Network Analysis

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Lecture – 43
Laplace Transform - I

So, welcome to lecture number 43 and as I told you in my last lecture that we are trying to solve network problems with the help of Laplace transform, that is another way, nice way and very popular way of solving network problems, we will see because I have not yet started Laplace transform.

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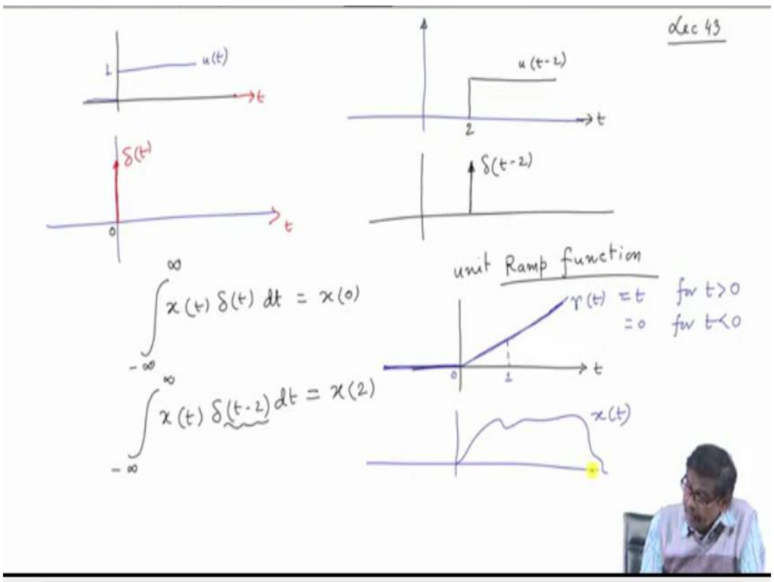

But I told you about some standard functions for like that unit step function, which is like this, this is ut, then I told you about unit impulse function, this is delta t, 2 delta t if somebody want to sketch 2 delta t; 2 into delta t, just this height of course towards infinity but 2 times, strength of the impulse is said to be 2 similarly, 2ut is the strength of the unit step is 2, you understand that.

Now and these axis are of course, time axis, then I told you that you can have a shifted version of unit step function for example, this is a signal which is ut minus 2, where this is 2, this is time shifted to the right, this is the value of the function similarly, you can have a shifted delta function here at 2, this one can be written as delta t – 2, so this is how, these functions were developed.

And properties of; one of the properties; there are several interesting properties, if it is needed for some problems I will further go into that but one property I told you that if you multiply xt with delta t and integrate it from minus infinity to infinity, it is going to give you the value of x0 that is wherever this delta function is situated, delta is situated at t equal to 0, so that value of time it will give for the value of x.

Similarly, minus infinity to infinity if you integrate, xt into delta t – 2, it will fish out the values of the function at t equal to 2, it will be x2, so you have to examine where this delta function is situated and if it is multiplied with any other general function xt, the corresponding values of x can be found out. Another function just I will mention is called a ramp function; unit ramp function.

It is like this, with respect to time the function will linearly go and it is called rt, slope is 45 degrees that is at 1, t equal to 1, this is also equal to; rt is nothing but t for, t greater than 0, so it is a 45 degree line and is equal to 0 for, t less than 0 that is the value of the function is 0 or t less than 0, that blue colour, then it ramps up as time passes, so this is; so these 2 things combined is called a ramp function.

So, all the functions we are considering is essentially a causal function, causal signal which exists for t greater than equal to 0 for, t less than 0 the functional value is 0 similarly, for any function which exists from t equal to 0, this is xt, this is the causal function, okay and the causal signals, we can say also like this.

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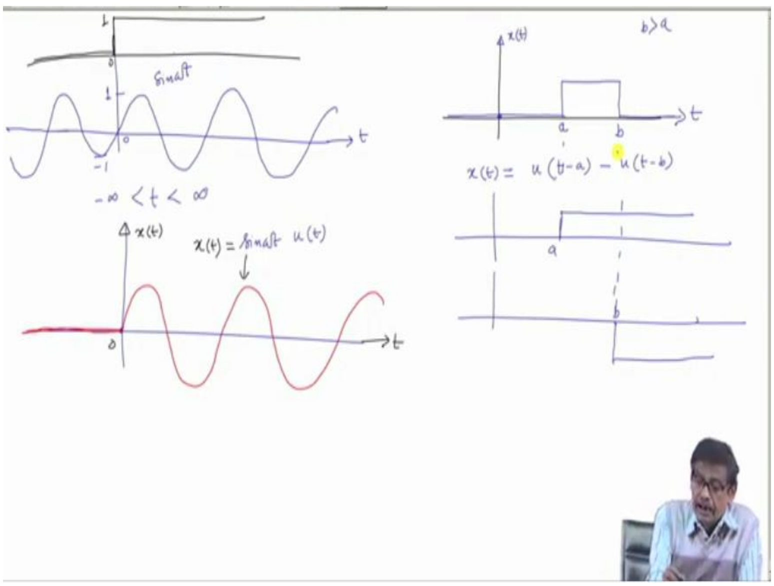

For example, if I ask you to sketch sine omega t; sine omega t, if I simply ask you to sketch, you will sketch like this, is it not and sine omega t, this is 1, this is minus 1 and it oscillates between minus and plus 1 with a regular frequency, angular frequency omega Radian per second, that is fine, this is the time axis, okay but if; but this is what I am telling is not causal.

Because here the function exists for all the time, minus infinity to plus infinity but suppose, somebody ask you to sketch a signal sine omega t, same sine omega t into ut, then he is going to sketch like this, sine omega t ut will be like this and for t less than 0, this will be 0, simply because you are multiplying this function with a unit step function, is it not, the functions; this is 1.

So, the functional value for t less than 0 is 0 here, so 0 into anything you multiply gives raised to 0 however, for, t greater than 0 you are multiplying sine omega t by 1, so it will be sine omega, so this is the function sine omega t ut, this type of signals will be mainly dealing with while solving circuit problems and try to use the Laplace transform. So, this xt is a general function for example, that I have sketched sine omega t ut, clear.

See you can play around with this functions for example, a function like this, a pulse signal; rectangular pulse signal, this is my xt, if you say that how can I express it suppose, this is t equal to a, this is t equal to b, b is of course greater than a, as can be seen, this is this one. Now, this function xt can be shown to be; easily shown to be u t – a, then – u t - b because ut - a is this signal.

So, the value of the function for t less than a is 0 and ut - b is this, ut - b but negative of that will be this one, so if you add these 2, this will cancel out and give you these pulse, this is very easy exercise you can easily see. So, many functions can be represented as a in terms of say, unit step function or say a ramp functions, things like that. For example, I will give you another interesting example.

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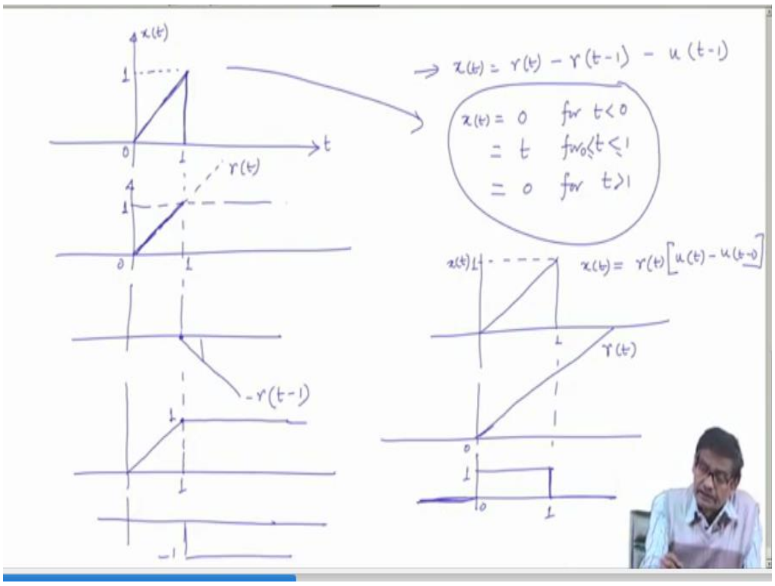

Suppose, I say that a function is like this, a triangular pulse okay, this is time, this is your xt, so this is suppose 0, this is suppose capital T, so I can describe this function and let us put some numbers suppose, this is 1 and the value of the function is also 1 here, how to represent this function in terms of either unit step or ramp functions like that. So, one thing is clear, this function looks like a; apparently, it looks like a ramp function rt okay.

Because at 1, this is 1, we have just indicated that, so up to this point it is fine, ramp function but after that the function drops down to 0 and maintains its 0 value for t greater than 1 for, t less than 0, of course the functional value is 0. Now, the question is; if you simply write xt is this one that will be wrong because xt is equal to rt if I write that is not the case because for t greater than 1, I have to make the value of the function 0.

Now, how can I make this function 0, so what you can do is this, at this point that is t equal to 1, please follow me; you add another ramp here in the negative sense, minus r but it is a negative ramp no doubt but shifted tr minus 1, therefore I will say xt is equal to rt - r - t – 1, I can write it in this fashion but if you add these 2, this plus this, it will now give rise to this function, it went up to the value 1 at t equal to 1, no doubt.

But after that this increment will be offset by this negative increment and it will remain flat, got the point, so xt is equal to rt – rt - 1 will be a function like this, which continues but once again this is not this function that is our original function I want to represent in terms of ramp unit step function like that, so that I have not yet achieved, what else I have to do; so at t equal to 1, I have to add a; so that this portion vanish, I have to add a negative step, that is minus 1.

So, this - ut – 1, this truly represents this pulse therefore, by visually looking at these curves, you can make it happen like that. Another alternative way of writing this; so ramp functions coupled with unit step function, you can manipulate them with your intuition and sketching the diagrams very quickly what it should be, how it can be represented in one line, in general this all these things means that, you are trying to tell xt in our older days, if I did not know the ramp function and unit step function, I would have described this function xt as xt = 0 for, t < 0, xt = t for, t > 0, < 1, > = 0, < 1.

And it is equal to 0 for, t > 1, this is how this whole thing you are writing in 3 lines can be written in 1 stroke in this fashion, that is what I want to mean, understood, therefore this is how one can write similarly, it is easy to see, this rt, another way of looking at it is this is the ramp function, it is certainly the ramp pulse 1, this is suppose 1, this is the original function, same problem I am doing, this is 1, it is a ramp.

You can tell that this is nothing but the multiplication of this; this is the ramp, all the rt and multiply this function with this pulse; rectangular pulse, what is this rectangular pulse; this is 0, t equal to 0, this is 1, so xt also could be written like this; rt into ut – ut - 1 because below this 0, so it is giving this part, during this portion this is unit step, this into 1, this into 1, it continues up to this.

After that rt is present, no doubt but you are multiplying with this function which is 0 here, ut – 1, so anyway there are several nice ways of expressing functions in terms of some, these are called basis; basics functions you can represent some standard waveforms like square pulse, triangular pulse, trapezoidal pulse and so on.

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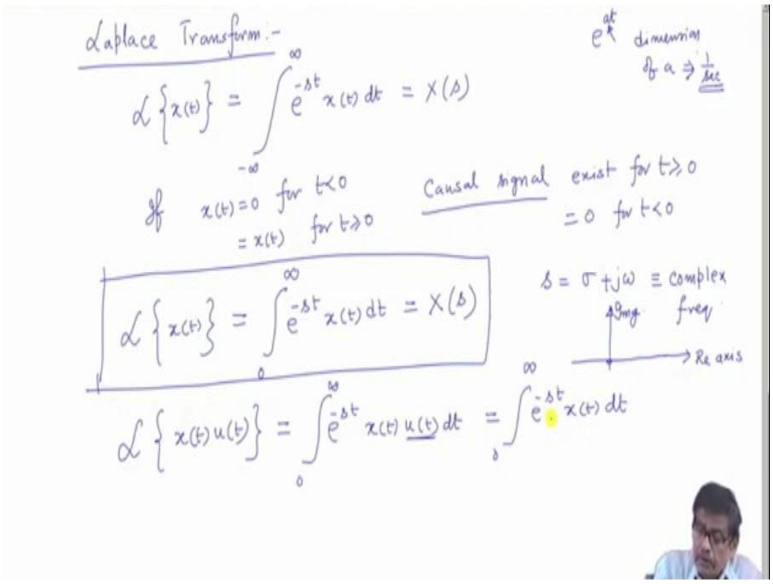

So, we can do exercise on that but anyway, now I will be telling you coming to the Laplace transform now, Laplace transform is generally written; we will not go into deep detail, we will start with the formula of Laplace transform because here only the solution of circuit we will be examining using Laplace transform. Now, what, how it is written as that if you have a time varying function xt, the Laplace transform of xt is defined as minus infinity this integral e to the power minus st into xt into dt, that is all.

And this is called double sided Laplace transform okay, in general this is the definition of Laplace transform and since if you carry out this integration with respect to time, right hand side you will be written with a function which is function of s only, so that is written as xs that is the thing, okay but as I told you if xt is equal to 0 for, t less than 0 and is equal to xt, the function for, t greater than equal to 0, then this integral Laplace transform that is causal signal exists for, t greater than equal to 0.

And equal to 0 for, t less than 0, if that be the case, then the integration; this is the general definition of Laplace transform, then the this integral then can be written as 0 to infinity only e to the power minus st xt dt and we call it xs okay. So, in circuit problem we will be doing like this in other words, what I am telling is we are really finding out Laplace transform of xt into ut.

Because xt may be a general function for all time, I can compute the value of xt but make it a causal function, how can you make it a causal function; multiply xt with ut, then only the value of this whole function xt into ut will be become equal to 0 for, t less than 0 and it will be equal to xt for, t greater than 0 because the ut being 1 and this is written as then 0 to infinity e to the power minus st xt into ut dt.

But this ut now, no point in writing because I am only considering the value of the function for, t greater than 0 so, this is as good as 0 to infinity e to the power minus st into xt dt because ut is always equal to 1 for t; between t equal to 0 to infinity, it is 1 and the same thing. So, this is how Laplace transform is defined, I am not talking about any circuit now, I am just telling okay, given a function you can find out its Laplace transform.

Similarly, if the Laplace transforms which will be function of s only, you can find out its inverse and come back to time domain. Now, the question is what is this s; this s is called a complex frequency, sigma plus j omega, mind you e to the power you write, e to the power anything what is the dimension of this a; dimension of a is 1 by second, is it not, 1 by second, sort of frequency, 1 by second is frequency.

So, this s is called a complex frequency and it is a complex number, so s can be represented in the complex plane, this is the real axis and this is the imaginary axis, some complex frequency plane, okay and we can then find out Laplace transform of some useful functions and make a table of them, so that whenever I wish, I will find out its Laplace transform and we will see later how it simplifies the solution of differential equation problem.

Now, without; now, let us take a simple example, suppose and we will come back to this sigma plus j omega business once again here.

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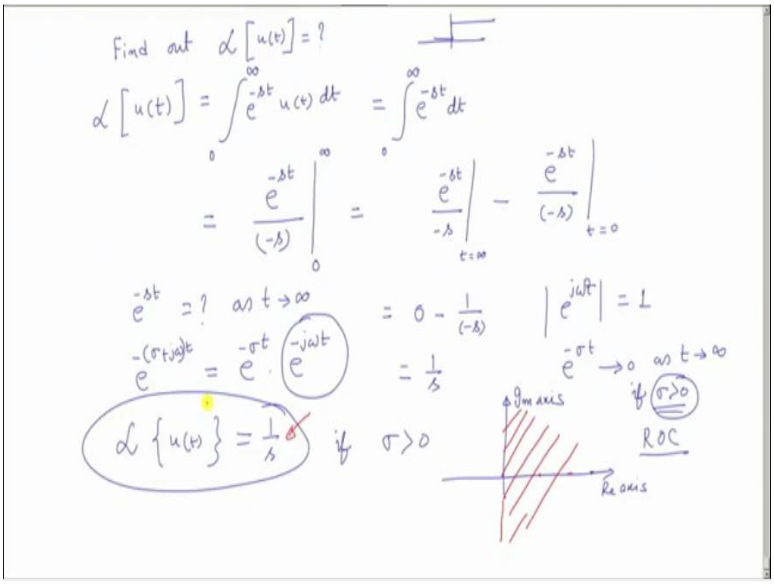

Suppose, I asked you that find out Laplace transform of unit step function, how much it will be that is what I have been asked to find out, then I will say Laplace transform of unit step function is equal to this integral 0 to infinity e to the power minus st into ut into dt, anyway ut is a causal function that we have seen for, t greater than 0 it exists, so it is like this. Now, this integral if you look at it carefully, it then can be written as 0 to infinity e to the power minus st into dt.

Because ut is equal to 1 Acha, now this integration can be carried out, e to the power minus st and divided by minus s, this will be the integration and then I have to carry out these values at t equal to 0 and infinity and take the difference. Now, that means I have to find out the value of e to the power minus st by minus s at t equal to infinity minus e to the power minus st by minus s at t equal to 0, this is what I have to do it, is it not.

So, I have to find out these values now, look at this first term; e to the power minus st, what does it do, what is it as t tends to positive infinity that is what we have been asked to calculate. As I told you e to the power minus st will be minus sigma plus j omega into t, is it not, that is what I told, yes, I have to evaluate the value of this at; as t tends to infinity now, this can be written as e to the power minus sigma t into e to the power of minus j omega t, clear.

Now, this quantity as t tends to infinity, I am sure about one thing, what is that? This quantity is some cosine omega t minus j sine omega t, therefore whatever will be its value, its value will be like sine omega, cosine omega has got finite, pick values between that 2 only it oscillates, is it not, this will be finite values, no matter what is the value of t because see, e to the power j omega t, what is the magnitude of this function?

Magnitude of this is equal to always 1 therefore, no matter what is the value of time, the magnitude of this portion is 1, sigma is a real number and if sigma is positive, then e to the power minus sigma t will tend to 0, as t tends to plus infinity, if sigma is greater than 0, is it not, that is what it means, got the point, so this is the thing, at t equal to 0 of course, the second term is very simple, t equal to 0, put t equal to 0 e to the power 0 that will become 1.

So, it looks like it will be equal to 0, the first term and minus it will be equal to minus s, that is equal to 1 over s, so we say Laplace transform of ut as expected, it will be only function of s is 1 by s that is what the final result is. Now, only one point I will tell you; in this derivations, when I talked about this I saw that this integral will become this 1 by s provided sigma is greater than 0, any value of sigma will do; 5 but it must be greater than 0; 2, 9 whatever it is.

We will see in the analysis if you want to apply Laplace transform, this sigma thing will be in the background, you need not worry about that and that is called the region of convergence; ROC, region of convergence. So, for unit step function for Laplace transform that is this integral will converge to 1 by s, if the real part of this complex frequency is greater than 1 and in the complex plane, we say that this is your real axis, this your imaginary axis.

And I will say the Laplace transform exists for sigma greater than 0, mind you, sigma greater than 0, not sigma equal to 0, just excluding the imaginary axis, any value of sigma will do, so we will always presume that whoever is taking Laplace transform, he has in his mind that this condition is satisfied fortunately, it will be always in the background, you need not worry, you need not find out, you have to specify the sigma values specifically to find out the Laplace transform of a given function.

No matter what is the sigma value which is greater than 0, this result will be unique that is what I want to tell, got the point therefore, this is the Laplace transform of unit step function. Similarly, let us take another Laplace transform of say, Laplace transform of a shifted function, if you have understood me correctly.

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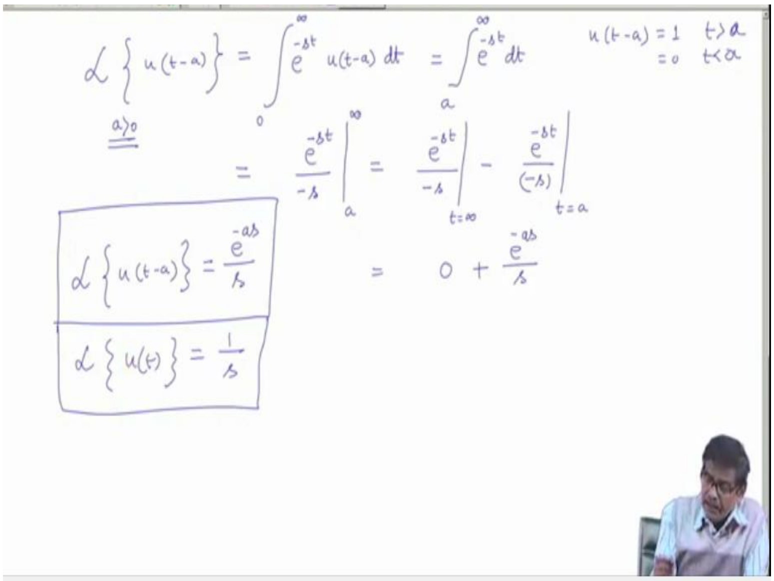

Suppose, I say I want to take Laplace transform of ut minus a, this time this will be equal to as we know this integral is 0 to infinity and suppose, a is greater than 0 so, e to the power minus st into u that the value of the function ut minus a into dt, this is what I have to do, is it not, so this will be this thing. Now, obviously this integration you have shifted the function to ut minus a, which means into xt minus a, whatever it is, it is like this.

So, this function ut minus a, as we know is equal to 1 for, t greater than a and is equal to 0 for, t less than 0; t less than a, I am sorry, this is known so, this integration can be replaced by a to infinity e to the power minus st into dt, is it not, this will be the integration, because ut minus a is 0 for, t less than a, so 0 to a nothing will be there, this function into 0 so, this will be equal to then e to the power minus st divided by minus s integration.

And limit is a to infinity like this, so this will be once again equal to e to the power minus st by minus s evaluated at t equal to infinity minus e to the power same function, minus st divided by minus s evaluated at t equal to a once again, the first integral, this one at t is equal to infinity, this will tend to 0 because s is sigma plus j omega, so it will tend to 0, so this will be 0.

And this will become plus of; here it is simple, e to the power minus as by s, so Laplace transform of ut minus a is equal to; you can see it is e to the power minus as by s, so and above this and we have seen that Laplace transform of ut is equal to 1 over s, so this integral I have to only calculate to find out the Laplace transform of a given function.

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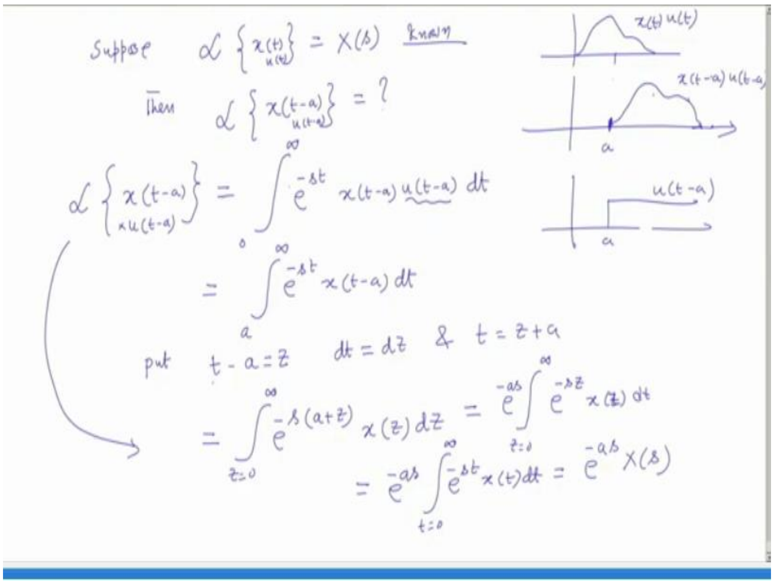

So, in general now next thing what I will do is this, suppose Laplace transform of a given function xt, it will be some function of s, so I write it as xs okay, then Laplace transform of the shifted function is how much, suppose I want to find out so, I will do it like this. So, this is known suppose, this is equal to known, then what will be the Laplace transform of this function, I want to find out.

Then, how can I find it out so, I will say Laplace transform of xt minus a by definition it is like this single sided Laplace transform, please forgive me, xt into ut, Laplace transform of xt into ut is this one, I want to find out the Laplace transform of this function, xt is a function like this suppose, where will be xt minus a? Suppose this is a, the same function, its form will remain same like this but it starts from here, so this is xt into ut that is what it is.

And this is xt into ut, sorry xt minus a into ut minus a, got the point same function, I am sorry here it is slightly looks like different, we have understood the point, so it shifted to right. So, how it can shift to right? So, shift xt; xt minus a, the curve bodily shifts to right by an amount a in time domain and then you multiply it with ut minus a, so that this part vanishes causal signal, so this is the Laplace transform of this into ut minus a, I mean that.

So, what does this mean; it means that e to the power minus st into xt minus a into ut minus a dt, this integral I have to calculate obviously, ut minus a for, t less than a is 0, is it not, this is ut minus a, so for t less than a, this is 0 for, t greater than a only this will become 1, so that this integration then will be a to infinity e to the power minus st into xt minus a into dt, this will be the thing, clear.

Now, to evaluate this integral put t minus a some variable say, z, so that dt will become dz and t is equal to z plus a, so put it here, so this equation will then become equal to e to the power minus s and for t you write a plus z into xt minus a, so xz; t minus az and this is tz and what will be the limit, when t equal to az is equal to 0 and when t equal to infinity, z is equal to infinity, this will be the thing.

So, you can write this integral as z equal to 0 to infinity e to the power minus sz into xz into dz and this e to the power minus s, since I am integrating with respect to z, that can be taken outside this integral, is it not. Now, this one is e to the power minus as, this is z is a dummy variable here it can be changed to once again to t, how does it matter, integration value will not change, xt dt is it not, got the point.

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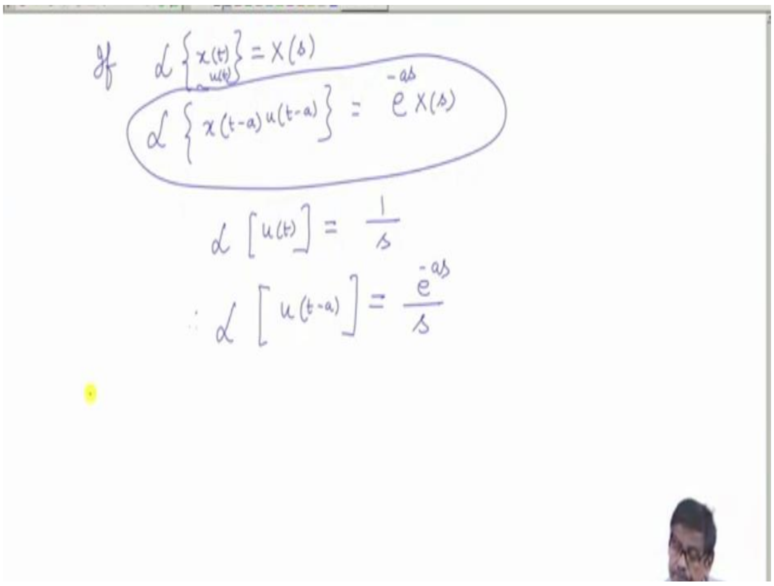

So, this is equal to e to the power minus as into Xs, therefore in general, if Laplace transform of xt is equal to Xs, Laplace transform of the shifted function I mean, xt into ut, I should not forget that too right and shifted version of it will have same Xs but multiplied by e to the power minus as, if a is a positive number it will be positive number and so on, got the point, we will continue with this.

And this is just we have verified earlier, Laplace transform of unit step, I found out it to be is equal to 1 over s, then what I am telling Laplace transform of ut minus a will be how much; e to the power minus as by s that is exactly we found out earlier anyway, we will continue with this and this is very interesting, if you know this rules you will be able to gain much more in solving network analysis in some later lectures, we will come to that, thank you.