Network Analysis
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Lecture - 18
General Method for Solving Linear Differential Equation

Welcome to lecture number 18 and in our last lecture as you know I told you how to translate an inductor with initial current into a linear inductor in parallel with the current source and similarly a charged capacitor initially can be also translated into an uncharged capacitor in series with voltage source.

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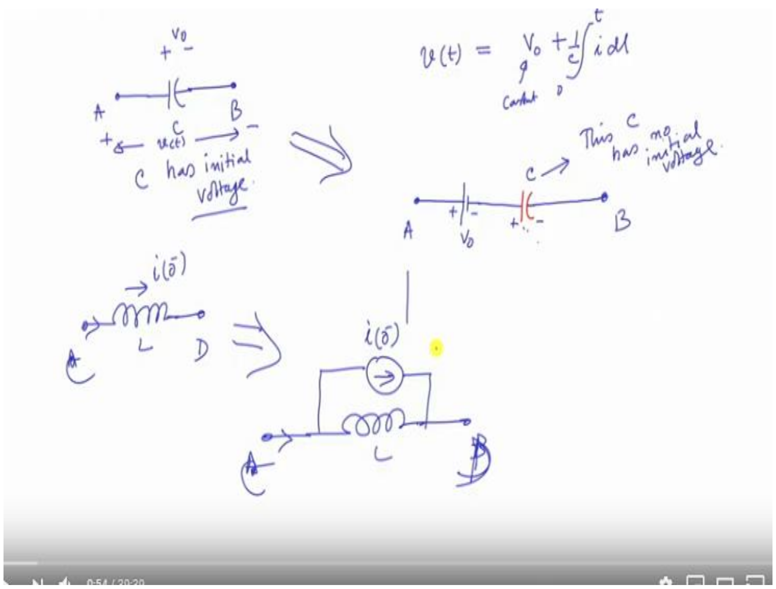

Like this A and B and we have to invoke a battery so another two sources will be introduced and well solve several problems in tutorial. And I will also ask you to solve several problems using this concept.

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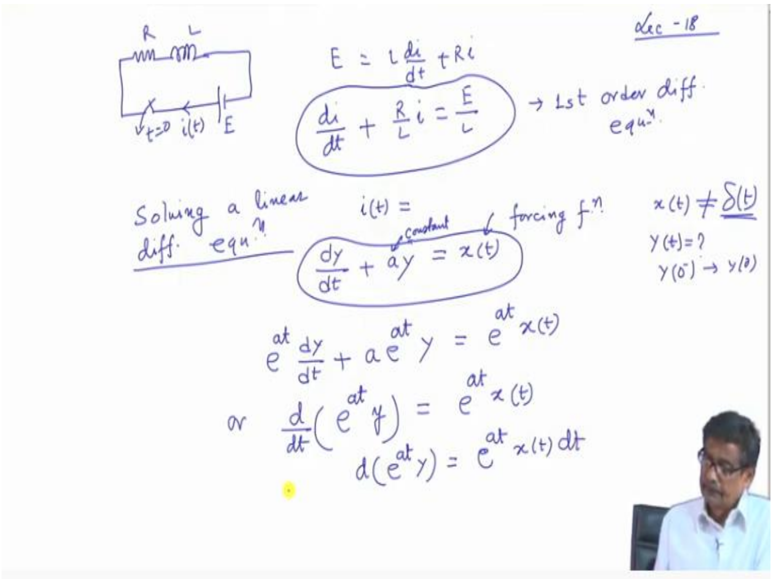

Now today what I will be doing is see while solving this circuit problems say RL.RC circuit discharging or charging circuit of both RL and RC type. Essentially what we are doing is we are writing down the differential equation and trying to solve that differential equation. Fortunately, the differential equations with constant value of RL and C will always give rise to a linear differential equation.

And also, I told you and we have seen that if in a circuit there is only one energy storing element is present. For example, RL circuit okay. And we have studied this, and you have a battery like that E and switch is closed at t=0 then I can solve for the current how did I solve this current? I wrote down the KVL equation which was equal to L didt + Ri and I divided both sides by L to get this R/l into i = E/L.

So it was a differential equation which is first order differential equation first-order because only ddt is present no d2 dt2 is present. So first order differential equation and then I multiplied with integrating factor then integrated the got the value of i in my previous lecture similar with the capacitive circuit. Now I will digress a bit from this circuit and try to tell you today that okay it is essentially in a circuit when current will be functions of time etc. and the energy storing elements will be present.

Then it is bound to give rise to a differential equation linear differential equations the question is can I avoid those steps of multiplying with integrating factor and then get the solution? Or is it possible also to find out the solution that is it the solution it whether can it be written just by inspection. I will have this differential equation can I write the solution in one stroke without every time multiplying with integrating factor and so on.

That is the whole idea that will also give us a better insight into the problem. Therefore, today and maybe in the next lecture also I will concentrate on how to handle linear differential equations given the linear differential equations how to write down the solution of that differential equation just to by inspection without spending too much time on integrating factor this that, that is the whole idea.

So first let us then take a differential solving a linear differential equation So take a general differential equation like this dy/dt +ay =xt where xt is called the forcing function which maybe a function of or it may be constant whatnot it may be anything except xt I rule out this possibility is not equal to an impulse function. We will talk about this later xt is a general reasonable function it can be anything.

Now what will be the solution of this? A is constant and we are looking for this solution yt is how much xt is given and may be some boundary condition is given. It is a first order differential equation so one boundary condition will be needed that is yo or y0- from which I will get y0 same if it is a inductor current y0- is y0 +y0 similarly capacitor voltage okay so this is the statement of the problem. This differential equation I have to solve.

Okay here of course I will use that fundamental thing that is I will multiply both sides with the integrating factor e to the power at very simple stuff but very educating a into e to the power into y = e to the power into xt, xt could be a function of time as I told you. So this is the thing then or we can write ddt of this is product up to function we have done several times before y= e to the power at, xt into dt no dt this is this, this will be the thing.

Now then we say that d of dt you bring it to this side so that you can say it is equal to e to the power at into y = e to the power at, xt, dt this is the thing.

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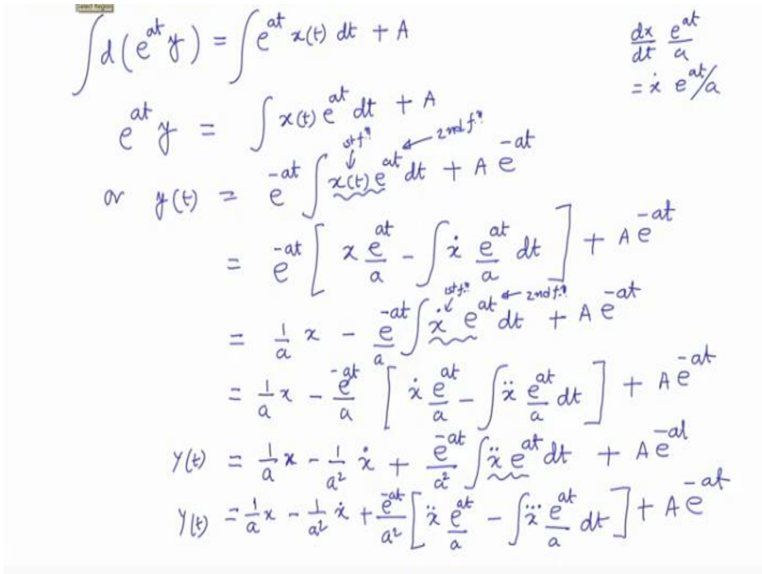

Now this one let me go to next page and write this d into e to the power at into y is equal to what? e to the power at into xt, xt is the input forcing function into dt. My target is to get to y so I will integrate both the size I will integrate the moment you integrate you generate a integrating constant A. Then left hand side will be e to the power at ito y = this integration which can be also written like this e to the power at, dt + A.

I am not putting any limits that is why these integrating constant has surfaced A okay. So this is the thing, or I will say yt will be multiplied both sides with e to the power -at then this integration xt e to the power at into dt + A into e to the power -at this is the thing. Now what we will be doing is that this integration if you look at this is fine some term but here is an integration which is product of 2 functions. So this integration I will expand so e to the power – at was there already. So what I will do is this I will apply the rule of integration of product.

So first function that is xt into integrate the these it test first function first function and this e to the power at as second function. So first function into integration of the second which will be this by this then minus of integration of differentiation of first function that is dxdt into integration of the second e to the power at/A this will be this term but for dxdt what I will do I will write x dot okay x dot is dxdt

So differentiation of the first function that is x dot into integration of the second e to the power at/a, a is a constant number into dt and plus of a into e to the power -at this will be the thing. So the solution will then look like 1/a put this e to the power -at inside. So this will cancel out It will be x- e to the power minus at it comes before this integration. And then this integration x dot into e to the power at and this 1/a can come outside this integral. So this I have written into dt and plus this term A is that integrating constant we have already written.

So this will be the thing clear. So you used the product of integration property first function integration of this second minus differentiation of the first function. That is x dot dxdt e to the power at and so on. So this is the thing now in this type once again I find that okay this term is fine 1/a into x – e to the power-at /a that is also fine. But here is once again product of 2 functions and it is to be integrated.

So I will apply once again the product rules of integration and expand this term and try to see what do I get. For example, I will say this is now my first function and this is second function. So I will say first function that is x dot into integration of the second e to the power at/a minus integration of differentiation of the first function that is d2xdt2 which I will write x double dot into integration of the second.

That is e to the power at/A into dt that is all. And then of course this term is always there a into e to the power -at is it not? This will be the thing so once again open this bracket and you will be getting it as 1/a x- this e to the power- at e to the power at will go and you will be left with 1/a square into x dot then plus off this minus plus and you will be left with e to the power minus at/a and this a will also come out make it a square and you will be left with x double dot into e to the power at dt and plus of course this term e to the power -at this we will get.

Okay now once again I have not really, I have got this solution but this integration once again appears x double dot I will do once more Okay so I will say and this time I will do very quickly this is once again this first function, second function I expand that. So what it will be 1/x-1/x square x dot then + of e to the power-at by let me do one step please bear with me. I am doing some mundane thing, but it is very useful when I finally draw the conclusion. You see how nice it is so it will be.

Then once again you integrated by parts integrate it by parts. Then it will be differentiation of x double dot so it will give you x triple dot. First function differentiation into integration of the second that is this by a oh no first function. That is x double dot into integration of the second minus integration of differentiation of the first function. That is x triple dot into integration of the second dt is not it will give like that integrating by parts going on. I mean first step I did then once again I expanded this.

Now I am expanding this, and this is this and this is plus A into e to the power -at. Now this is equal to your yt what we do is this. I will copy this last step here copy this here where is copy?

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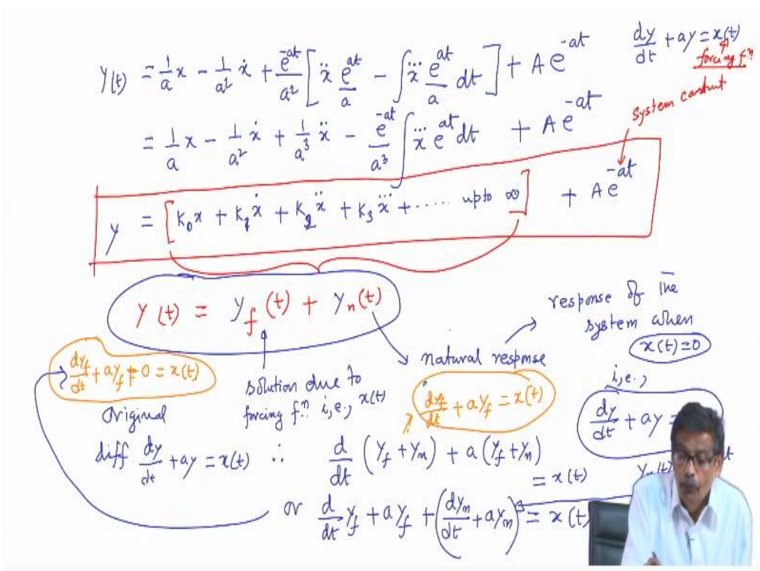

Go to next page and paste it okay so this is the equation last I got. Okay only thing here that I forgot to copy this -at it was there other things are correct I got this. Now this if you expand it comes out to be 1/a x-1 a square into x dot. Then if you put it inside it will be +1/a cube into x double dot this e to the power at cancels out this e to the power - -at/a cube because a square and this integration of x triple dot into e to the power at dt and then plus finally this one.

Okay I have now understood what is going to happen if I further expand is it not but nonetheless this solution looks like if you go on expanding this it will be like this some constant this is important x plus another constant into x dot plus another constant let me write it k0x k1x dot plus k2 x double dot. Similarly, if you proceed like that it will give rise to x triple dot and it looks late. It will go forever up to infinity go on splitting this one and you will get like that.

But nonetheless up to infinity in finite terms +a into e to the power -at that will be the solution of what of this differential equation dydt+ay=xt what is xt, xt is the forcing function got the point. So this is the solution. Now in language if you say if you look at this one you will see that this solution has got 2 parts.

One part depends on the input and its higher order derivatives up to infinity plus another term which does not depend on input signal xt but it depends on the system characteristics R/L for example in case of RL circuit that system this a is system constant. So this part of the solution I can say that the solution yt will comprise of two parts.

One is except xft, xt is the forcing function that is and another part which depends upon only system constant and that is called natural response. I will write it as xnt so every differential equation whether it is first order or second order we will extend this result to second order as well. Do not worry about that at least for the foster system I can very confidently say logically thinking that okay this is the differential equation.

It is a forcing function it could be anything any function of time maybe a sinusoidally varying function of time maybe some pulse whatnot whose equation is known or maybe a constant number as in case of an RL circuit excited with the DC voltage at right hand side is a constant but whatever it is the first part of the solution is called solution due to forcing function that is xt. It depends on that only xt and its derivatives up to infinity that looks like a daunting task.

But we will see how simple it will become. And this part of the solution is called solution due to natural response solution natural response let us write like that natural response or response of this circuit with no I am so sorry I should write it in this way. This solution has got 2 parts yft+ynt because I am looking for y. So yft is solution due to forcing function and ynt is the natural response y next is 0.

So what is natural response? It is response of this system of the system when xt = 0 put xo all are zeros e to the power -at is it not that is response of the system when x=0 that is the solution of the system when dydt from this equation +ay =0 no input is given. What is the solution? Can there be a solution without input? Yes, because a differential equation means so far you are doing circuit analysis you can easily correlate it oh there was a either some inductor capacitor was present and that might have some initial stored energy.

So you can get always solution because of that or in physical system mechanics say mass spring system. So if mass is present even without any excitation there maybe movement because of initial kinetic energy of the mass. Although external force applied is 0. So this is the thing mind you, so the goal is that okay differential equation where somewhat solving mechanically there is differential equation is there find out the integrating constant each time I did in my previous lecture while solving RL circuit transient or RC circuit.

I wrote the equation I multiplied with integrating factors solve the put boundary conditions got the solution. Nothing wrong in that but what I am now trying to tell you, you know think physically okay solution now I identify that the total solution of this system has got distinctly two pieces. One piece depends on the forcing function on the right hand side and the other part is natural response when xt is 0 and when xt is 0 natural response will be like this that is what we have found got the point.

Now when I say that this is the solution all of these different this the solution of what differential equation although so original differential equation very interesting, I am telling you if you please follow me dydt+ay=xt this is my original differential equation and we have got this total solution now two observations obviously this if you claim this is the solution. So if I substitute this here it must satisfy this equation.

Solution means that whatever solution you have got if you put it here to it will satisfy that first thing is, therefore, I will expect that ddt let me put this as yf+yn+a of yf+yn and this must give rise to xt is it not. Now this I will write it as ddt of yf +ayf +dyn/dt+ayn and that will has to be xt is it not. But the solution due to natural response we have found out earlier this ynt is this one a into e to the power -at.

So what will be this if you substitute here what is this addition? It will be equal to dyndt that is -A into a e to the power -at +a into yn A into e to the power -at and it will give rise to getting therefore the natural part of the solution. I am substituting the full solution I find that yn part will and therefore I will say I will not go to next page because this is so important conclusion then I will say from this to this that let me use some other color.

I will say that it will be equal to dyfdt+ayf+ this is this has to be 0 these two terms =xt excitation. So the conclusion of this statement of this equation is this solution consists of two parts natural parts which does not depend upon excitation and natural part if you substitute on the left hand side do law always give rise to 0? And I find that the part the piece which I am saying is the solution due to forcing function.

Forcing function alone will also satisfy the differential equation. This is the statement that is yf+yn satisfies the differential equation. It has to =0 also dyfdt+ it is obvious from this ayf will also satisfy this, this is crucial to understand ddt of yf+yn+ into ya+yn=xt. So ya+yf satisfies the differential equation so also forcing function alone solution due to forcing function yft alone will also satisfy the differential equation. Now what is the then in which way it is going to help me I will stop here.

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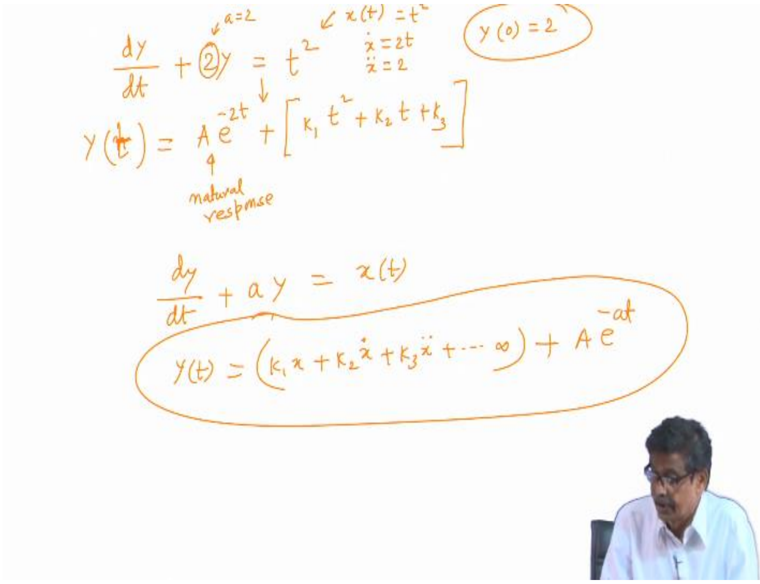

Before that I will tell you one thing. See for example let me take an example so that you keep your interest alive in this particular topic because at the end things are so simple. For example, you have a differential equation dydt+2y= say what? t square this is my xt I have to solve this differential equation and the boundary condition is giving y0= suppose 2 solve for yt. That is what I have been told.

Now I know I will write down in one stroke I am not going to use now any integration like that I identify xt and then y total solution yt will be what? This 2 comes in as A into e to the power-2t that is what we have seen. This term will be there then there will be, so this is the natural response straight. I mean no further thinking straight away I write, or this is the thing A e to the power -2t A value =2.

In this particular case this term is natural response then plus the solution due to forcing function this I am putting a bracket. That solution I know it will be proportional to k1 into x xt=t square so k1 into t square +x dot into some k2 bar into x dot what is x dot 2t. So 2 ito t and this 2 I will absorb it in this constant. So k2 into t + because I have to go up to infinity +k3 into x double dot which is =2 d2xdt2.

So k3 into 2 k3 bar into 2 is k3 +k4 inti x triple dot oh that is 0 then k5 into 0. So although it looked a formidable task in general whatever functions will be present on the right hand side it vanishes. So all the terms I have taken into account only these two terms are present. Got the point that is all that is the total solution. So differential equation is given the nature of solution of this equation I will write in one stroke no wastage of time. I know this constant these the natural response this is this one that is the thing.

Now the big question is how to find out these constants say k1, k2, k3 so here it ends here. So people say that given a differential equation I will carry on with this in the next class. But I the crux of the matter is if you are given a differential equation dydt + some constant =y into some xt we say that solution total solution will be a linear combination of the forcing function and its higher order derivatives that is k1x+ you understand the term linear combination.

That means k1 into x k2 into x dot k3 into x double dot these are constants up to infinity of course do not forget to write that and plus solution due to forcing function and that will be equal to e to the power -at called this is the thing I wanted to tell you and I will also discuss in the next class is there any exception to this rule? I will discuss and then I will also try to solve higher order differential equations.

So please go through this lecture it is interesting is it not now I am not integrating after I obtain this pattern I know what is waiting for me for yt and this can be easily written linear combination of x and its higher order derivatives this term of course up to infinity it may slightly disturb you but in almost all the cases that you will find it is not tough to tackle. For example t square t,t square t and then t and then constant after that all differentiation should be 0.

So linear combination of the forcing function and its higher order derivatives is responsible for this solution due to forcing function. This part and natural response is of course is a system property which is decided by this fellow which is mind as at whatever coefficient is there. Some constant into e to the power -at.

Now at the end therefore the solution can be written in one stroke straight. Only thing is you have to put numbers wat will be the values of k1, k2, k3 and A those things I will discuss in the next class and we will use the boundary conditions and some conditions of identity to determine this constants K and A. Thank you.