Network Analysis
Prof. Tapas Kumar Bhattacharya
Department of Electrical Engineering
Indian Institute of Technology - Kharagpur

Lecture – 19
General Method for Solving Differential Equation - II

So, welcome to 19th lecture and we are discussing on the very important topics that is the how to solve a differential equation because differential equations, you cannot avoid whenever you want to analyse the circuit in time domain, having energy storing elements.

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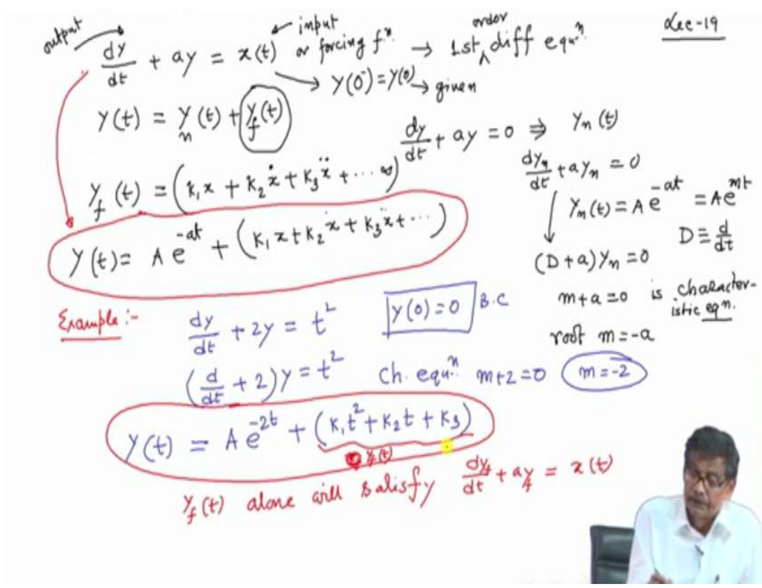

So, in our last class, I told you we took this general first order differential equation which whose output is yt, so dy dt plus ay is equal to on the right hand side, there is a forcing function, so this is input or forcing function and y is the output, is that or may be the current or voltage across a capacitor and so on in circuit analysis. So, this yt is this one that is the output we are interested in.

So, you have to we have formed a differential equation, this is a first order differential equations so, we know how to solve it by multiplying with integrating factor and so on but then we found out that the solution of this equation; total solution and whenever this differential equation is giving, you are also provided with first order differential equation 1 boundary condition will be necessary that is y0 minus or y0 that is same as that one for reasonable good function xt, this is given, okay.

The value of y0 is known, so the solution will consist of a natural response plus solution due to forcing function which I am writing as t, these are the 2 points. Now, this natural response alone if you substitute that is a natural response will be the solution, if were the right hand side is 0, then you get ynt and ynt if you substitute alone on the left hand side, it will always be raised to 0 because dyn dt plus ayn you has to be 0 that is why it is the solution.

And the question is about this forcing function and the solution of this one, ynt is equal to some constant into e to the power minus at that is the thing. Now, at this point I will tell you another important thing that this equation you know can be written as D + a into yn is equal to 0, where D is this operator d dt, then if you substitute D by some m, we say that m + a is equal to 0 is the characteristic equation of this one.

And the root of the characteristic equation is m equal to -a in fact it this this root which comes as mt, if the root is m, it is equal to A into e to the power mt and then I have substituted the value of A, therefore given a differential equation characteristic equation can be easily formed and the root of this equation eventually comes as the coefficient of t in this exponential time, anyway so that is the thing.

Now, the question is what should be the nature of yft; yft, nature of yft we have seen will be a linear combination of xt, k1x and its higher order derivatives up to infinite terms that is k1, k2, x dot plus k3 x double dot up to infinity, a linear combination of the forcing function and its higher order derivatives, this will be the nature of yft. So, the total solution yt, then will be A into e to the power -at plus of k1x + k2 x dot + k3 x double dot up to infinity.

This is the solution and the point I want to make it, now given the differential equation I am not going to integrate or things like that, straight away from this I should be able to write the; so here the nature of the solution is known, only thing this constants are to be determined. So, how to do that for example, last time I was trying to doing a problem suppose, I say that the equation is example, it was like this that dy dt plus 2y is equal to suppose some t square.

And it is giving that y0 is equal to 0, this boundary condition given; boundary condition and get the total solution of the system, so this states will be like this; ddt, you replaced by m, so this equation is ddt plus 2 into y is equal to t square. Now, I will say that characteristic equation will be m plus 2 is equal to 0 that is xt is equal to 0, like this, is it not. So, root is m equal to minus 2 that is what you get.

Then, I will say the solution is; total solution is natural response solution will be A into e to the power minus 2t plus this terms, now this terms although it is infinite but for this function, it will be like this, k1 into t; k1 into t square plus k2 into t because differentiation is 2t; 2 will be absorbed in this constant k2 plus differentiate it once again k3, after that no terms exist, so that only goes, I mean it is like this, so this is the total solution.

Now, after you get the total solution, you have to find out so many constants, see once you write these, you have solved 90% of the problem without wasting much time, only thing you have to find out the characteristic root and say that the nature of the solution will be like this. Now, okay this constants are to be determined now, I told you one thing that solution due to the forcing function, this point is to be remembered; yft alone will satisfy the differential equation; dy dt plus ay is equal to xt, it has to.

Because yn part will always give rise to 0, so it will be; so this one is true, if that is true, then I will say that the this part is the forcing function except not except t; yft, so yft alone will satisfy these differential equation, so I will put it here in the differential equation.

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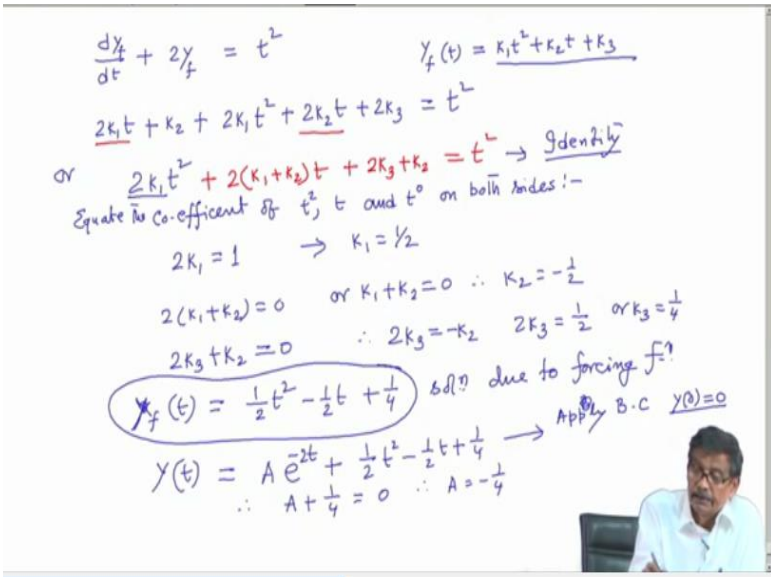

And I will say that the differential equation dy dt plus 2y, it was equal to t square in this case, forcing function will alone satisfy this therefore, yft will have seen that its nature has to be k1 t square plus k2t plus k3, only 3 terms, so put it here, if you put it here, so ddt of yft will be 2k1 into t plus k2, dyf dt, 2k1 plus k2, dt of k3 vanishes, only 2 terms will be there, okay and then plus 2yf; 2yf is 2k1 t square plus 2k2 t plus 2k3 and that has to be equal to t square, this is the thing.

Or you collect the terms, if I make a mistake, point out that so, it will be 2k1 t square term, t terms are; what are the t terms; this is one; this is one plus 2k1 plus k2 into t and plus the constant term that is 2k3 plus k2 and this I am telling it is t square. Now, this thing has to be true for all time t, no matter what is the time, so this is an identity, therefore coefficients; equate the coefficients of t square, t and t to the power 0 on both sides that is 2k1 must be equal to 1.

On the right hand side, there is t square, here also it is t square, so its coefficient is 2k1, it must be equal to 1, therefore it straightaway gives you k1 is equal to 1/2, there is no terms here, so coefficient of t terms on the right hand side is 0, so I will say 2k1 plus k2 is equal to 0 or k1 + k2 is equal to 0, therefore I know k1, k2 must be equal to minus 1/2 like this. Finally, 2k3 plus k2, constant term there on the right side no constant, so that is also equal to 0.

Therefore, you know 2k3 is equal to minus k2 or 2k3 is equal to 1/2, is it not, minus 1/2 or k3 is equal to 1/4, therefore except t, yft, sorry, yft will be equal to k1 t square, put the values there, so k1 was 1/2 t square plus k2 into t that is minus 1/2 into t plus k3 which is equal to 1/4 this will be the solution due to forcing function mind you, solution due to forcing function. Therefore, total solution now I know is A into e to the power minus 2t that was a natural solution of the system.

And plus this thing; 1/2t square minus 1/2t plus 1/4, to find out this constant, now apply the boundary condition, what is the boundary condition given? Boundary condition; now, apply boundary condition to the; so boundary condition is to be applied to the total solution mind you, so apply boundary condition which is given to be y0 is equal to 0, therefore I will say that A at t equal to 0, y0 is 0.

So, put t equal to 0, so it will give you A, this will give you 0 plus 1/4 is equal to 0 because boundary; so A is equal to minus 1/4.

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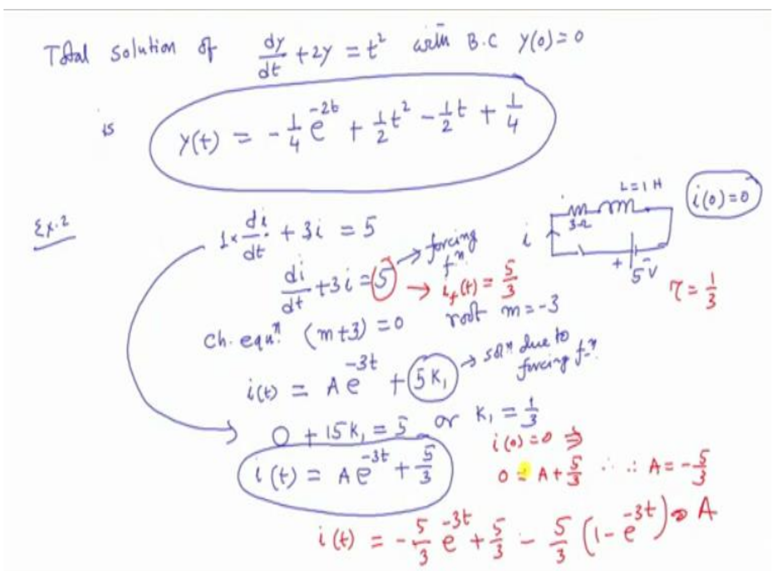

And hence finally, I will say that total solution of dy dt plus 2y is equal to t square with boundary condition y0 is equal to 0 is yt which is equal to I have found out now, A is equal to minus 1/4 so, minus 1/4 into e to the power minus 2t, then k1t square, k1 was minus 1/2; minus 1/2 plus then, k2 into t minus 1/2t plus 1/4, this will be the solution, you can verify at t equal to 0, it must give rise to 0, is it, minus 1/4, minus 1/4, t equal to 0, it will vanish.

And this 2 terms are already 0, t equal to 0, so at least it is satisfied, hopefully it is correct, therefore you see to solve the differential equation at least first order differential equation, it is so easy, you and this way if you solve, it will give you a better insight into the problem and I am not integrating anything, if I know that rule that the total solution will be 1 exponentially decaying term depending up on the characteristic root.

And then the rest of the thing is x dot x plus x dot plus x double dot linear combination of those terms of course up to infinity. For example, let us say another example, let us say example 2; dy dt plus 3y is equal to y, a constant DC voltage, suppose this circuit; RL circuit you have connected a 5 volts apply to it. Suppose, the current in the circuit I am telling y, so voltage drop and suppose the value of R is 3 ohm, L is 1 henry or I will write it i, let us write i anyway, differential equation matters.

So, what is the differential equation? Differential equation will be this Ldi d, suppose the L is 1 henry, 1 into di dt plus Ri; Ri is this and that is 5, so it is a first order differential equation and it is in this form di dt plus 3i is equal to 5, we have solved it earlier but now I will apply this method to solve it very quickly. First thing is characteristic equation; m plus 3 is equal to 0, characteristic root is equal to m equal to minus 3, this is your forcing function.

So, if you so, total solution it will be equal to A into e to the power minus 3t that is that part is over plus forcing function; linear combination of forcing function and its higher order derivatives; higher order derivatives will be 0 because it is constant number 5, d5 dt is 0, d25 dt2 is 0, so only one term will remain here and that will be proportional to 5 plus k1, you know xt is 5 into k1 that is the thing.

Then, this is; this will be the solution, okay and boundary condition given is suppose i0 is equal to 0 that is i0 minus 0, i0 is 0, so I will find out this solution here but one thing is this is the solution due to forcing function and forcing function alone will also satisfy this equation, so I will first find out this k1 by putting 5k1, which is constant that will be 0 plus 15k1 and that is equal to 5, is it not.

And obviously, here no complications, identity it is, so k1 is equal to 1/3, 5/15, k1 is over, therefore then I will say total solution is equal to A into e to the power minus 3t plus 5 by 3, this is the solution; total solution. Now, to find out the constant A, apply the boundary condition which is i0 is equal to 0, so i0 equal to 0 will give you what; 0 is equal to A plus 5 by 3 and this is the solution, up to this correct know; k1 is 5 by 15 1/3rd.

So, at equal to 0, it will be A plus 5 by 3, this is the thing therefore, A is equal to minus 5 by 3, so what will be the total solution; it will be equal to minus 5 by 3, sorry; minus 5 by 3 e to the power minus 3t plus 5 by 3 which is equal to 5 by 3 you take common and 1 minus e to the power minus 3 by t and it is consistent, RL circuit final current is 5 by 3 for this circuit and the time constant is 1/3rd so, this is nothing but this is okay, okay that is all, 1/3rd is the time constant.

Therefore, you get the same solution very quickly, in case I am telling you one another nice thing about it, if your excitation is constant number, all this you do that is fine further you can reduce your labour that is the if the right hand side is constant, the solution that is ift due to forcing function will be supress this term, di dt 5 by 3 will be this thing, got the point, you suppress di dt term, then solve for i it is 5 by 3, in one stroke I could also write that.

But anyway this is the most general way, therefore I told you how to apply but once again, I am reiterating this point very carefully that forcing function alone satisfies the differential equation, then it will give rise to an identity so, equate the coefficients of the like powers on the left hand side and right hand side to get the constants k1, k2, k3 etc., that is one thing and second thing is then write down the; so first determine k1's, forcing function constant.

And then you write down the total solution, then apply boundary condition, this is very important step, it is not that you start applying the boundary condition only to the solution due to forcing function that point should be noted, I hope you have understood this.

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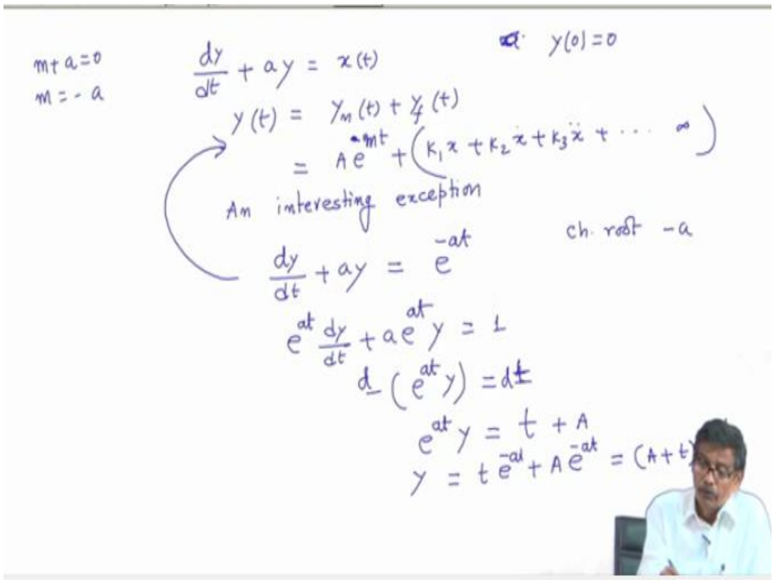

Now, so far, so what we have seen is this that so, we have seen that dy dt at least the first order differential equation ay is equal to xt, it may be voltage, y may be current and so on, in mechanics y may be displacement, x may be applied force anyway whatever it is and boundary condition is giving that is output boundary condition is suppose, giving some value it will be given, okay, it is 0 I am assuming.

And I told you, let me write many a times, so that you understand this solution consists of a natural part plus solution due to forcing function which will depend on xt alone and the solution due to natural response, you get from the characteristic equation, you find this constant and say that it is A into e to the power minus mt, m is equal to minus a that is all plus solution due to forcing function will be k1x plus k2 x dot plus k3 x double dot, linear combination of x and its higher order derivative, that is the thing.

And now, I know okay with the functions will be handling a network analysis, this term one need not think one has to differentiate indefinitely to get the solution, okay that is there then, you get that. There is one exception that is why I have written this one, an interesting exception to this rule what is that? Suppose, I say that differential equation is like this; ay and the forcing function is e to the power minus at.

See, characteristic root is minus a, if on the right hand side, the forcing function is e to the power minus at, what will be the solution, so from this it looks like e to the power minus at and this I will do but if you do, you will be making a mistake, why; I am telling you because all this equations where found out from fundamental things, what was that fundamental thing; you multiply with integrating factor on both the sides, is it not.

You got after all these thing by multiplying with integrating factor starting from the first principle and here if you multiply this becomes 1, then may be some other constant multiply there; 5e to the power minus at, that does not matter, so this will be the thing. So, what you get here is d dt of e to the power at into y is equal to 1. So, what will be e to the power at into y; it will be t.

Because if you differentiate t, you get 1, integrate both side, take dt that side, is it not that is I have taken this dt to that site and integrated, so you will get like this and plus a constant of course constant of integration, then your y will be t into e to the power minus at, multiply both sides with e to the power minus at plus A into e to the power minus at. So, the solution will be A plus t into e to the power minus at.

And this is not the natural part elements undisturbed but it is not the forcing function solution due to forcing function is e to the power minus at but it will be not like up to infinity you differentiate like that so, this rule of exception one must understand, then of course it will be very easy, you just integrate and get that, therefore if the root characteristic root happens, the excitation function or input functions happens to be same as the characteristic root, e raised to the power same mt.

Then, one has to do like that, okay, this will just remember, be careful about xt okay, now so in the first order system things are pretty simple. Now, to complete to the discussion on this, we will take up now higher order differential equation.

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$$\frac{d^2y}{dt} + a\frac{dy}{dt} + by = x(t)$$

$$y(t) =$$

And try to see that whether similar things can be done that is suppose, a higher order differential equation is like this; plus ady dt plus by, it is the second order differential equation and it is excited by input, output however, by inspection looking at this differential equation whether can I write yt, the nature of yt in the same way as I am done in case of a first order system. So, in the next class, I will be doing that, thank you.