Principles and Techniques of Modern Radar Systems
Prof. Amitabha Bhattacharya
Department of E & ECE
Indian Institute of Technology, Kharagpur

Lecture - 55
Statistical Detection Theory (Contd.)

Key Concepts: Determination of PDF of phase in presence of an echo signal, concept of false alarm, introduction to the probability of false alarm, false alarm time

Welcome to this NPTEL lecture on Principles and Techniques of Modern Radar Systems. We were discussing the envelope detector, what will be the distribution at the output of the video filter, so, video amplifier.

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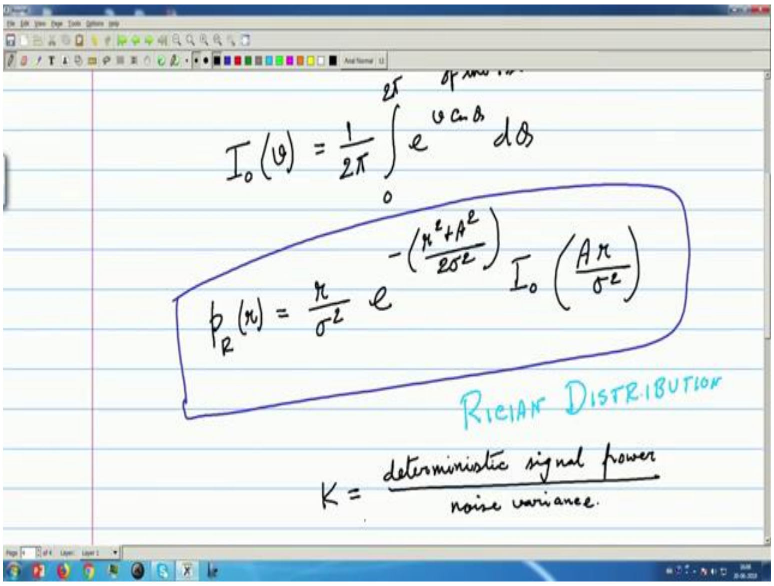

So, we have found the Rician distribution when the signal is present Rician distribution, now to gain more insight into these, let us define a parameter K. So, what is K? Deterministic signal power divided by the noise variance. So, in terms of this parameter determine so that means, K high; that means, the signal is dominating over noise K is small value that is the noise is dominating.

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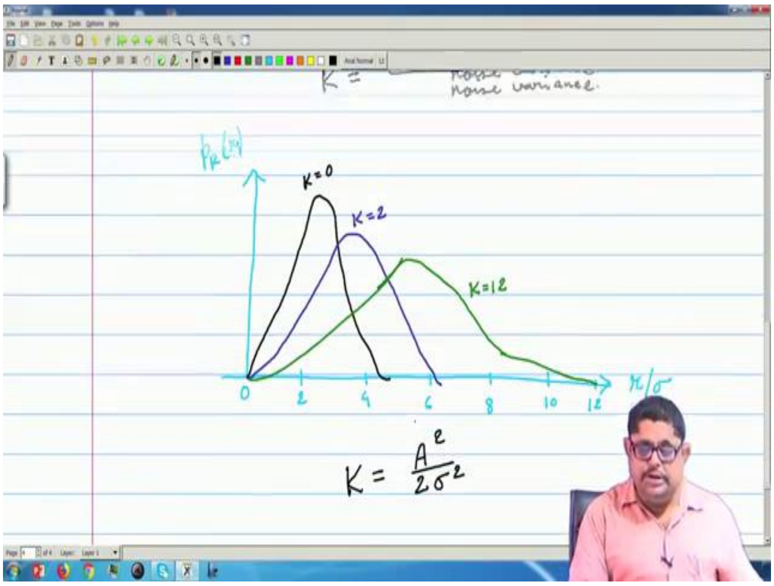

So, we can see that if we plot the envelope detector versus the envelope value then you see that for K is equal to 0; that means, the signal power is not there or noise is very high. You see this the blue curve that is small, K very small and this is some K is equal to 12, the green curve. Now; obviously, this K is equal to 0 it looks like a Rayleigh pdf and it is true that when K becomes very small the Rician pdf degenerates to Rayleigh pdf.

Similarly, if you see K is high then it looks like a Gaussian pdf, our standard bell shaped curve. So, that is also true that Rician pdf goes to Gaussian pdf when signal is high. Now, if suppose the signal is a cosinusoidal signal; that means, sine or cosine type then we can write this expression also that what will be the value of K? The signal we see amplitude A, if the a thing the cosinusoidal variation then deterministic signal power is A square by 2 and what is noise variance, that is sigma square. So, A square by 2 sigma square is basically K. So, by that actually these plots are obtained.

$$K = \frac{A^2}{2\sigma^2}$$

So, as I already mentioned that when K is equal to 0, Rician distribution degenerates to Rayleigh and which physically means that the signal is absent on the and mathematically means that both the constituent Gaussian, both the constituent random variable are Gaussian 0 mean. So, that was the Rayleigh case that is why you are coming here.

When K is large approximately it becomes a Gaussian thing, physically the meaning is deterministic signal dominating over noise and mathematically it means the envelope distribution is Gaussian with mean equal to the signal amplitude. You see it is not a it is Gaussian, but it is not 0 mean Gaussian. So, that amplitude is given by the signal and variance is noise variance, ok. So, this is about the Rician distribution a very famous distribution very useful distribution we will be using it; next, also bypassing I am just mentioning because in the previous case we have done.

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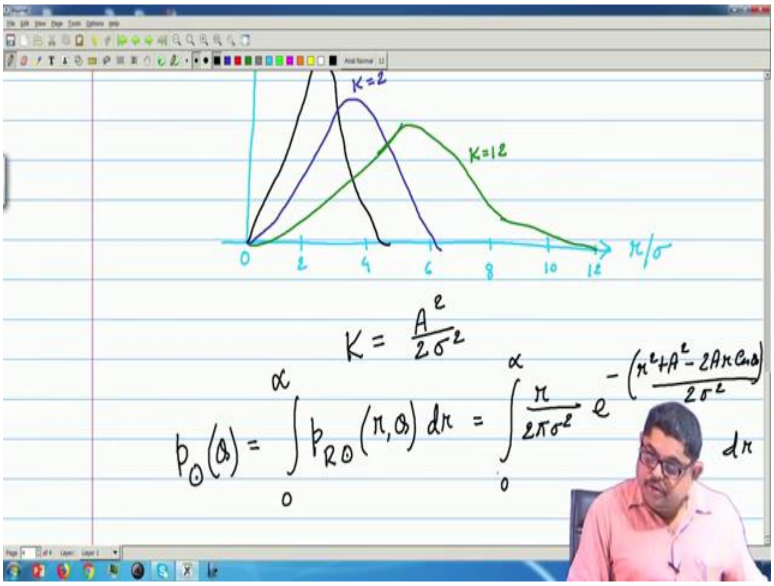

So, what will be the phase distribution, p theta, again from the joint distribution you put the whole range of the envelope variation and p R theta r, theta d r. So; that means, that you have 0 to infinity r by 2 pi sigma square e to the power minus r square plus A square minus 2 A r cos theta by 2 sigma square d r.

$$p_{\Theta}(\theta) = \int_{0}^{\infty} p_{R\Theta}(r,\theta)\, dr = \int_{0}^{\infty} \frac{r}{2\pi\sigma^2} e^{-\left(\frac{r^2+A^2-2Ar\cos\theta}{2\sigma^2}\right)} dr$$

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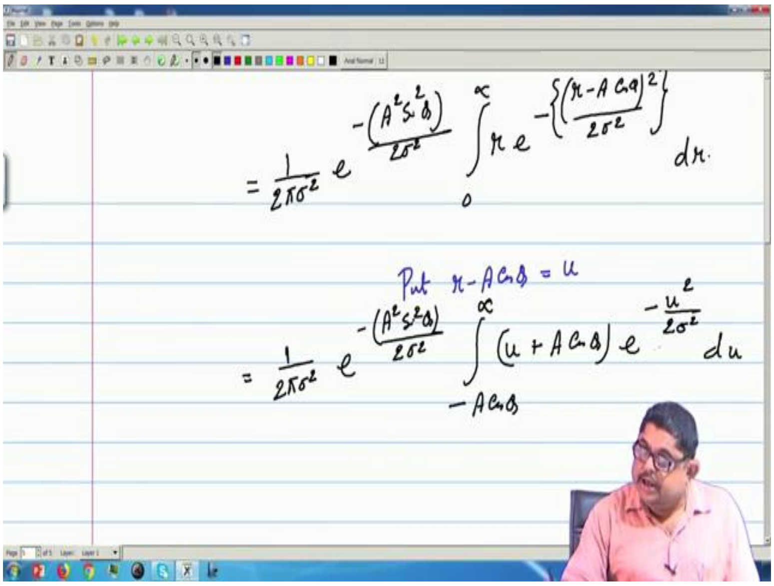
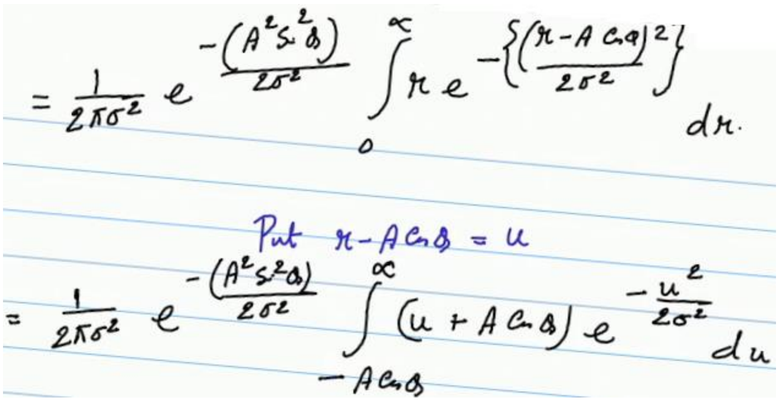


Now, here this I can write as 1 by 2 pi so, take out all the terms which you can take out outside, you will see r e to the power minus r minus A cos theta whole square by 2 sigma square this whole thing we should give it here d r. Now, to solve this integral, I will suggest that you use a change of variable put r minus A cos theta as some new variable and then you will get that this becomes minus A cos theta to infinity u plus A cos theta e to the power minus u square by 2 sigma square d u.



So, there are two integrations, this first integration is obvious many times in the deriving Rayleigh thing we have done this and this also is easier because a cos theta whole thing will be a thing. So, you will be; so, this is basically integration of Gaussian function the second one; so, that can be expressed in terms of Q1.

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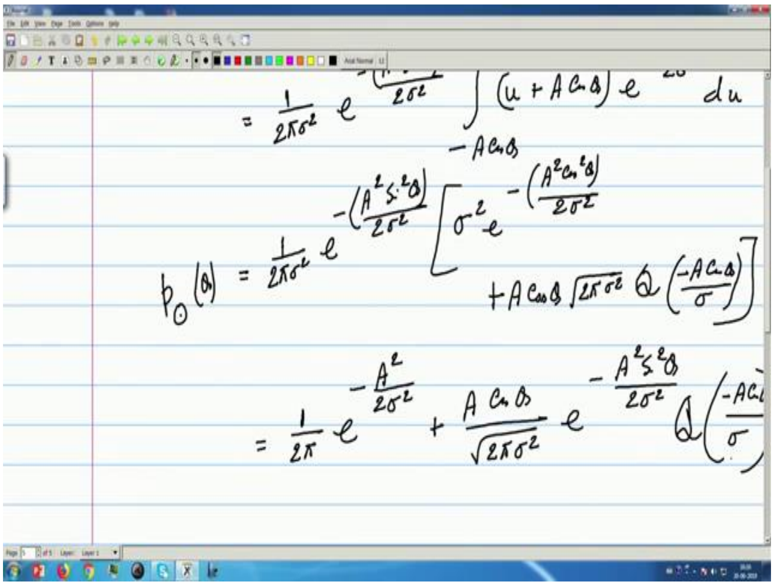

And so, I can the steps etcetera you will get in many standard books in my Digital Communication book also you I have given the whole derivation you can refer to that. So, I can write the final answer as 1 by 2 pi sigma square e to the power minus A square sin square theta by 2 sigma square, sigma square e to the power minus A square cos square theta by 2 sigma square plus A cos theta root over 2 pi sigma square then Q function minus A cos theta by sigma.

So, this will be the phase distribution and if you proceed further. The next step will be 1 by 2 pi e to the power minus A square by 2 sigma square. So, I am multiplying plus A cos theta by root over 2 pi sigma square e to the power minus A square sine square theta by 2 sigma square into the Q function minus A cos theta by sigma.

$-A\cos\theta$

$$p_\Theta(\theta) = \frac{1}{2\pi\sigma^2} e^{-\left(\frac{A^2\sin^2\theta}{2\sigma^2}\right)}\left[\sigma^2 e^{-\left(\frac{A^2\cos^2\theta}{2\sigma^2}\right)} + A\cos\theta\sqrt{2\pi\sigma^2}\, Q\left(\frac{-A\cos\theta}{\sigma}\right)\right]$$

$$= \frac{1}{2\pi} e^{-\frac{A^2}{2\sigma^2}} + \frac{A\cos\theta}{\sqrt{2\pi\sigma^2}} e^{-\frac{A^2\sin^2\theta}{2\sigma^2}} Q\left(\frac{-A\cos\theta}{\sigma}\right)$$

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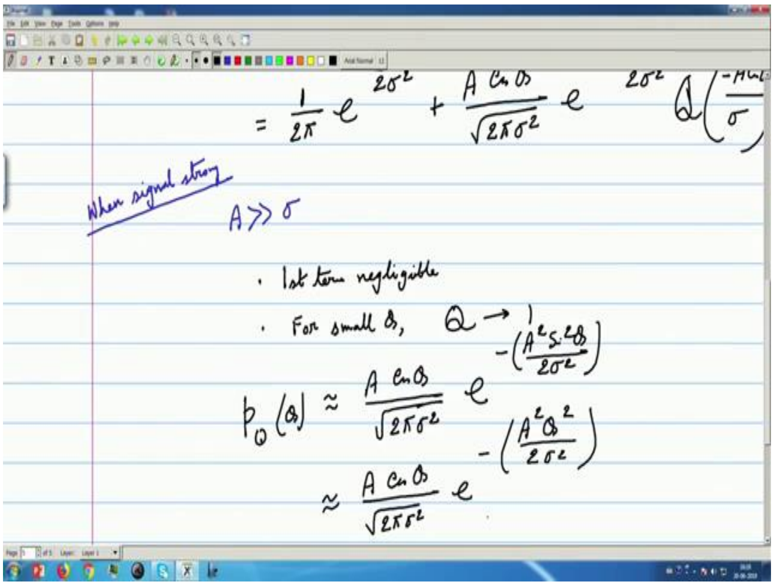

So, the phase distribution is sum of the two parts, here again I write that when signal strong; signal strong so, that time it will be high SNR; signal strong means A is much much greater than sigma. So, we can say that the 1st term negligible in comparison to the second term for small theta, this Q of minus A cos theta, cos theta is 1 so, minus A by A. So, that Q function almost for small theta it goes to 1.

So, this time we can say that when signal is strong, phase distribution is something like. So, first one is negligible A cos theta by root over 2 pi sigma square into e to the power minus A square sine square theta by 2 sigma square. Again, the sine square theta we can approximate for small theta, we can write this that this is again approximately A cos theta by root over 2 pi sigma square e to the power minus A square theta square by 2 sigma square.

So, as can we see something like Gaussian with mean is equal to 0 and variance is sigma square by A square, so; that means, this is something like a Gaussian type of thing.

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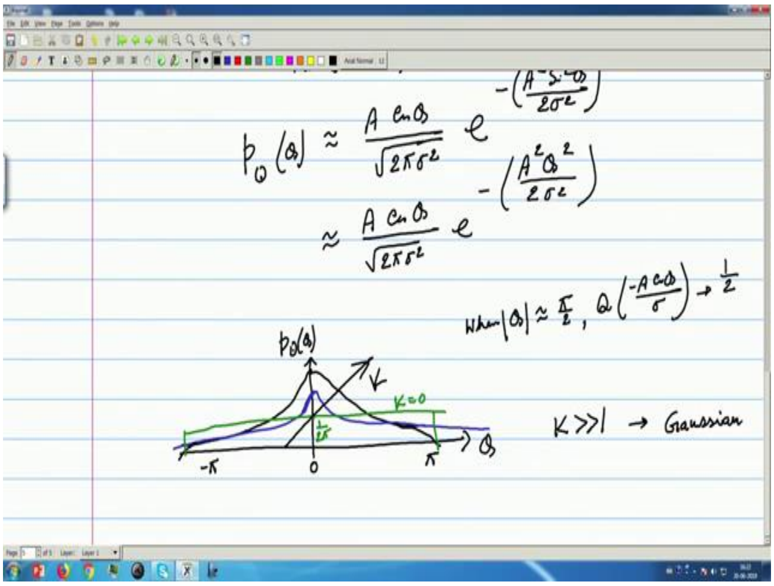

So, I can say that if this is theta versus this is p theta. So, then the curves will be something like this and so, this is variations it is both sided curve so, I can say the whole range of theta means from minus pi to plus pi. But here one caution I have assumed small theta, but let us say when theta is near about pi by 2 I cannot have this approximation that sin square theta square.

So, that again needs to be checked that when theta is of the order of pi by 2; what happens to the Q function, Q of minus A cos theta by sigma, we know that if theta goes here. So, it is basically Q up you can say almost 0 so, that is goes to half Q function goes to half. But, remember that the main thing is this is still negligible also for this theta is equal to these, the overall probability very small. So, we are not at a fault to have these type of curves and so, this is for a high SNR.

Now, as we decrease the SNR, the curves become less peaky. So, less peaky means, but the area should be same so; that means, they goes to broader ones, they should have written that this comes here. But, as I make it suppose less SNR so, actually you will see that basically this is the; that means, almost K is equal to 0 that time this same thing coming down. So, it becomes almost like the uniform case.

So, in the Rayleigh case we have seen that is an uniform graph it is also true there. So, and that this value will be 1 by 2 pi etcetera and I can say then that these are various parameter is K; so, K is becoming higher and higher. So, K is equal to 0 boils down to uniform case and K much above 1, K greater than 1 the phase pdf also becomes something like Gaussian curves. So, this is about the distribution Rician distributions properties, but let us come back to our probability concept we have seen.

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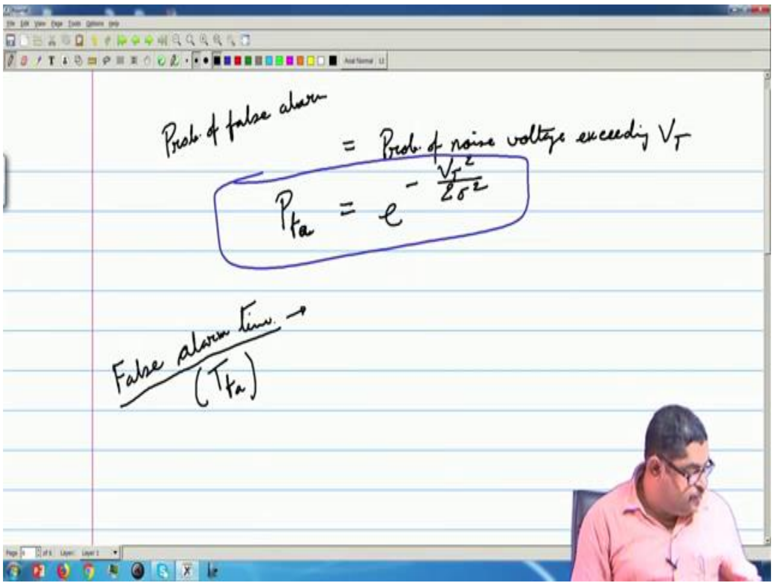

Now, let me define what is probability of false alarm probability of false alarm, what is a false alarm? That no signal is present, but noise is substantial and radar is saying that no there is a signal. So, what is that in our mathematical terms I can say it is nothing, but the probability that the, so, it is a signal is absent; that means, it is a Rayleigh distribution case and probability that.

So, before that I probability of false alarm means probability of noise voltage exceeding a threshold value V T because based on that threshold rather is saying whether there is. So, that is already you see in case of Rayleigh we have done that this probability we have found out.

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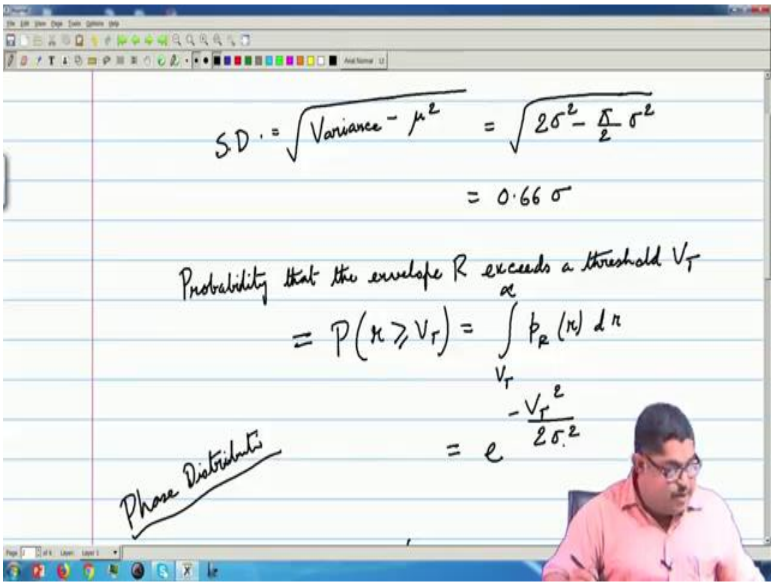

You see probability that the envelope R exceeds the threshold shall be this is P r greater than equal to V T and this is V….. So, this is the nothing, but the probability of false alarm. So, I will write that. So, that is a beauty if you understand mathematics properly then putting physics is not a problem and this is nothing, but so, I can write e to the power minus V T square by 2 sigma square that is all, probability of false alarm.

= Prob. of noise voltage exceeding V_T

$$= e^{-\frac{V_T^2}{2\sigma^2}}$$

So, probability of false alarm is designated by P fa. So, this we have found probability of false alarm. So, based on where I put my threshold I get the probability. If I raise the threshold, the probability of false alarm comes down as it is obvious, if I lowers the threshold noises we will be crossing it more often and so, probability will increase etcetera.

Now, one thing is this is we understand the concept of probability etcetera, but generally you see that if I say that probability of false alarm is 10 to the power minus 5, 10 to the power minus 7 something like that. Now, what exactly that means for a radar operator, more meaningfully I can say that can I not say it in terms of time that on an average every 20 second a false alarm comes.

So, that is why instead of working with probability because probability is to the educated persons probability is meaningful, but rather gets operated by many persons who are not so highly educated for them, more important thing is how often I the system raises false alarm. So, that is why there is another parameter that is called the false alarm time it is usually written as time of false alarm T fa, we will understand this false alarm time now.

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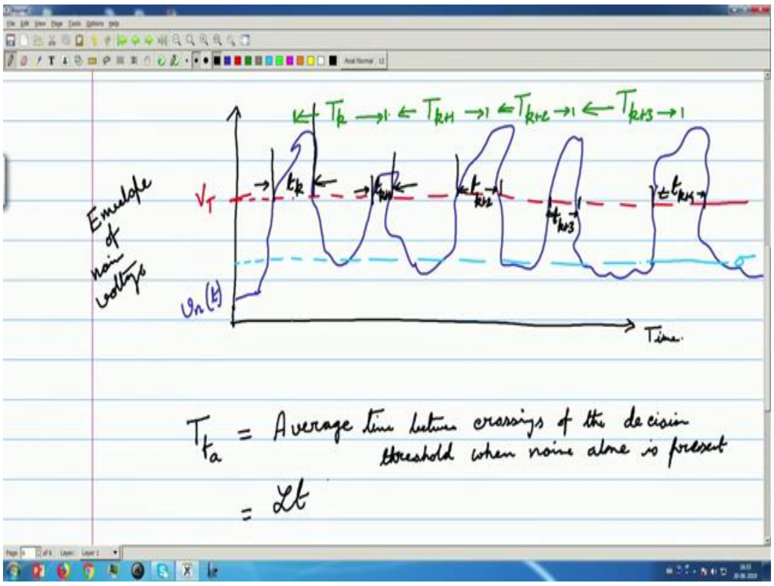

For that I just draw a simple picture that time domain picture and here I am writing envelope of noise voltage, I am observing over time so, I am just finding the thing. So, it will be something like this, I do not know I am arbitrarily just drawn like this. So, I am calling that this is let me call again v n t please remember that this is not a function, these are some sample values I have just made the locus of that and something like these it will be looking and let us say that I have a threshold put here V T.

So, you see if I put a threshold, these are the instances you can see easily that the noise voltage is crossing that threshold. So, let me define here two things, one is that you see this is let us say that I am observing for a long time and this is the k th instant where the noise has overshoot the threshold. So, what is the time here for t k amount of time the noise is above the threshold? Here I again notice that how long it is there.

So, this time I am calling t k plus 1, this time I am calling t k plus 2, this time I am calling t k plus 3 this time, this time I am calling t k plus 4 ok. And also I make another time I keep that so, almost midpoint to these so, let us say that these. So, this is the I can say that almost midpoint of these intervals where the noise is above the a.

So, I am just noting them and I am saying that here I can say that noise has crossed here also noise has crossed so, this time I am calling capital T k. So, what is capital T k? The time between two successive noise crossings so, this is T k, this is T k plus 1, this is capital T k plus 2, this is T k plus 3 etcetera. And also I can have that based on the statistics of the noise that what is the noise power, so, this thing let me call sigma the standard deviation. So, square of that so, this a voltage graph, so, square of that will be power.

Now, you see that I will say what is my T f a falls alarm time. So, this is defined as average time between crossings of the decision threshold when noise alone is present. Average time between crossings of the decision threshold when noise alone is present; average time between crossings of the decision threshold when noise alone is present, correct.

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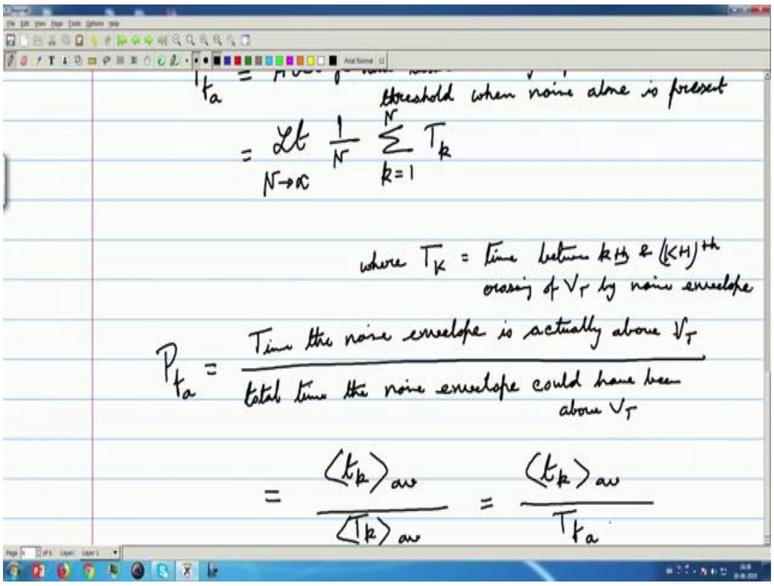

So, what is it; mathematically in terms of limit I can say that limit N tends to infinity average so, 1 by N sigma k is equal to 1 to capital N T k.

$$T_{fa} = \text{Average time between crossings of the decision threshold when noise alone is present}$$

$$= \underset{N \to \infty}{Lt} \frac{1}{N} \sum_{k=1}^{N} T_k$$

Where what is T k, capital T k; please remember capital T k, T k is time between k th and k plus 1 th crossing of V T by noise envelope. So, let me write what is T k, where T k is equal to time between k th and k plus 1 th crossing of V T by noise envelope, fine ok.

Now with respect to this diagram let me tell what is probability of false alarm; you see, what is probability? I toss a coin number of success by total number of outcomes. So, with that; so, here what is the success, let us say the success is noise is crossing the threshold and total number of outcomes that noise could have always did that, but out of that sometimes it will do. So, those are successors others altogether all tossing number that is the outcome.

So, from that standpoint I can say what is probability of false alarm? So, I can say time the noise envelope is actually above V T by total time the noise envelope could have been above V T, that is all; that means, here if I put it you see suppose I am observing from here to here. So, this throughout noise could have cross the threshold, but it actually did not in this instant it stayed there T k, stayed there for T k plus 1 it is there. So, these are successes and this is the total. So, this is that in writing I have written, time the noise envelope is actually above V T, total time the noise envelope could have been about V T.

So, in the parlance of averaging I can write this the upper one is t k average and the lower one is T the capital T k average. Again I refer you to this diagram; so, what I am saying, these are the success these small ones so, sum all of them and then divide them by the number of occurrence. So, that is that t k average small t k average and what is capital T k average that it could have been there so, the denominator is T k average capital T k average numerator is small t k average, that is all.

And this is as it is I am reading, but what is t k average? You see the definition of T f a, average time between crossings of the decision threshold when noise alone is present. So, that is nothing, but in my diagram capital T k average time between successive crossings. So, I can say this is nothing but T f a. So, I can relate P f a with T f a provided I can find this T k average ok, that you think how to do it, we will do it in the next class this is a simple thing.

$$P_{fa} = \frac{\text{Time the noise envelope is actually above } V_T}{\text{total time the noise envelope could have been above } V_T}$$

$$= \frac{\langle t_k \rangle_{av}}{\langle T_k \rangle_{av}} = \frac{\langle t_k \rangle_{av}}{T_{fa}}$$

So, conceptually you see from the basics of probability we could relate the probability of false alarm with time between false alarms, but we need to tackle these that what will be this t k average. Again, I am showing what is the t k average, that now noise is here present for this time next time it success existed for t k plus 1, next time t k plus 2, next time t k plus 3, next time t k plus 4 etcetera. Now, how this average time, average this success time how we can relate that with some radar parameter that will you also think and next class we will do that.

Once we do that we can relate P f a and T f a; T f a will be more meaningful for normal people that how often the false alarm is there. Suppose you have put a theft alarm in your a thing. So, you should know that, in a day how many false alarm that can generate because that is important to know. So, similarly instead of telling you that probability for that is so, instead you will be more meaningful if we can say ok, on an average everyday one false alarm is generated that is more meaningful. So, that for that we are doing this exercise from our basic knowledge of probability.

Thank you.