**Evolution of Air Interface towards 5G**
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**Lecture - 37**
**Mimo Signal Processing ( Capacity )**

Welcome to the lectures on Evolution of Air Interface towards 5G. So, we have been discussing about the multiple antenna signalling schemes which enable us to have high reliability as well as provide better spectral efficiency. And we are looking at the diversity schemes, we have looked at receive diversity, we have looked at transmit diversity; both without channel state information at the transmitter as well as with channel state information at the transmitter.

And then, we have started looking into the diversity schemes where both the transmitter and the receiver has multiple antennas. The scheme we have been discussing is dominant Eigen mode.

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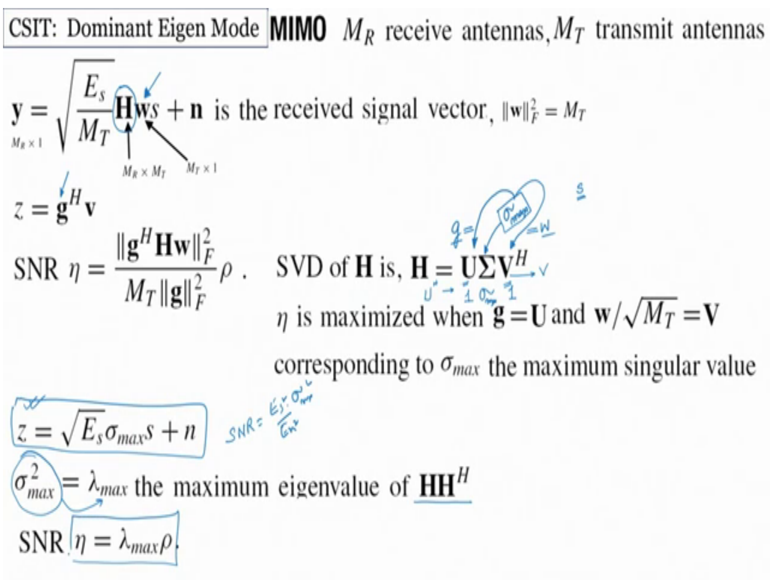


In the previous lecture, we have described all the details procedures; where we said that we look at the channel, in terms of its eigen value decomposition, whereby at the transmitter side we pre code using V and we post process using U hermitian and since it is a diversity mode that is only one value of signal is said or one signal is sent from the diagonal eigenvalues or the diagonal singular values.

We take the one corresponding to the maximum eigenvalue for this, we choose the vectors from U and V which correspond to the maximum eigenvalue and use them for processing. So, that particular one is used in the g whereas, this one is used in the transmitter so, the vectors are formed accordingly. So, at the transmitter you have w, which is defined below and at the receiver you process with g; so, g hermitian that is what we said with U makes it one for this entry because that is what we are doing.

At the transmitter side, since we are sending only one value and there is only one column of the vector being unitary. This also leads to one and hence, there is sigma max; so, the received signal is written in this form where n indicates noise, and sigma max. So, basically the SNR is equal to E s into sigma max squared upon En squared right; that is what we have. So, sigma max squared is the one, which influences the SNR and sigma max squared is equal to lambda max which is the maximum eigenvalue of HH hermitian and hence SNR can be written as lambda max rho.

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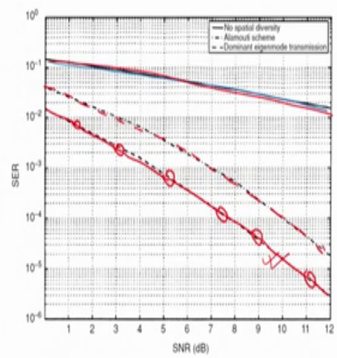


$\{M_T = M_R = 2.\}$

So, let us look at the performance of such a scheme. So, this flat line is the one for the flat curve rather is the one for SISO link then, we have the result for two cross two alamouti scheme and then finally, we have the result for dominant Eigen mode for a two cross two system; so, which clearly proves that, this particular mode of transmission provides the best reliability in terms of error probability compared to other mechanisms which use two cross two transmission receiving system.

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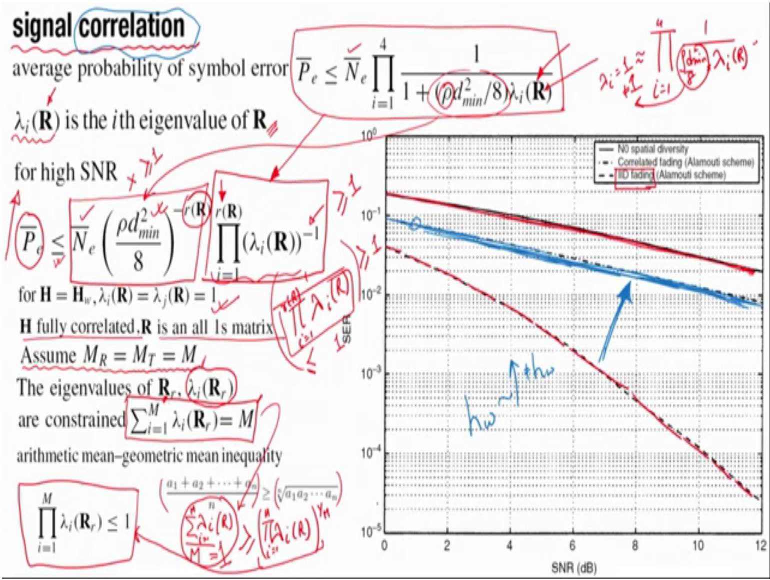


So, after studying the error probabilities, we now look at the signal correlation model; that means we have talked about the channel correlation or spatial correlation and see how does it affect the performance of the system. So, the average probability of error, we have described this in all our previous discussions is given by particular expression over here and what we see is that, there is the eigen values of R matrix which is present. So, we have a two cross two system in this case, and the eigen values of R are determined or are influenced by the correlation which is present in this R matrix.

In case of H w channel, we have stated that lambda i's are equal to 1; whereas, in case of correlated channel this will be not equal to 1 right so, let us see that. So, under high SNR approx assumption; that means, when rho is significantly high under that case, this expression can become N e as we see over here; remains as it is, and from the denominator term we can get this is we can get this out of course, it is kind of upper bounded. So, this term comes over here for high SNR approximation. This is a constant term; so, basically under high SNR, you will not get this product, i equals to 1 to 4 under high SNR. This approximately equal to rho d min squared by 8 multiplied by since you have the i over here lambda i of R ok.

So, that is; so, let us clean it and write it again. So, what we have there is a rho d min squared upon 8 multiplied by lambda i of R and since, this is a constant term it can be brought out; that is what has happened and in case of which is non identity of R. So, what we will get is this raised to the power of rank of R because this product will only be to the rank of R right otherwise the rest of them are 0. So, you are using only those within the rank of it.

So, for H equals to H w of course, we have said this is identity according to which we have derived the earlier result and when R is fully correlated, R is all 1 matrix so; that means, there is only one Eigen mode and hence, you do not get any diversity. So, to see the effect of covariance or correlation, which is non-identity; you take a situation for simplicity that MR equals to M T equals to M. So, that the analysis becomes easier and then, the eigen values of R; capital R matrix are represented by this expression which is kind of standard based on what we have been doing.

And they are constrained to this right, that is the constraint that we bring into the system meaning that the channel is restricted to a power of M right, channel does not provide any extra power than M. And using the arithmetic geometric mean inequality, which states that the arithmetic mean is greater than or equal to the geometric mean so, we apply it over here. So, in this case we have sum of lambda i over M lambda i of R i equals 1 to M is greater than or equal to product of lambda i i equals 1 to M raised to the power of 1 by M.

So, from this, what we find that this term is equal to 1 and hence, what we have is the product of eigen values is less than or equal to 1. So, that is the result that is shown here right. So, now, what we see is that, the product of eigen values that is less than 1. So, we have that term here which is 1 upon pi lambda i R i equals 1 to rank of R right. So, that is what we have.

So, what it means is that, this denominator term because there is this inverse over here. This term is greater than or equal to 1; that means, the whole ratio is less than or equal to 1 right. So, if the denominator is greater sorry, we wrote it wrong; this denominator is less than or equal to 1; that means, this ratio is greater than or equal to 1 so; that means, this multiplicative factor is something which is greater than or equal to unity, to be equal to unity under the case of Hw channel.

So, this probability of error expression, which we are outlining with the box is multiplied by a factor which is greater than 1 which in turn means that the average probability of error increases if R is not an identity matrix. So, if R is not an identity matrix in that case, we see that the error probability increases. So, to check the performance what we see that, under no special diversity we get the SISO link, what we see over here is that, this line that I am drawing is the one which is without any special diversity.

And the new curve that I am tracing is the one which is with IID; that means, Independent Identical or Identically Independent fading, which means that is independent there is no correlation. So, because of a certain amount of correlation, what we find is that the error probability curve has shifted upwards. So, there is an upward shift in the error probability curve; which I am tracing by the blue coloured ink and thickening the line; so, that clearly shows that the error probability increases because of correlation present.

So, correlation is not beneficial for error probability and whatever error probability one receives under Hw channel becomes only worse; so that means, it increases when it is not an Hw channel right. So, that is the important summary that we get in studying the signal correlation. So, once we have discussed about the diversity.

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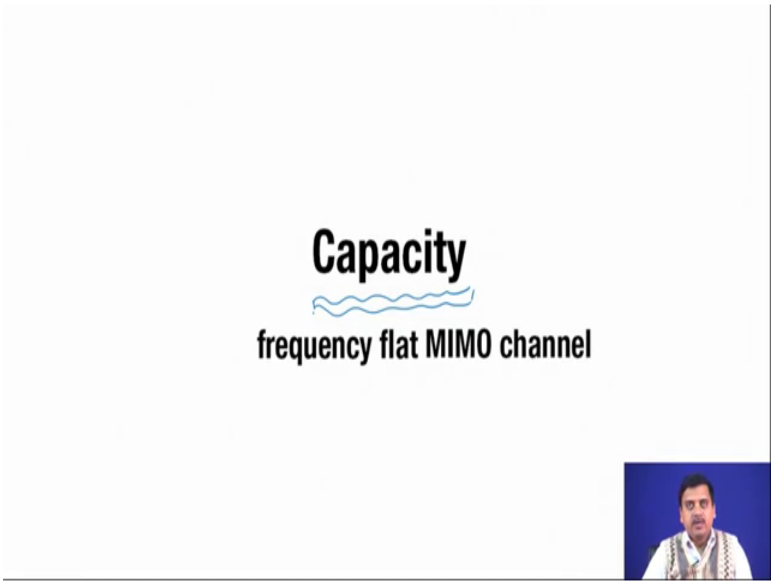


It is very important we move into the next set of things that is the capacity, which is one of the most interesting aspect why MIMO is so popular.

Of course, there is one more interesting aspect in the new generation that is beam forming, But the biggest advantage that MIMO has brought in over the last few generations of communication systems is enhancement is in capacity. And we are going to study the system under frequency flat fading conditions; this is what we have mentioned earlier. So, also slow fading condition and all the MIMO assumptions that we have made before. So, for a typical MIMO link a M T matrix and what we have is M R cross M T matrix.

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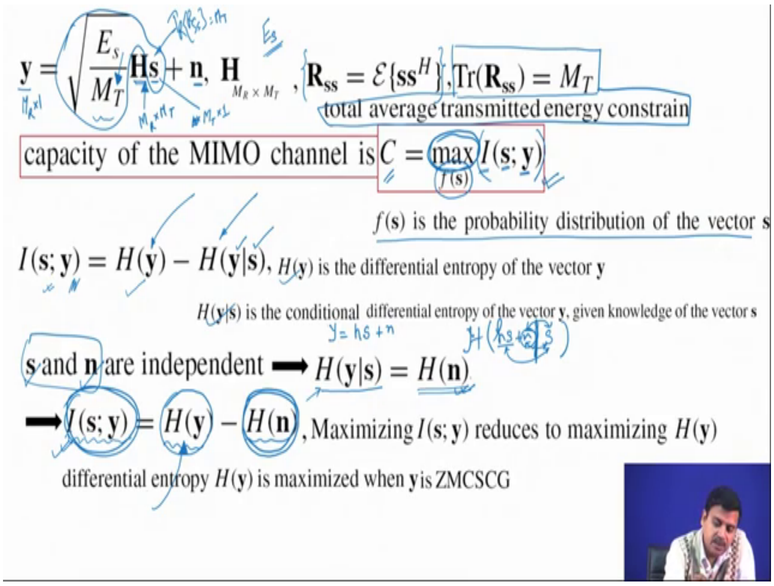


y is a MR cross one receive vector s is an M T 1 cross sorry M T cross1 signal vector ok. So, what we have is the received signal in its linear equation from y vector is some scaling H matrix s vector plus noise and it is also given that R ss is the receive or the signal covariance matrix with a constraint that trace of R ss is equal to M T, this is an important constraint. So, the total average transmitted energy constraint; that means, we do not want to use excess transmit see, ss transmit power and what we see over here from this part, is that the transmit power is equally divided amongst the M T transmit antennas that is what we have over here.

And then, if we constrain that trace of R ss is equal to M T, then we will ensure that the total transmit power is restricted to Es; that means, we can compare the performance with SISO link. So, the capacity of a MIMO channel is the one given by maxim, which maximizes the mutual information between the received signal and the transmitted signal over the distribution of the transmitted signals. So, f s is the probability distribution of the vector s.

So, this is a standard result which we will accept, we cannot afford to go through the derivation of this it is available in standard textbooks. So, now let us focus on this mutual information expression. The expression for mutual information is given as the entropy or the differential entropy of y because this is a continuous random variable, take away the conditional differential entropy of H; that means, the differential entropy of y given s right. So, that is the expansion of the mutual information.

And we have also defined both the necessary terms s, the signal s and noise are independent. This is one of the assumptions, it is kind of obvious, but still it is important which leads to the condition that H of y given s because y is equal to h s plus n. So, we could write that differential entropy of y given s is equal to that of h n right.

So, because we are saying that h s plus n conditioned on s this is what we are trying to evaluate. Now, since n and s are independent here you do not have any uncertainty. So, what is left with is uncertainty between h of n given s. So, h of n given s is basically Hof n right that is what we have over here. Now, we get back to the original equation that is this one. So, we have I s semicolon y, which is the mutual information between transmit and received signal can be expressed as H y minus H n, where H indicates the differential entropy of the received signal and H n receives the differential entropy of noise.

So, now, if we see the capacity expression, it is maximization of the mutual information right; that means maximization of this term. If we look at the right hand side, we do not have any control on the differential entropy of noise it is a natural event. So, we only have control possible control over differential entropy of the received signal. So, therefore, we say that the mutual information is maximized by maximizing the differential entropy of the received signal because the received signal is connected or is controllable through the transmitted signal s with this we proceed.

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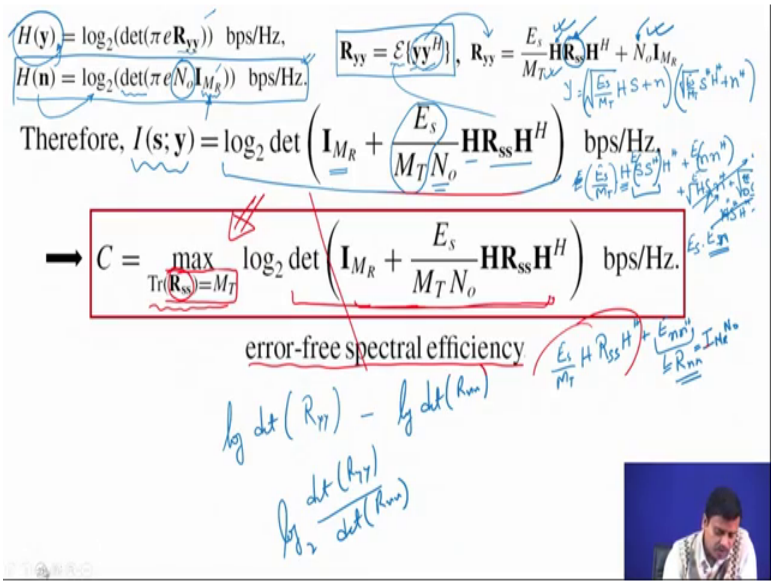


So, the differential entropy of noise is mentioned over here which is fundamentally controlled by N naught which is the noise power spectral density, IMR or you can say that MR is another parameter, but we have seen that as MR increases, the received signal to noise ratio increases. So, I would like to have MR as much as possible.

So, therefore, we use this definition of differential entropy for noise. Similarly, the differential entropy for y can be written as given over here, where R yy is the term is the covariance of the received signal that is expectation of yy hermitian. So, now let us expand yy hermitian over here that is, if you take y equals to square root of root over E s by M T H S plus n. So, you want to multiply this by the hermitian of the same term. So, you are going to get noise hermitian; you are going to get root over E s by M T s hermitian h hermitian. So, this product is what you are going to get.

So, if you expand the terms, you are going to get E s upon M T and then you are going to get H S S hermitian H hermitian plus you are going to get n n hermitian and you are going to get to the cross terms that is HS n and of course, the root part is there and the root over s by n naught times H S sorry, S hermitian H hermitian noise and of course, the hermitian. So, these are the terms that you are going to get and then, you have this expectation operator.

So, if you apply the expectation operator, it would not apply on this that is constant. H is given that means, for a particular value of H. So, E would operate on SS hermitian and E would operate on this as well as E would operate on this; so, E what we have stated that S and n are independent. So, what we would get is E s times En, we have said that En is 0 and hence, this term would go to 0; the third the fourth term would also go to 0.

So, we are left with the first and second term that is E s upon M T H. So, E of SS hermitian we have defined earlier as Rss H hermitian plus E of nn hermitian. Again you can write this as R nn which you can write it as I N naught of course IMR times N naught. So, what you see over here this term is available here and this term is available here. So, we have got the expression of R yy.

So, now if we have to maximize the differential entropy of y, what we are left with is; we have, we are left with this expression where H is something, which is not in our control it is from the channel. Noise is something which is not in our control, it is again from the channel. The only thing that is left with us in our control is R ss; therefore, we can say that as you are seeing that differential entropy is given as log determinant.

So, the log determinant of R yy so, what you have is log of determinant of R yy minus log determinant of you can say R nn you say that way. So, you have determinant of R yy upon determinant of R nn and a log outside that. So, if you expand this in terms of these expressions you will end up in the expression over here, where there is IMR E s by M T N naught H R ss H hermitian.

And therefore, you can state the capacity as the one, which maximizes the mutual information that is maximizes this entire expression over trace of R ss or over R ss with the constraint that trace of R ss is equal to M T. So that means, the capacity expression to make it look clean, we have as given there so; that means, it maximizes this expression just change the pen colour; it maximizes the expression over here. Of course, it is the log determinant over R ss with a constraint that trace of R ss is equal to M T all right. So, this is also called the error-free spectral efficiency. So, in all the analysis of MIMO that we do here on will be using this particular expression for all our right.

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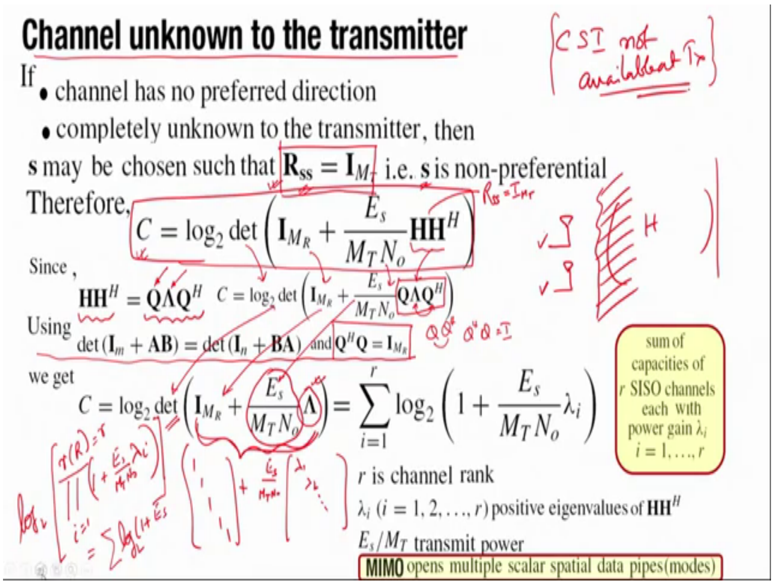


So, the first thing that we discuss is the situation, where channel state information is not known to the transmitter; that means, CSI is not available at the transmitter right. So, if CSI is not available, then that means, at the transmitter side one, does not have any information about the channel this is opaque, one does not know what is going on in the channel. So, there is no specific information about the channel. So, the best that one can do is set R ss equals to IMT right; that means, you just divide the power equally and you have the; you have no other option to do and s is non-preferential; that means, you do not have any partiality over the selection of s.

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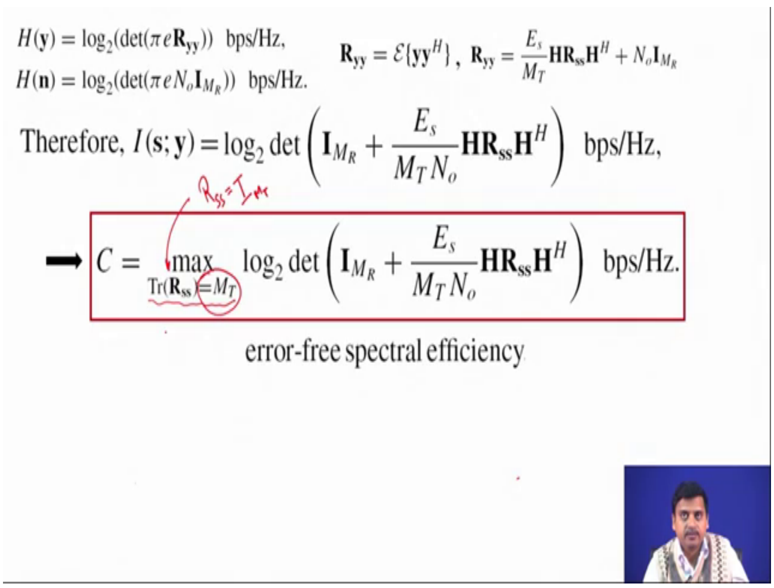

So that means, you have already chosen a condition on R ss. If we go back; we find that your trace has to be constraint to MT, but R ss can be of any structure. So, the particular structure of R ss that we have identified is IMT. All the diagonals are one and the matrix is of size M T cross M T or M T order identity matrix whose trace is definitely equal to M T.

So, now, we see that so, the constraint is gone. So, C is equal to log determinant of; so, in this we had R ss and since, that is set equal to IMT it kind of vanishes from the equation and the rest of the equation as it appears over here is the expression for the capacity under such situations. What we now do is HH hermitian is a symmetric matrix and therefore, it can be factored into a structure like Q lambda Q hermitian, where Q are orthogonal matrices and lambda contains the eigen values of HH hermitian right.

So, now we can write the capacity as log determinant; that means, we have not changed this part IMR also remains as it is, E s by M T N naught remains as it is, and HH hermitian gets replaced by QQ hermitian; Q lambda Q hermitian. Using an identity of determinant and also using the condition that QQ hermitian equals to IMR; that means, we will swap these two positions, you are going to get QQ hermitian and then, again you are going to swap the positions. You are going to get Q hermitian Q that will be let equal to identity.

So, what you will be getting is log determinant IMR E s by N naught, you will be left with lambda. This entire matrix as you can see is a diagonal matrix because this is a diagonal matrix with constant multiplicating terms. So, IMR is all ones and this matrix is E s by M T N naught and lambda 1 lambda 2 so on. So, that is it so; that means, this whole matrix is a diagonal matrix, whose determinant is a product of 1 plus E s by M T N naught times lambda i, i equals to 1 to rank of R. So that means, the rank of R you can set it equals to R.

So, determinant of this and a log base 2 so determinant sorry, determinant gets changed to a product right. So, this would translate to sum of log base 2 1 plus Es; well of course, it is these things.

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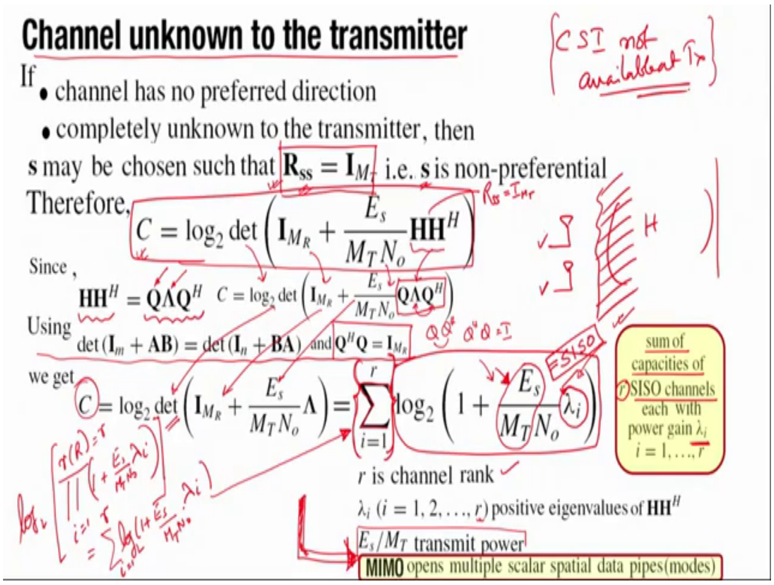


So, that you get a cleaner place E s by M t N naught multiplied by lambda i and i goes from 1 to r which is the rank of the matrix and this is the expression that you have over here right. So, where r is the rank of the matrix and lambda i go up to r and E s by M T is the transmit power right. So, what we see is that, if we look at this particular part; this particular part is a SISO link that is; this is the capacity of a SISO link and we are summing over the capacity of a SISO link each of the SISO link has a strength corresponding to lambda i. So, that means, we can say that in this case it is the sum of the capacities of r number of SISO links or SISO channels each with a power gain of lambda i.

That means, the very important situation that we see is MIMO opens up multiple scalar special data pipes or modes and this is exploited in providing high amount of spectral efficiency. In SISO, you have only one data pipe where your SNR is inside the logarithm. Here what we see is that this is broken down; the signal power is broken down and we have added them; that means, the summation is outside the law so; that means, now the power is distributed to different SISO links. Each SISO link having a certain amount of gain and the transmitted power against each SISO link is an equal power that is E s by M T.

So, all one needs to do is to compare this and see whether it gives an increase in power and the clear cut answer is this gives a much increase in signal in spectral efficiency.

There is much larger value of spectral efficiency than a SISO link, which has all the power entrusted inside the logarithm. So, this helps us grow beyond the logarithmic growth of spectral efficiency.

So, this is a very important result that we have arrived at, we will continue to discuss the capacity of MIMO channel, when channel is known to the transmitter as well as look at the beam forming techniques which help in providing high capacity for MIMO with advantage of millimeter waves where you can provide much more focused beam in the next lecture.

Thank you.