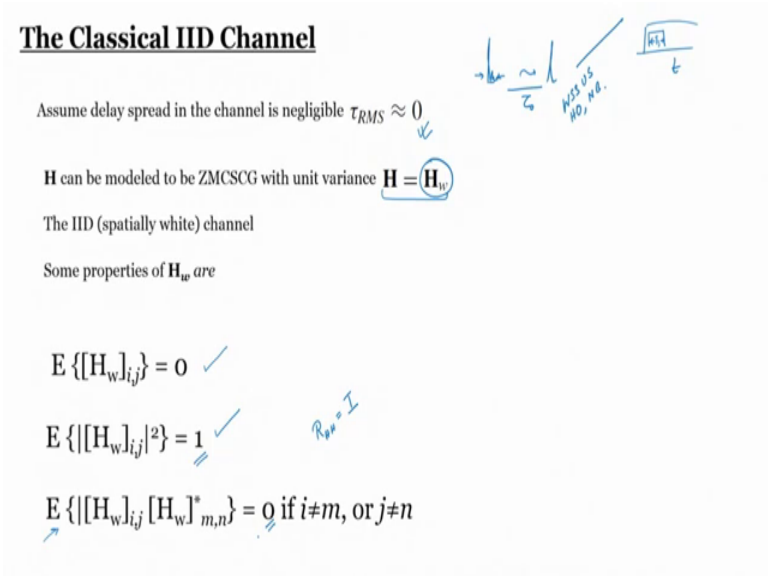

So, we have been discussing about the classical IID channel in the previous lecture and briefly we will mention it once again. So, that there is continuity. So, one of the important things is we assume that the delay spread is negligible; that means, there is the channel impulse response is very very narrow. So, it is almost approximated to a delta function with only a delay. And that means it is flat in frequency.

And we will also assume that it is slow fading; that means, over time the channel is fluctuating at a rate which is much much smaller than the symbol duration. So, these are some of the important assumptions. And then we talked about wide sense stationarity, uncorrelated scattering and we also introduced the homogeneous channel and then the narrow band antenna area assumption. So, these things have been discussed in the previous lecture.

And we also talked about the classical IID channel where it means where we note the classical IID channel with the H w indicating it is spatially white. So, this H w channel has certain properties which define H w channel. So, some of the common properties with the other situations are that the individual elements are 0 mean of unit power, while when we take the covariance we will find that the R HH should be equal to an identity matrix because the diagonal elements will be one from this and the non-diagonal elements would be 0 from this, right. So, that is what defines the classical IID channel which we will be using.

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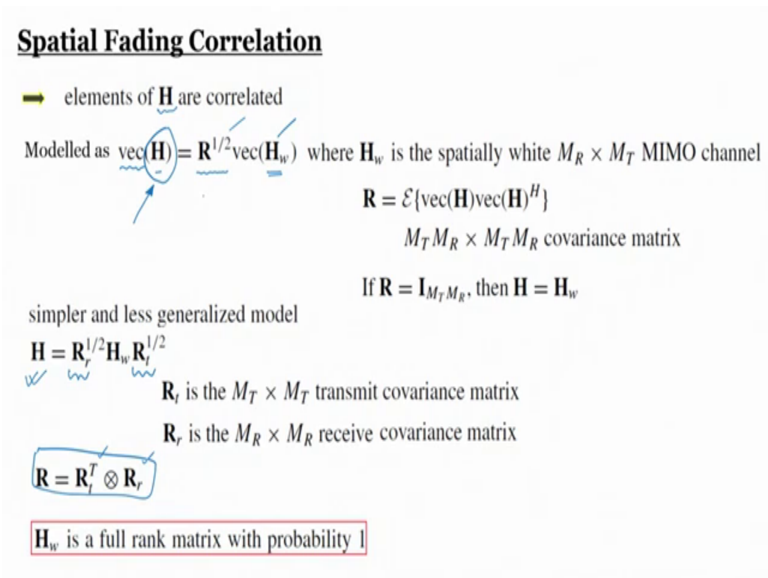

We also talked about the spatial fading correlation, where we said that if H is a correlated channel it is usually modeled in form of vectorization of H which is provided through the relationship R raised to the power of half; that means, our half and wake of H w. So, you generate a spatially white channel given a spatial covariance matrix, you can generate the matrix of H coefficients which are correlated. And we will see the impact of correlated channel coefficients.

Although this is a general model, we also said that is simpler and less generalized model is this where the correlation is split between that at the transmitter and receiver where the entire covariance matrix is related to the Kronecker product of the R t and R r, right. So, that is how we have described it and we also mentioned that H w is full rank with probability 1. So, these are some important things that we should remember while continuing with the description.

So, we continue with this description and we move forward with a few more essential things.

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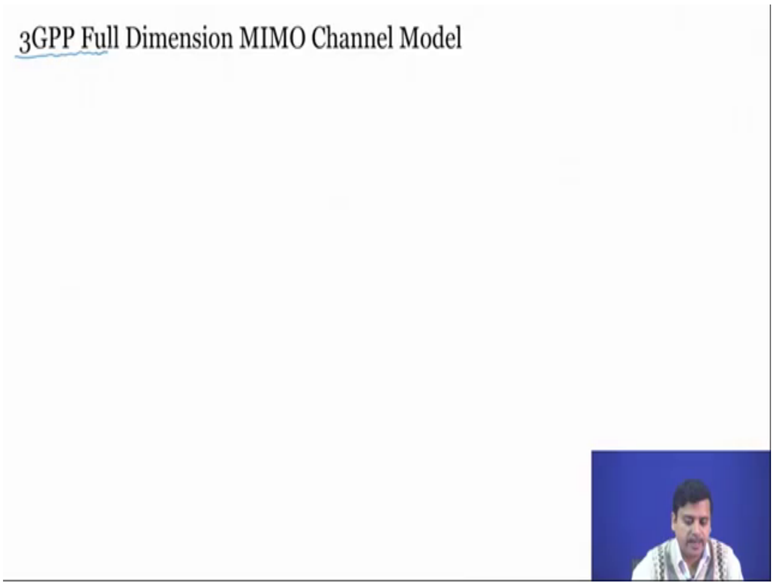

So, just a side note the 3 GPP has provided a description of the full dimension MIMO channel, ok. So, with the description that we have given, now one should be capable of going through the details and understanding all the propagation aspects that are provided for MIMO.

And we will also try to provide some of the generic results that we have obtained from that particular model at an appropriate time.

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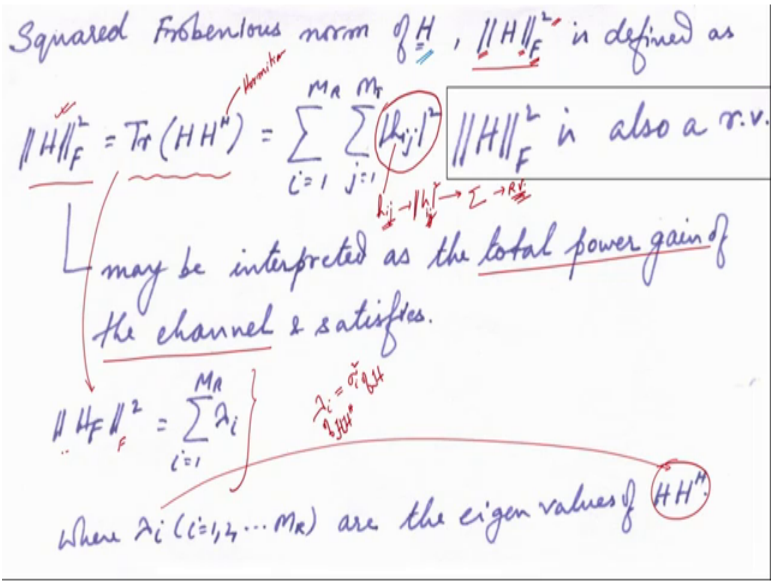

A few more interesting important ones that needs to be defined is the squared Frobenius norm of H. This is important this is what will be used throughout in the next part of the analysis, where it is denoted as H with double line on both the sides and a subscript of F and squared, and its meaning is it is the trace of HH Hermitian.

So, this is the Hermitian operation, which in turn means that you are essentially adding up all the elements squared together. So, which is can also be interpreted as the total power gain of the channel. So, that is a critical factor. And what we can also see is that mod H F square or the Frobenius norm squared of the channel which is the trace of HH Hermitian is composed of the square of the power of individual terms. Now, h i,j are random variables and hence mod h i,j squared should also be random variable which in turn means that the summation would also be random variable, ok.

So, that means, H F squared is also a random variable. And it can be also seen that mod H F squared since it is the trace of HH Hermitian can also be written as sum of the eigenvalues where of HH Hermitian. If lambda is are the eigenvalues of HH Hermitian then from this definition one can also write that Frobenius norm squared of H is equal to sum of the eigenvalues of HH Hermitian. And the eigenvalues of HH Hermitian would be square of the singular values of H. So, in other words we are kind of connecting the singular values to the Frobenius norm or whatever way you want to look at it. So, this is something that we will be using very soon.

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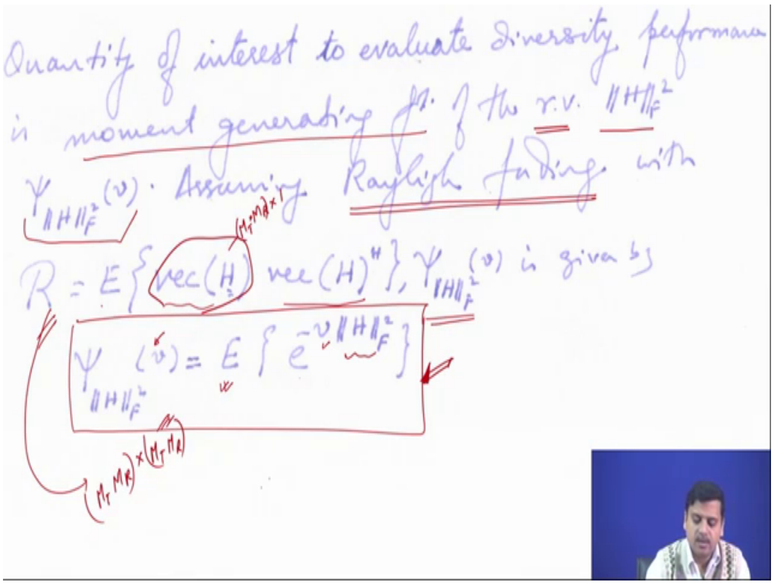

And the quantity of interest to evaluate diversity performance and that is what is written over here is the moment generating function, ok. So, this structure will be used and we have already established that H F squared is a random variable. So, we need the moment generating function of H F squared. And it is denoted in this particular case as psi sub H F squared of nu.

Now, assuming Rayleigh fading, we have described the Rayleigh fading condition. R there is a covariance matrix is expectation of the vec of H times vec of H Hermitian that is what is already defined in previous set of discussions. So, in that case with all the other above assumptions H F squared nu is defined as; that means, the moment generating function of H F squared is defined is this value. With this expression there is expectation of the exponentiated nu which is the parameter and H F squared the random variable. So, this particular structure will be used throughout whenever we are discussing the error probability. This only helps us in getting an easier expression for error probability when we are talking about diversity game.

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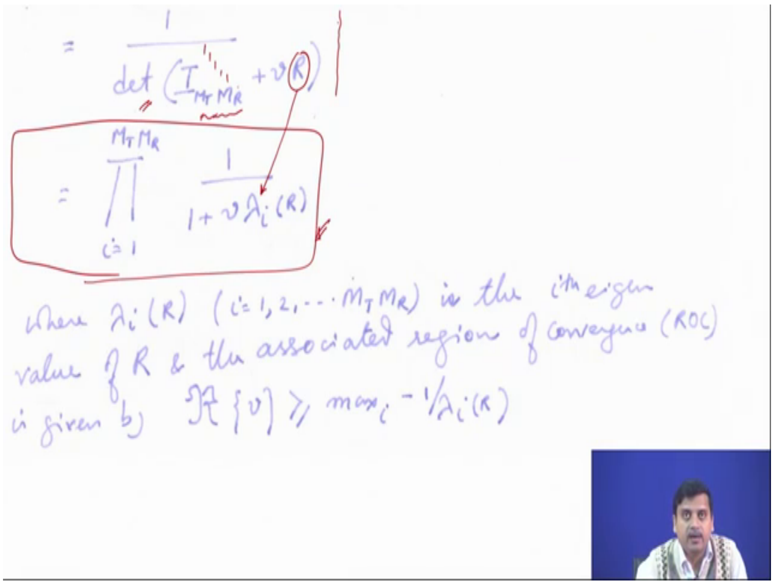

So, this term as we are seeing can be written as 1 upon determinant of I M T M R, we have M T M R because we have this HH Hermitian and H F square. So, H squared as you are clearly seeing that it contains of M T M R components, nu times R, R is the expected value of vec. So, basically if you look at R, R is of size M T M R cross M T M R because each individually these are M T M R, M T times M R, basically M T times M R cross this thing cross 1, ok. So, what you can see is that R is an M T cross M T M R plus M T M R matrix and hence you have the determinant of this quantity where the i is added of the same order.

And the determinant since this is an identity matrix; that means, all diagonal elements are 1 and this has eigenvalues which are lambda i of R, you can write the same through this expression which will be used. Again, as of now we will just use this expression we will take it for given. I mean if you expand this you are going to get these results. And we will be using these set of results in calculating the error probability, ok.

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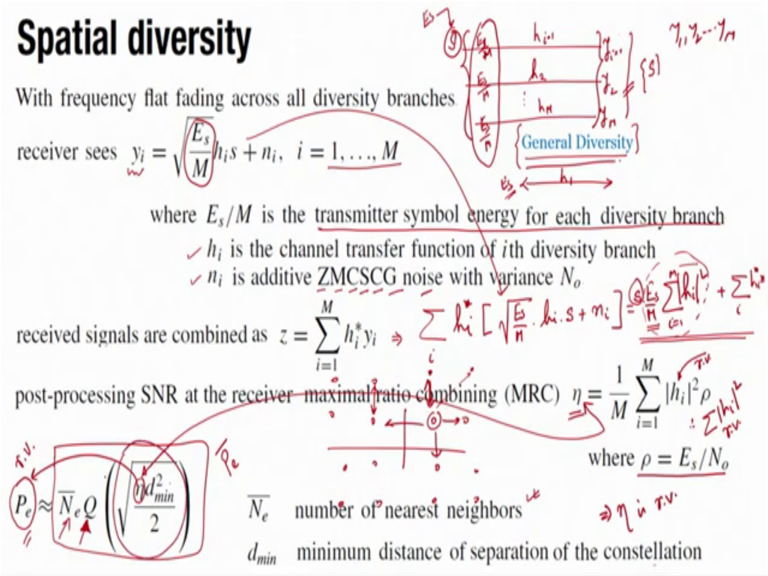

So, then we move forward to discuss the spatial diversity which is of our main interest at least as of now. And we begin with the description of general diversity. The general diversity means that there is some transmission transmitted signal s over some channel h i and what is received is y i rather i equals to 1. It is sent through another channel h 2 and what is received is y 2 and so on and so forth. And it is resent through m number of channels you are receiving h m.

So, it is the same signal s which is being sent over multiple paths or multiple received signal is there belonging to the same information s. And that is what is captured over here, that the receiver sees y i which is the received signal, i is the index which runs from 1 to m. So, one can translate this to receive antenna branches, transmit antenna branches, time slots, frequency slots. So, that is why we are doing the general diversity discussion.

And we have E s over M, because E s over M is the transmitter symbol energy for each diversity branch. That means, if the total energy is E s for s for each of the branch you would be having E s by M, E s by M, E s by M. So, that the total energy at the transmitter is E s is not violated when comparing against a single link which has a total power of E s. So, we are comparing two situations where the transmit power is divided into M parts sent over parallel channels compared to the situation where you have E s being sent over one single channel. Of course, h i is the channel trans function and noise is the 0 means circular symmetric complex Gaussian noise, right, ok.

So, the received signal are combined, so since we have all these different receive signals y 1, 2 up to y m, we would like to combine them and one of the process of combining news as the MRC combining, maximal ratio combining. So assuming channel knowledge available at the receiver you would take h i, conjugate it, and multiply by y i and add over all the receiver branches.

So, if you do that that is what is written in the expression next to it. What you are going to get is, this is going to give you some over i, h i conjugate. If you expand y i, y i from this you are going to get root over E s by M h i s plus n i and thereby if we look at the desired term E s upon m goes outside the summation you are going to get mod h i squared times s, s will also be outside the summation and you are also going to get summation h i conjugate times n i, yeah, so times n i, right.

So, if we look at this term over here this is the desired signal with a certain weighted power and you are adding it over 1 to M. A very gross view, if all of these values are 1, if these are 1 then the total power is E s and then we have the total received power that is the same as the SISO case.

So, now, if you look at the post processing SNR from this expression, if you calculate the post processing SNR you are going to get the sum over h i squared that is from here you can clearly see that 1 upon M, 1 upon M over here some over h i squared is here, right. So, these are the two things that you can clearly see which describes the received SNR.

And then we have the next term rho which is E s by N naught has given over there, right. So that, so that we now have the entire expression of the SNR of the received signal. We have seen how the signal is processed at the receiver as well. And from this one can calculate the probability of error using the expression as given there, where any bar is the number of nearest neighbors from the constellation. So, you can have QPSK constellation or a 16 QAM constellation, right. So, what you will be concerned is with the number of nearest neighbors. So, in this case these are the number of nearest neighbors.

And d min squared is the minimum distance of separation of constellation. So, if this is the minimum distance of separation. This is not the minimum distance these are the minimum distance of separation, so that is d min squared. Eta is the SNR of our concern. So, eta is the one which is going to be there. So, we have all the terms now, and then we can calculate the probability of error. So, now, one can clearly see that once again h i is a random variable and therefore, as said earlier some of h i mod squared is also a random variable which implies, but this is a constant term that is rho is a constant term that eta is also a random variable. So, eta is a random variable which comes in here that in turn means that probability of error is also a random variable.

So, since if the probability of error is a random variable then there is hardly much that you can do about it, except that you can provide the statistics and in this case what we would be interested in is the average probability of error. So, let us look at calculating the average probability of error for this particular situation.

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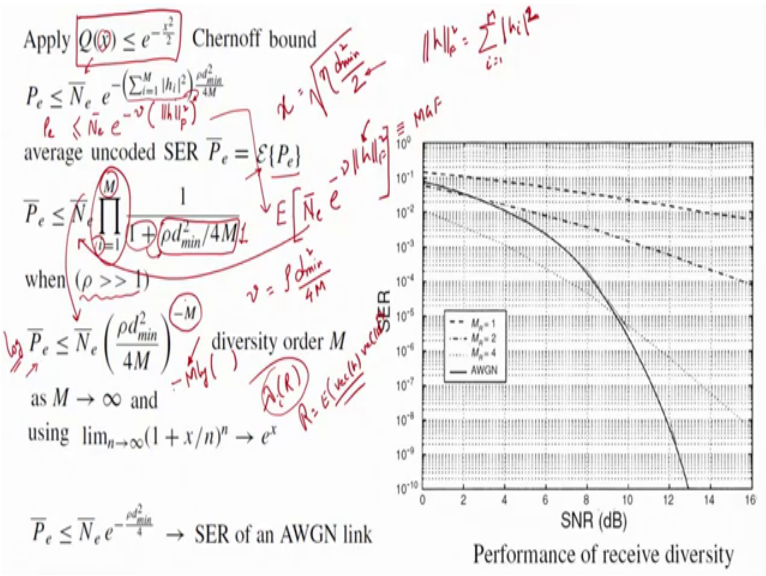

So, now to calculate the average probability of error we will use the Chernoff bound where what we find over here it is in terms of Q function and Q function is in terms of error function, so that is in the integral form. So, we use in the Chernoff bound and provide this approximation for Q function. That is Q of x is less than or equal to e to the power of minus x squared by 2 and x that is over here is all the terms that is over here there is square root of eta d min squared by 2. So, x is equal to square root of eta d min squared upon 2. So, that is what is x, ok.

And so, we now have the approximation when it is applied, we are going to have any bar. Instead of the Q function we have e to the power of minus this whole term squared. So, that voltage squared means d min squared by if you look at the thing over here it is 2 so, eta that that is what we had over here, eta is E s by n naught ok, and there is a row term and we have 1 upon M mod h i squared.

So, from that we get this summation mod h i squared and 1 upon M and we have rho which is E s by N naught and this 4 is because of 2 is getting multiplied with this 2, so we have this 4 term. So, since we have now identified all the terms of this expression we move on to calculate the average probability of error.

So, the average probability of symbol error is given by P e bar which is expectation of probability of error. So, now, one would be able to connect to this expression and see that we have e to the power of minus nu times mod h F squared, right. So, this is the expression N e bar P e. So, from this we have to next go into expectation of N e bar e to the power of minus nu mod h F squared because here h F squared is equal to sum over i equals 1 to M mod h i squared, ok.

So, since we have that, so we can easily see that this entire summation is now replaced by the term here that is below this entire summation is replaced by this term. And nu the next parameter that we have is all the other terms rho d min squared upon 4 M, right. So, now, you recollect that this is like the MGF of h, ok. So, the same expression that we had seen earlier that is it looks like this expression. So, we use the result from this, so it is basically the MGF of proven is norm squared of H.

So, if we use the result, we will be applying it over here; that means, nu would come as it is and lambda i of R, right. So, what we see that nu has come in its entirety, ok and when we go back this determinant gets translated to this product term and in our case here, we have only M we do not have an M T and M R, we only have M. So, you have an M term over here N e bar comes out over there and now comes the lambda of R, lambda i of R corresponding to this is. So, R is the expected value of vec of h times vec of h permission.

If we assume that these branches that is what we are going to take R uncorrelated; that means, they are independent, I mean if we take independent that would result in uncollected branches in that case, we will be getting the eigenvalues as 1 and hence you have a 1 multiplied over here and the expression fits in. Now, if we let the SNR becomes very very high, what we will find is that we can neglect this 1 with respect to this term and you are going to get P bar which is the N e bar comes here and this term which is a product of the terms inside this which has a constant term raised to the power of M and if you bring them to the numerator you get a minus M. So, effectively what this means is that if we take the log of it and then this minus M is going to come on the outside minus M log of this expression indicating this is the slope of the curve in the log scale of probability of error.

So, in other words when we talk about the diversity gain; so, let us release erase all the ink on this slide, yeah. So, when we talk about diversity gain what we mean is that the exponent that is associated with the SNR term that is inside the bracket, ok. So, that is the diversity gain.

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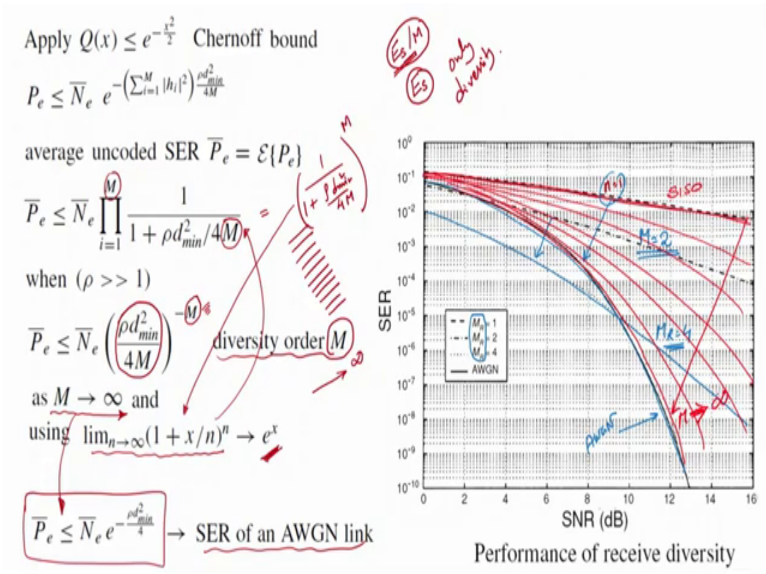

So, now, let us look at a few other interesting outcomes of this expression. So, as we let M tends to infinity; that means, as we let the order of diversity become higher and higher and high, so we have described the order of diversity as M. So, as we increase the number of receive branches or number of independent transmission we can apply the limit that 1 plus x upon n to the power of n can be approximated as an exponential. So, if we apply it over here, right, we see M in the denominator and we also see M in the, because this term is you can write it as 1 by 1 plus rho d min squared by 4 M, whole raised to the power of M, right.

So, that now is what we are approximating over here to get an exponential. So that means, under M tends to infinity the error probability expression can be approximated to an expression which looks like this which is the approximate simulator probability for an AWGN link. And what we have from this result is that as we increase the order of diversity towards infinity what we get is the symbol error rate which goes towards the AWGN link.

Now, a careful note we remember we have not increased the power per branch of diversity; So, per branch of diversity is E s by M and hence the total received power is E s, it is not more than that. So, we are talking about the pure diversity, only diversity case. So, if there are other gains the results would be different. So, when there is only diversity with just by making by increasing diversity you can achieve the error probability of a AWGN which is the best situation that one can think of, ok.

So, what we have over here is a set of results which indicates the curve that I am tracing is for M equals to 1 in other words it is for the Rayleigh fading channel one can think of this as the Rayleigh fading channel with a SISO link, ok. And then what we have is the next line this is for M is equal to 2; that means, 2 order diversity and this curve is quite visible. And the next one that we have over here is for M R equals to 4; that means, there are 4 receive antennas it is slightly a different figure and then what we have over here is the AWGN curve. So, this is the one for AWGN.

Now, why this crosses over? Because this particular result is for received diversity which we are going to see shortly nut what we find is that M equals to 1 is there and AWGN is over here. So, if we have pure diversity or curves are going to bend in this manner for M equals to 2, 3, 4, 5, and as you increase slowly, they are going to merge with AWGN as M tends towards infinity, ok.

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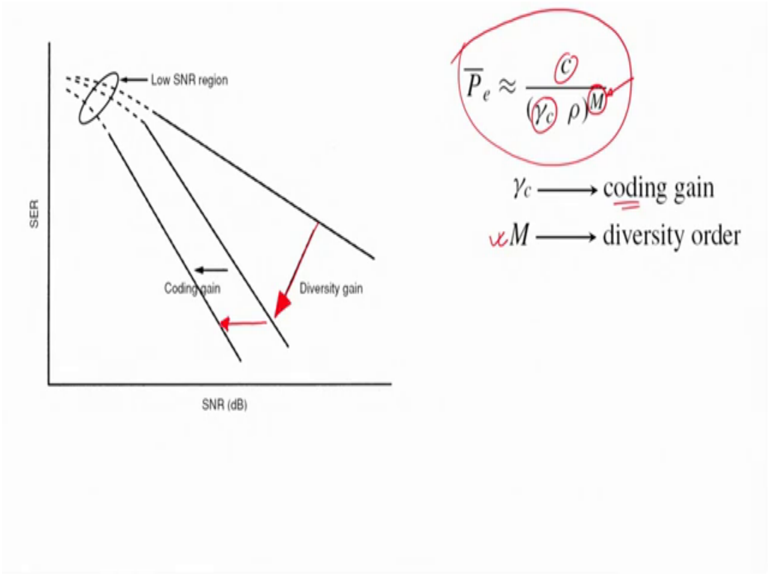

Moving forward so, we can see that the error probability average error probability expression is written in this form where this M exponent of M indicates the order of diversity is given over here and the multiplicative factor is the coding gain, right. So, sorry this should be the coding gain not that one that is we need to correct this particular part, ok. So, we will correct that particular this is a constant sorry, yeah. So, we have the coding gain associated with it all, right.

So, what we see is that diversity gain effectively gives you a increase in the slope of the curve and coding gain gives you a lateral shift of the error probability curve. So, any expression which is bringing your increase in the slope it is the order of diversity and that component which is giving you a lateral left shift is basically the coding gain part.

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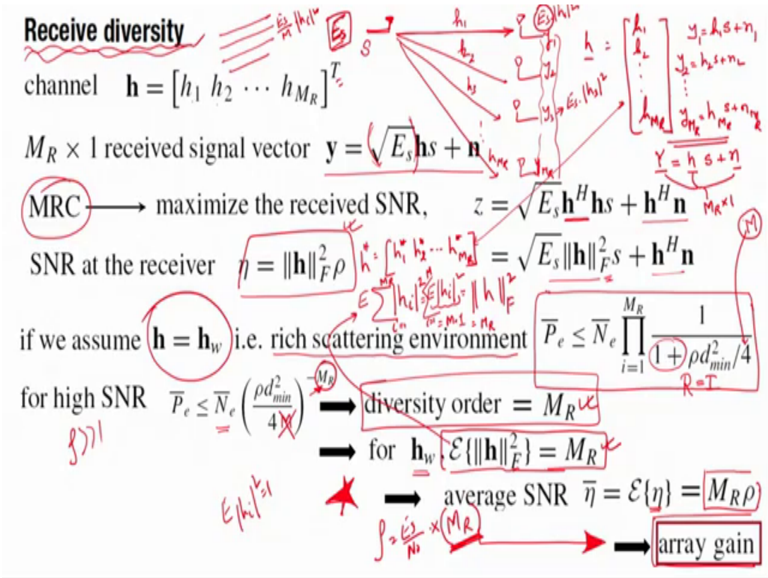

So, now we move on to the receive diversity. So, in case of receive diversity what we mean is that there is a transmit antenna and there are receive antennas, ok. And these signals are received whereas, only s is sent this is h 1, this is h 2, h 3 and so on up to h M R and this is y 1 that is received, y 2, y 3 up to y M R that is received. And hence the channel vector can be written as h 1, h 2 up to h M R transpose meaning you are having h vector is equal to h 1, h 2 up to h M R like that, ok.

So, to maximize ok, the received signal again what do we have y 1 is equal to h 1 s plus noise 1, y 2 is equal to h 2 s plus noise 2, like that y M R is equal to h M R s plus noise M R, M R indicating the received branch number. And if you write these equations in a vectorial form you are going to get y equals to h s plus n, these are all vectors of order M R cross 1, right. So, that is written over here in this expression in a vectorial notation, bold, small indicating vectors and this is the normalized transmit power. So, we have a single transmit antenna hence the total transmit power through that antenna is E s which is the square of which is the square of this particular term.

To maximize the received SNR, MRC combining issues maximal ratio combining which is given by h Hermitian times h; that means, if you look at this h, h Hermitian would be h 1 conjugate, h 2 conjugate up to h M R conjugate. So, when we multiply this with this, ok, what we are going to get is sum over h i mod squared i equals to 1 to M R which is nothing but the Frobenius norm squared of h, and that is what we have got over here this is the one that we had seen earlier also. So, the next expression at the receiver is this and we also have h Hermitian multiplied by noise, so that term continues. So, from this we have to calculate the probability of error.

So, if we assume that h is equal to h w, that means if we assume a rich scattering environment in that case again, we will be able to calculate the probability of error as an expression which is given over here, right. The difference what you see with respect to the previous thing in the denominator term there was an additional term of M which is missing over here. And the reason is at the transmitter now we have E s the total power being transmitted from one of the branches the power that is received in this branch is also E s times h 1 squared.

The power that is received in this branch is this E s times h 3 squared, right. So, the difference with the previous mechanism is that in the previous mechanism we said that each of the branches receive a power which is E s upon M, but here it is receiving the E s upon M multiplied by h s squared. So, that term is not over here the entire power E s is received in each of the branch.

So, naturally one can think that we are actually increasing the total received power, and that is pretty obvious because you are having more number of antennas, you are accumulating more amount of energy that is a natural translation compared to the previous situation. So, hence that is the difference in this equation.

So, for high SNR that means, when rho is greater and greater than 1. The approximation; that means, this is this term is neglected again just as we have done in the previous case we get N e raised to the power of minus M R. So, the difference is we do not have the M term over here, that term is missing compared to the previous term. So, diversity order is M R because we have this thing and for h w that means, for especially white; the reason we have talked about h w because we have again taken R is equal to identity matrix, ok.

For h w expected value of h F squared is M R, that one can see. If one takes the expectation over here, so basically go back and take the expectation over here that would mean you are taking the expectation of you are taking the summation outside and h i squared. So, we have seen earlier for h w it was mentioned that E of h i squared equals to 1. So, each of these elements are equal to 1 and hence this is equal to M because you have M summations i equals 1 to M, M times 1 which is equal to M and here it is M R, so you have M R, all right.

And the average SNR is expectation over eta. So, we have the expression of eta over here ok that can be calculated directly from this. So, again since we have h F squared E of h F squared is M R. So, we have M R times rho. So, which means that rho which is equal to E s by N naught is now getting multiplied by M R; that means, the average received signal power has increased with respect to noise power by a factor of M R and hence there is an array gain is the thing that we add in this picture.

There is an array gain in the picture, in this scenario which increases the average received signal strength compared to the previous case which we were talking about general order of diversity. So, we have two aspects now one is the array gain and the other is the diversity gain. So, we have both these things when we are talking about receiver diversity.

We stop this particular lecture over here, and will continue with this general framework of analysis for all the things in future.

Thank you.