Network Analysis
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Lecture – 44
Laplace Transform - II

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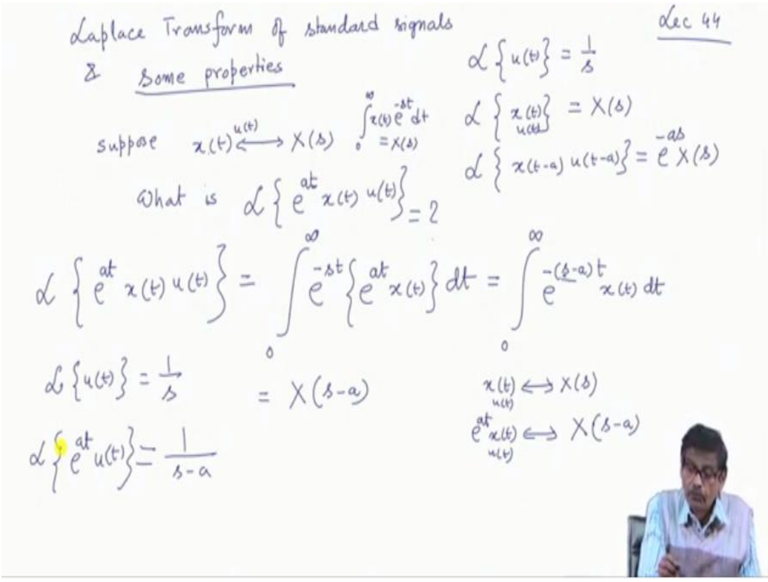

So, we were discussing about Laplace transform, very interesting topic and useful practical applications, Laplace transform of some standard signals and some properties, for example we found out last time Laplace transform of a unit step function is equal to 1 over s and the Laplace transform of e to the power; no, not e to the power, in general Laplace transform of xt, I will be noted by capital Xs.

And the Laplace transform if you multiply this function by; if you time shift the function, xt into ut, mind you because one sided Laplace transform we are doing xt into; ut into ut is ut that is why, so xt into ut is access, then xt minus a into ut minus a shifted version, this will become same access without a but multiplied by e to the power minus s that is what we have seen, okay.

Now, there are next thing is; if a function is multiplied by e to the power at, suppose xt has a Laplace transform Xs, this time I will write it like, this is an unique pair, xt in time domain it has got a Laplace transform Xs, if Xs is known I will be able to find out xt, inverse way.

Inverse way I have not talked about till now but that will be done from the table as we will proceed further.

So, xt is Xs, then I say that what is Laplace transform of e to the power xt into ut, Laplace transform of this is this, then what will be the Laplace transform of e to the power at xt ut, that is what I want to find out, got the point. So, we start from fundamental so, I say that Laplace transform of e to the power at xt into ut, this multiplication of ut allows you to use the limit of integration from 0 to infinity, you must have understood by this time.

So, it is equal to e to the power minus st into this function e to the power at, this is the function I am writing, e to the power at xt and this ut I am not writing because between 0 to infinity, ut is 1, it is 1 and the same thing into dt, this is what I have to do, is it not. So, this integration is equal to 0 to infinity e to the power minus s minus a into t into xt into dt, is it not, that is what you will get.

Now, obviously this I have got the result 0 to infinity is e to the power minus s minus at xt dt, now but I know that Laplace transform of xt is Xs, which means that 0 to infinity xt e to the power minus st into dt, this integral is Xs, is it not, so here instead of as, it is s minus a, so this must be written as the same Xs but it should be replaced by s - a that is all, there is the matter, e to the power, if it were minus pt, I would have written X of p, I think you have got the point.

So, essentially what I mean to say, if Laplace transform of xt is Xs, then if you multiply this function with e to the power at xt and all our causal signal multiplied by ut into ut, even if I forget, I mean that so, e to the power at, xt, ut will be nothing but X of s minus a, whatever was a, you multiply with replace s by s minus a for example, Laplace transform of ut, I have already calculated it is equal to 1 over s.

Then, I am saying that Laplace transform of e to the power at ut will be nothing but replace s by s minus a, so if you multiply function with exponential e to the power at, its Laplace transform will be e to the power at into ut. So, practice it and you can find out anything okay, so this is the thing. Now, some property; so we have seen for a given function if Laplace transform is known, if it is time shifted how to calculate the Laplace transform.

If there is a given function; causal function, whose Laplace transform is known, if it is multiplied with e to the power at, then what will be its Laplace transform that also we have calculated, we have seen that. Now, what will be the Laplace; now these are now topics which will be directly applied to the solution of differential equation.

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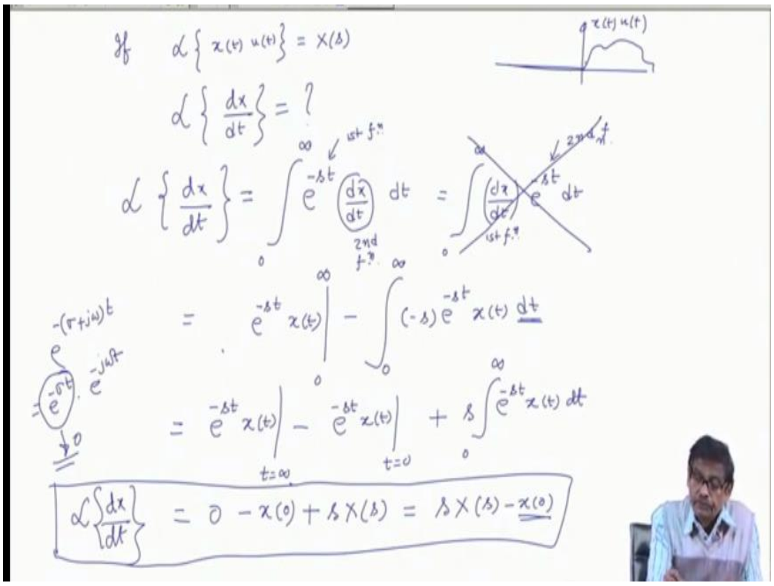

The idea is; if suppose I say, if Laplace transform of xt obviously, ut is Xs, I am repeating; xt, ut you understand, it means that the value of this whole function xt ut is equal to 0 for, t less than 0 and it is equal to xt for, t greater than 0 that is clearly understood. So, if this is the thing then what will be Laplace transform of dx dt, that is the question. Now, as you can see the Laplace transform of dx dt of this function, so here is the function; xt into ut if you sketch it, it will have it say only for, t greater than 0 for, t less than 0 it is 0.

Therefore, dx dt for t less than 0 is 0, a constant 0 value, its differentiation is 0, so the dx dt also does not exist for t less than 0 therefore, when you differentiate a function x tut, it also become a causal signal like here therefore, I can start from this thing that Laplace transform of say, dx dt into ut; ut is ensured here because xt ut that is what I told, so dx dt will be 0 for t less than 0, it will be equal to this integration you have to calculate, 0 to infinity e to the power minus st into the function that is dx dt into dt, is it not.

Laplace transform of dx dt, I want to find out so, this will be the thing what is known; Laplace transform of xt is known now, this I will write it as integration of 0 to infinity, first I will write this function dx dt into e to the power minus st dt. Now, this I will integrate by parts, this is the first part, first function and this is the second function okay, this correct, I will take and let me see if something comes out wrong, we will see.

So, this is the first function if you take, so first function into integration of the second, then differentiation of the first function, then d2x dt2 will come. So, what I will do is; I will treat this you cancel out, I leave it here and this I treat as first function and this as second function, so this one will be first function into integration of the second, what will be the integration of this second function?

It will be xt, first function into integration of this second and then I have to put the limits between 0 to infinity minus integration of differentiation of the first function that is minus s into e to the power minus st into integration of the second that will be xt and dt, this will be the thing and this is s equal to 0 to infinity. Now, here comes this first integral once again, this is equal to e to the power minus st into xt to be evaluated at t equal to infinity minus e to the power minus st into xt to be evaluated at t equal to 0.

This will; these 2 terms and this will become s, this s can be taken outside because you are integrating with respect to time, so this will become 0 to infinity e to the power minus st xt into dt, this will be the thing. Now, once again if I want to evaluate this function for t equal to infinity and once again, as I told you this s means minus sigma plus j omega into t e to the power, this you have to evaluate at t equal to infinity.

What will be the fate of this term? Fate of this term will be so far it is equal to e to the power minus; some time before I told this argument once again, I am repeating, j omega t, as t tends to infinity, no matter what it is; what will be the magnitude of the e to the power j omega t, even if t is very large, can it be infinite? No, because it is cosine omega, its amplitude is 1 only, bounded by 1.

But as t tends to infinity and sigma is correctly chosen as ROC, then this fellow that is sigma is a positive number, this fellow will tend to 0, so that is why this will become zero, this first term but this term, put t equal to 0 straight away e to the power 01 and this is x0 and so far as this term is concerned, it will be s into Xs, is it not. So, Laplace transform of dx dt is nothing but s into Xs minus x0, this is the thing, ultimate final result, got the point.

Therefore, you see the as if it looks like this ddt term is replaced by s with the Laplace transform Xs and it becomes; differentiation becomes an algebraic multiplication of s into Xs, this is the first thing to note because after all in circuit analysis with energy storing elements we have to face differential equations and we know from our previous studies how it can be done by classical way that is different story.

Now, here is something very interesting, differentiation of a given function, if you take its Laplace transform it gives rise to some sort of an algebraic equation. What is x0; x0 is the initial value of the function at t equal to 0, so one can extend this idea.

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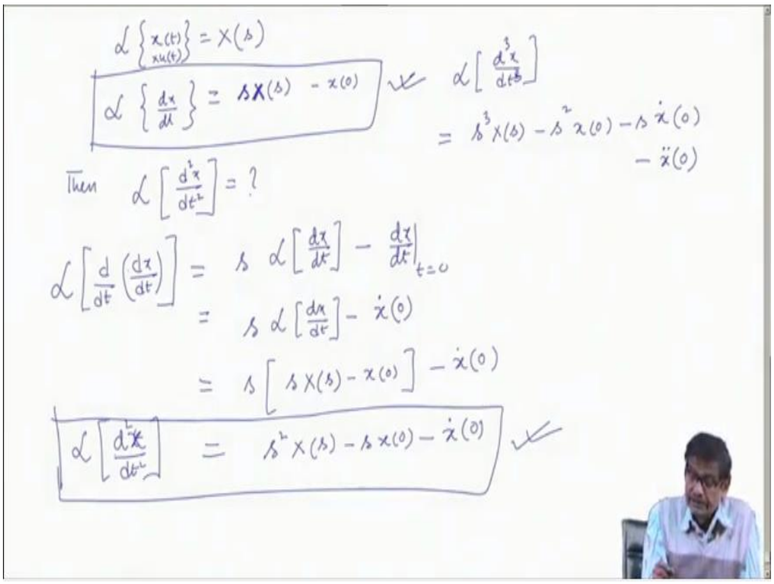

So, we have seen that Laplace transform of xt is equal to Xs with this we started, then I am saying Laplace transform extend to ut, always I forget is excess, then Laplace transform of dx dt, we have just established, it is equal to s into Xs okay, s into Xs, this Xs minus x0, this is the value of the function at t equal to 0. So, as if this ddt; ddt has been replaced by s like in sinusoidal phasor notation, ddt was replaced by j omega.

Laplace transform is a more general frequency, not only j omega but e to the power sigma plus j omega, one can find out such a correlation I mean, think in terms of that correlation but this is the story now, therefore Laplace transform let me not spoil it, so we have got this one dx dt, this is one interesting result. What about higher order derivatives? Then, Laplace transform of d2x dt2, suppose second order differential term, it will be how much?

Answer is very simple now, I have established this, no one can stop me for example, I will say okay, look here Laplace transform of d2x dt2, you can write it as this, is not it, Laplace transform second derivative, I can write it like this, so here this is the function whose but Laplace transform of dx dt is known, this to be this, so what it will be; it will be s into Laplace transform of dx dt and minus of functional value of dx dt at t equal to 0, is it not, this will be the thing, by this argument.

So, here x is taken care of by dx dt, its Laplace transform is known yes, I have right now derived; sX; s into sX minus x0, so ddt of this function will be Laplace transform of dx dt, this one minus the initial value of dx dt, that is this, so this will be equal to in short, I can write Laplace transform of dx dt minus this is in short I will write it at x dot at t equal to 0; x dot means dx dt.

Then, s into the Laplace transform of dx dt will be how much; s into Xs minus x0 and then minus this x dot 0 and this if you open this bracket, it will be a square into Xs minus s into x0 minus x dot 0, so this is the thing you see Laplace transform I will say, then d2t; d2x dt2 is equal to this one, so one can go on extending this to find out the Laplace transform of differentials of any order.

For example, now after this I will say, I will not derive but I will just I can write down, can you write down this d2x dt2; yes, I can, it will be s cube Xs - s squared x0 - s sorry, x dot x0; sx0 – x x dot 0 and minus x double dot 0, this will be the order in this, s square into Xs minus; because in you know, in a third order differential equations, 3 boundary conditions will be necessary, x0, x dot 0, x double dot 0.

Second order differential equation; 2 boundary conditions will be necessary, x0, x dot 0, so anyway so, this is one of the very important finding so far as Laplace transform is concerned that the differential terms okay, fine, acha. Now, another interesting thing is; so then the next another interesting property I will tell that is naturally comes as the integration, although will not be using integration.

Because I told you whenever, I will be solving circuit problem better write it in terms of differential equations, no integral differential equations.

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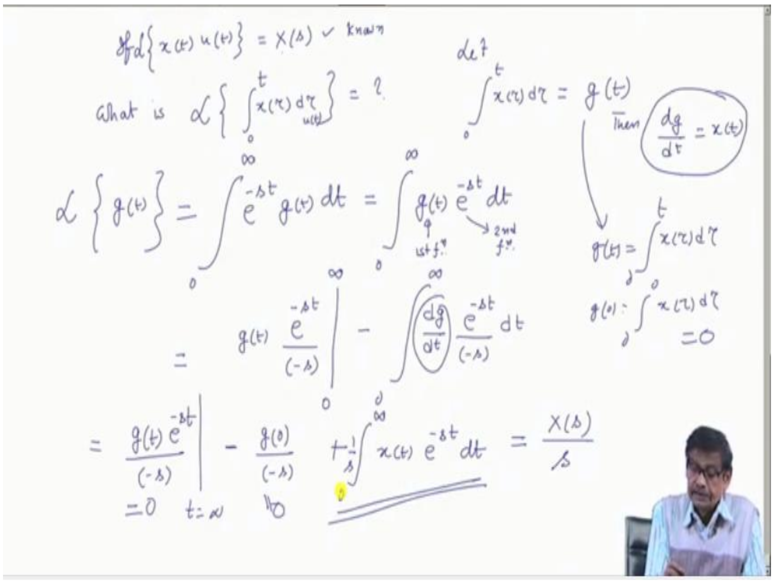

But nonetheless, it is worth noting what happens that is the second thing is; if xt Laplace transform of xt into ut is equal to Xs, then the question is what is the Laplace transform of integration of x tau d tau, 0 to tau; 0 to t, mind you this way why I am writing because integration we want to take the function, take the integration but ultimately, it has to be a function of time, so this integration will confirm that.

When you carry out this integration, you will get the area under the curve up to time t, so what is the Laplace transform of this one, I want to find out, clear. The answer is to find out this and this is known, first step what I do is this; let this integration 0 to t, this function whose Laplace transform is necessary is this one; x tau d tau and obviously, it will be a function of time this t and let this function be gt, let this be this.

Then, what I am trying to calculate is Laplace transform of gt, I am trying to finding out, where gt is this obviously, gt is this, then dg dt must be equal to xt, integration of this function is gt, so if you differentiate this you will get xt, so dg dt is xt, we note it down there. Then, a gt and of course, this is multiplied by ut that stuff remains, which allows me to integrate it from 0 to infinity.

And this is equal to e to the power minus st into gt into dt, is it not, that will be the thing, e to the power minus st dt now, here what I will do is this; this I will write it as, I will treat this gt as first function and this as the second function and integrate it by parts, so this will be equal to first function into so, this is the first function, treat it and this you treat it as second function and integrate by parts.

So, first function into integration of the second that is e to the power minus st by minus s, this will be this one and this you have to evaluate at t equal to 0 and infinity, you take that difference, minus integration of 0 to infinity, differentiation of the first function that is dg dt into once again integration of the second and that is into dt, got the point. Now this, so this is once again is this one, gt into e to the power minus, which I will not further elaborate now, minus s, this I have to evaluate at t equal to infinity.

And the result I know it will be 0, why; gt is a reasonable good function nothing like that, so as t tends to infinity that minus sigma t will vanish, no matter what is the value of functional value of gt at t equal to infinity, this will make it 0, so that will be 0, in the same way as I have explained it, minus this I have to evaluate at t equal to 0, that is very easy, 0 divided by minus s, is it not, put t equal to 0 and get this.

And minus of integration of 0 to infinity dt dt is nothing but xt, so xt this one is xt, so write xt e to the power minus st and this minus 1 over s will come outside make it plus and this s because you are integrating it with respect to time not with respect to s, so this s term can be taken outside. Now, the question is gt is this, so what is the value of g0? So, since gt is equal to; since gt I am rewriting, 0 to t x tau d tau is what; area under the curve.

And g0 is 0 to 0 that is no area will be covered so, the g0 is any function if you integrate with same limit what it is; 0 so, you will be left with, so this term is 0 because e to the power minus sigma t goes to 0, as t tends to plus infinity and this term g0 is 0, so this is also 0, it goes and you will be left with only with this term and this by definition, xt into e to the power minus st dt is nothing but Xs.

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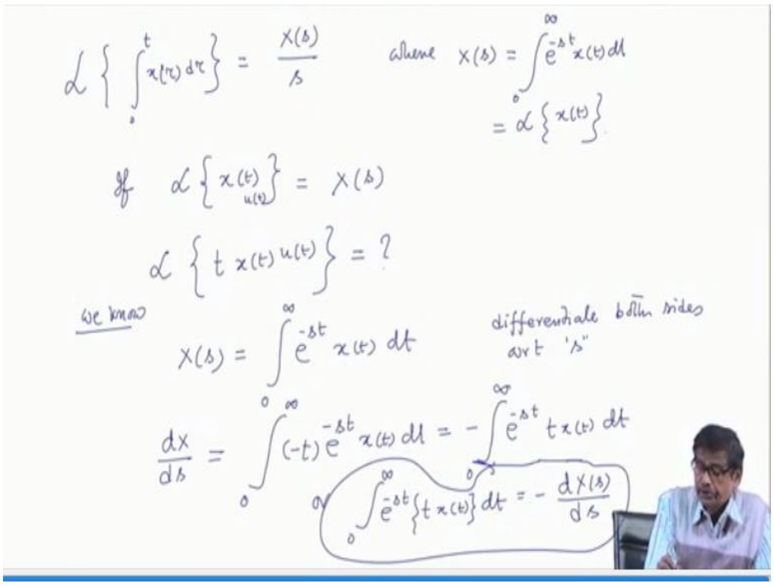

So, Xs by s therefore, we say that finally, Laplace transform of 0 to t x tau d dau tau will be nothing but Xs by s, where Xs is nothing but the Laplace transform of xt, got the point or where Xs is equal to Laplace transform of xt, so this is the one thing, another small result I will just tell, so these are some of the properties of Laplace transform and how it can be used. One last property I will just show you if I say, if these one line proof, if Laplace transform of a signal xt, ut is equal to Xs.

What will be the Laplace transform of this function are XT ut, you recall t and t, ut is rt that ramp function, so if you multiply a function with a ramp function, what will be its Laplace stuff so, we want to find out. Now, to do this the proof is very simple and interesting way it can be done, we know that Xs we know that Xs is equal to 0 to infinity e to the power minus st into xt into dt, Laplace transform of xt into ut is this, we know that.

Now, you see what you do; you differentiate both sides with respect to s, s so you will get dx ds here, on the right hand side integration means summation of small, small numbers infinite whatever number, so and those terms has only e to the power function of s is this term, so differentiation of this term only so, it will be minus s into e to the power minus st, xt will be treated as constant, you are differentiating it with respect to s.

So, I can write it like that so, dx ds, oh I am so sorry, so if I differentiate it with respect to s, so it will be t, is it not, t should be treated as constant minus t, is it not, so this will be equal to minus, this integration sign can be taken outside and it is e to the power minus st into t into xt into dt or I can say if you bring the minus side to the left, I can say e to the power minus st into t into xt dt integrated from 0 to infinity is equal to minus of dXs ds.

Therefore, this is also interesting to note that if you know the Laplace transform of a function X in terms of s, then the Laplace transform of the same function when multiplied by t will be simple differentiation of that function preceded by minus sign, I leave it to you to find out; only indicate that, so that you can do.

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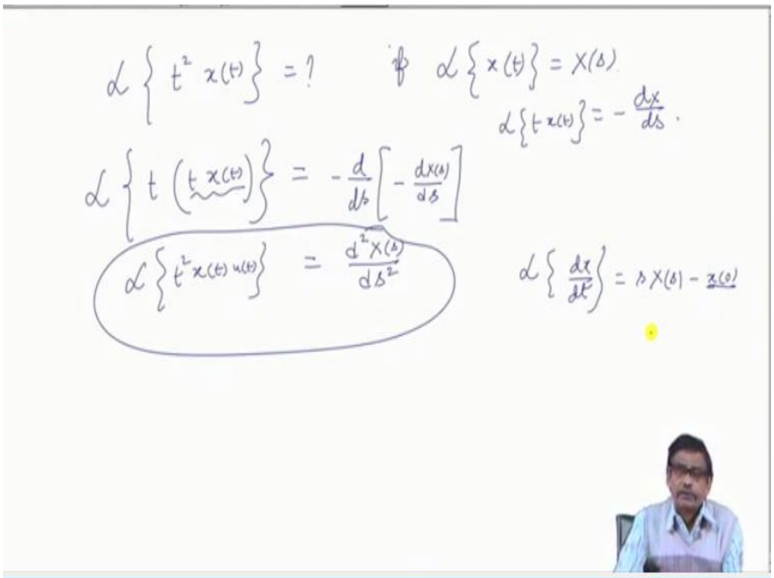

Laplace transform of suppose, I say t square xt, if I multiplied it by twice, I will not find out a general formula, t to the power nXn, okay that can be done but I am telling you the idea, this is the thing is how much; if Laplace transform of xt is equal to Xs, suppose this is known then what should be this? Then, I will go like the previous differentiation formula for higher orders; I will write it like this.

Laplace transform you are asking me for t and this function t into xt, we group this terms is how much I have to find out; this will be equal to, we know from the previous result; what was the previous result? Previous result was if this is this, then Laplace transform of t xt is nothing but minus dx by ds that was the previous result, so here it should be minus, so Laplace transform of t xt is known, which is equal to minus dx ds.

Therefore, it should be minus dds of the Laplace transform of this quantity, this quantity Laplace transform is minus dx ds, so put it here and which will come out to be d2 Xs ds2, so Laplace transform of t square into xt ut of course is there is this one, anyway I stopped here today and next class will be very interesting, we will be applying this whatever we have learned to solve network problems.

And there we will see Laplace transform method if we adopt, it does several things, it will essentially convert the differential equations into algebraic equations and also the initial conditions will be automatically inherent in the Laplace transform expression for example, Laplace transform of dx dt, as we have seen is nothing but s into Xs minus x0, what is this; this is small x, no capital X, this is the functional value at t equal to 0, more on this in the next class. Thank you hopefully, you are following me, hope you are following me, thank you.