Network Analysis
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Lecture - 50
General Second Order Circuit Analysis with L.T - II

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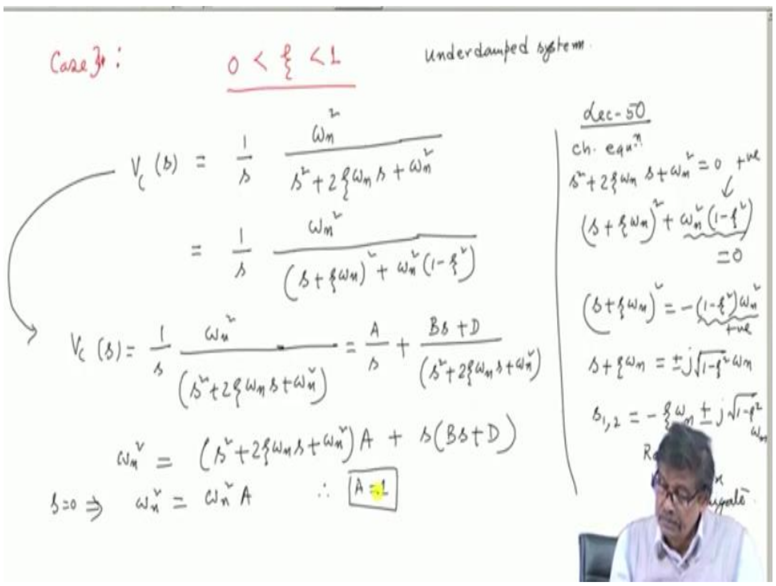

So let us continue with this under damped system of calculation, so what do we do? Same steps Vcs is equal to 1 over s and the system thing is this one omega n square plus s square plus 2 zeta omega ns plus omega n square, that is the thing, is not. If zeta is a number less than 1, but greater than 0, then the roots of this equation can be easily shown to be complex conjugate if you want to, because what you can do is this. Try to understand these steps.

That is the characteristic equation. Let us do it side by that s square plus 2 zeta omega ns plus omega n square equal to 0. This is nothing but s plus zeta omega n whole square plus omega n square into 1 minus beta square. That is you have added omega n square zeta square and subtracted and that is equal to 0. Mind you, this is always a number greater than 1. It is a positive number omega n square into 1 minus z square, it is positive.

Therefore s plus zeta omega n square is minus of 1 minus zeta square into omega n square. So if I want to find out the roots, what I will do is this. I will take square root of both the sides. If you take square root, square root of -1, this is positive, mind you positive number, because zeta is less than 1 so 1 minus zeta square is always positive. So this will be equal to plus minus j root over 1 minus zeta square into omega n.

So s12 will be equal to minus zeta omega n plus minus j root over 1 minus zeta square into omega n. Roots will be complex conjugate. So system roots will be complex conjugate. Anyway this is behind the scene. You try to understand. So this one therefore, I will write it as 1 over s omega n square and this one in this form only s plus zeta omega n whole square plus omega n square into 1 minus zeta square. The denominator, I can write in this fashion okay.

Now this can be written, but what I will do? I will come here that Vcs is equal to 1 over s omega n square divided by s square plus 2 zeta omega ns plus omega n square and let us do partial fraction expansion. So I will assume that this is equal to A by s plus Bs plus D, because square term is there. This is this and this is s square plus 2 zeta omega ns plus omega n square and we will apply this. I will multiply it as an identity and try to find out the values of A, B and D.

So I will write omega n square is equal to multiply this. It will become s square plus 2 zeta omega ns plus omega n square into A, s s cancels out and these things will give you plus s into Bs plus t, is not. This will be the thing. So the value of A can be easily calculated by putting s equal to 0. s equal to 0 will give you omega n square as this, this 0 omega n square into A. This term of course will be 0, s equal to 0 I am putting. Therefore A is equal to 1. A 1 we have got.

Now I have to get the values of B and D. So what we will be doing the same equation, I have to substitute that is this is the equation. So I will go to next page, copy it and go to next page and paste it.

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$$\omega_n^2 = (s^2 + 2\zeta\omega_n s + \omega_n^2)A + s(Bs + D) \qquad \boxed{A = 1}$$

$s = 1$

$$\cancel{\omega_n^2} = 1 + 2\zeta\omega_n + \cancel{\omega_n^2} + B + D$$

$$\text{or} \quad B + D = -1 - 2\zeta\omega_n$$

$s = -1$

$$\cancel{\omega_n^2} = 1 - 2\zeta\omega_n + \cancel{\omega_n^2} - 1(-B + D)$$

$$\therefore D - B = 1 - 2\zeta\omega_n$$

$$2D = -4\zeta\omega_n \Rightarrow \boxed{D = -2\zeta\omega_n}$$

$$2B = -2 \Rightarrow \boxed{B = -1}$$

$$\therefore V_c(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$V_c(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n)}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

So this is the equation. So we have already got A is equal to 1. We have already got this result. We have got A is equal to 1. Therefore, I will now put s equal to say plus 1. If you put s equal to plus 1, it will be omega n square is equal to 1 plus 2 zeta omega n into 1 plus omega n square into 1 whole thing into 1 plus B plus t. That is what I will get or I will get B plus D two equations I have two form involving B and D.

It will be if you bring it to that side B plus D omega n square goes, so it will be minus 1 minus 2 zeta omega n. This is one equation. Then put s equal to minus 1. If you put s equal to minus 1, it will be omega n square is equal to, this is 1, then this is minus 2 zeta omega n into s equal to minus 1 I have put. So this plus has become minus and then plus omega n square and s equal to minus 1 means, it is minus 1 outside into minus B plus D, this is the thing.

So once again this goes and arranging the terms you will get. Therefore, D minus B will be equal to 1 minus 2 zeta omega n. You bring this to this side, leave it here on this side, so this is the thing. So these two equations you have to solve. So add them and you get the value of D. So 2d by adding these two equations, you will be getting minus 4 zeta omega n, which implies that D is equal to minus 2 zeta omega n. So D will be minus 2 zeta omega n.

Similarly, if you subtract these two, you will get 2B, this minus this if you do, 2B it will become minus 1 minus 1 will be equal to minus 2 and these two will cancel each other, so which implies that B is equal to minus 1. It will be like this. So I now have all the values A, D and B. Therefore, voltage across the plate of the capacitor Vcs, which was equal to what A by s Bs plus D, etcetera.

So A by s, A was 1 A by s plus Bs plus D that is minus s plus D means minus 2 zeta omega n into s. As + Bs some mistake here As plus just one second let me satisfy myself. So this is A by s, then Bs plus D. So Bs, B is what - 1 plus Bs plus D minus 2 zeta omega n. This will be the thing and this below is s square plus 2 zeta omega ns plus omega n square. This is the thing, clear and what you can do this I will not rewrite. This minus I will put it before and both these terms become plus.

Next step, it will be like this s plus 2 zeta omega n plus omega n square etc. Now this is nothing but 1 by s plus, not plus minus, this I will write it as s plus zeta omega n. Now you see what I am doing plus zeta omega n and divided by this one is s plus zeta omega n whole square plus omega n square into 1 minus zeta square. This is the thing or Vcs will be equal to 1 over s minus, you now separate these two things.

So it will be equal to s plus zeta omega n divided by s plus zeta omega n squared plus omega n squared into 1 minus zeta square. This will be the first term, this by this then minus you have got zeta omega n by the same thing, s plus zeta omega n whole square plus omega n squared into 1 minus zeta square. So looking at the denominator it looks like sine and cosine terms will come and there will be also an exponential minus zeta omega nt term.

Now so far as the first term is concerned, it is already nicely placed. Laplace inverse of this term will be e to the power minus zeta omega nt into cosine of this number, I mean square, but here so that will give you cosine term looks like and this will give you sine term provided this term also comes here. So I will rewrite this in the next expression like this, copy, go to next page and paste it.

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$$V_c(s) = \frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} - \frac{\zeta\omega_n}{\sqrt{1-\zeta^2}\,\omega_n}\,\frac{\sqrt{1-\zeta^2}\,\omega_n}{(s+\zeta\omega_n)^2+\left(\sqrt{1-\zeta^2}\,\omega_n\right)^2}$$

$$= u(t) - e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n\right)t - \frac{\zeta}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n\right)t$$

$\sqrt{1-\zeta^2}\,\omega_n = \omega_d$ = freq of damped oscillation

$$= u(t) - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\sin\omega_d t$$

So this is the thing. We have got this now. What I will do is better I will write it, then this is equal to 1 over s fine plus, not plus minus this is s plus zeta omega n divided by s plus zeta omega n square plus this one is omega n. I will write it in this way, root over 1 minus zeta square omega n whole square. This way I will write this term and then this one is minus zeta omega n and here it is also like that s plus zeta omega n whole square plus root over 1 minus zeta square omega n whole square.

So far as this term is concerned, Laplace inverse if you take, this is ut straight away and this term is already fixed now. It will be e to the power minus zeta omega nt, cos or sine, cos because s is also on the top, cos of omega t. This is the thing, root over 1 minus zeta square into omega n into t. This is that omega terms and here it would have been sin, provided this term exists on the top, is it not? But there is no such term, so you bring it here and divide it.

Then, this will be minus this Omega n goes, it will be zeta by root over 1 minus zeta square. This will be this into sine of 1 minus zeta square into omega n into t. This is the frequency in which it will happen. What happens people say, that let this one root over 1 minus zeta square into omega n is called the damped frequency. Mind you zeta is a dimensionless quantity okay. So omega t is called the frequency of damped oscillation.

So the solution can then be written as e to the power t, e to the power minus zeta omega nt cosine omega dt minus zeta by root over 1 minus zeta squared e to the power minus zeta omega nt is also there here, is not. I forgot this was there. So e to the power minus zeta omega nt sine omega dt damped frequency of oscillation. This will be the thing clear. Now what you do is this, you can take root over 1 minus zeta square common. So this expression can be written as this. This one copy and we go to next page and paste it.

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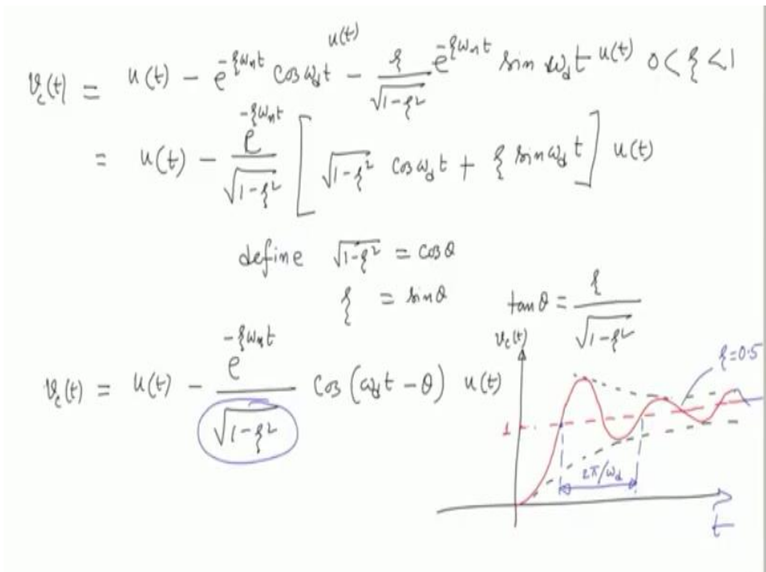

So this is the thing we have got and mind you, this is equal to Vct, capacitor voltage in time domain. Problem solution is over, but still we will further simplify it. Then this what do you do? Minus 1 over root over 1 minus zeta square, this thing you take common. Then you will be left with 1 minus zeta square and also take this outside zeta omega nt, which exist in both the terms, take that out.

So this will be cosine omega dt minus this will be also plus and zeta sine omega d into t, this will be the thing. Define, mind you root over 1 minus zeta is the number less than 1 in this case. What is the range of zeta, neither 0, greater than 0 and less than 1, but positive number. So root over 1 minus zeta square will be a fraction and define root over 1 minus zeta square to be equal to some cos theta. Let us define that. Then zeta sine theta has to be.

Because this square plus this square is one matching, so in that case I will write it as ut minus e to the power minus zeta omega nt over root over 1 minus zeta square into cos A cos B plus sin a sin B is minus theta, got the point. What is tan theta? Where tan theta is theta my root over 1 minus zeta square. So this will be the capacitor voltage Vct, finally. See omega d is called the damped frequency of oscillations and mind you, all things are multiplied by ut.

Because we are dealing with signals, applying signals for t greater than equal to 0, one sided Laplace transform. So this thing is also ut multiplied by, I think that does not create any confusion. So these are all multiplied by this into ut, this into ut individually, you get that. Now in this case, if you sketch this capacitor voltage output Vct. this is suppose 1, this is the level 1, unit step. DC voltage is 1 we will put.

Then the solution will be, if you sketch it, it will be like this, because the peak value of this cosine card will gradually decay, because of the e to the power minus zeta omega nt term. Mind you, zeta omega n is a positive number, so exponentially decaying. So this term, so these peak values will be modulated by this term. This dotted line I have just telling the effect of e to the power minus zeta omega nt that is coming into being.

Now in this case, so it will be oscillating. Capacitor voltage will first go up, there will be an overshoot, then undershoot, but the first undershoot will be less than the amplitude of the first overshoot and so on, but mind you this thing, this one, this interval like points, this is equal to 2 pi by omega d, where omega d is called the damped frequency of oscillations. I think you have got. It is for a certain value of zeta. Suppose I do not know, zeta maybe.

Suppose let us arbitrarily say, this is equal to zeta equal to 0.5. It is like this. Then if you reduce zeta further, suppose 0.2 you make it, what you will find? There will be many more oscillations and overshoot will be much more, because of this term 1 by root over 1 minus zeta square. So if you make zeta further less 0.2, is not, so amplitude under damped case oscillations will further grow and finally when zeta is made 0, it will be sustained oscillations.

And if zeta is equal to 1 it will be exponentially attaining this value, no oscillations, no overshoot understood and it will continue like this. So depending upon the relative values of R, L, C of a second order system, system capacitor voltage may sustain the oscillation forever, provided there is no resistance present. Similarly, it can also have oscillations, but with reduced amplitude as time process.

Mind you, this is timed axis and finally it will settle down to one capacitor voltage or it may be critically damped. That is exponentially, it will go neither overshoot, no undershoot like that. Generally systems are made slightly under damped. Zeta is equal to say 0.8 or things like that okay. So this is the classical example or application of Laplace transform to find out the response of an RLC circuit, which is excited by sinusoid.

See, it does not mean that if suppose I say same RLC circuit, I excite it with a sine wave, I will only just indicate this. You should not think that always, I should excite.

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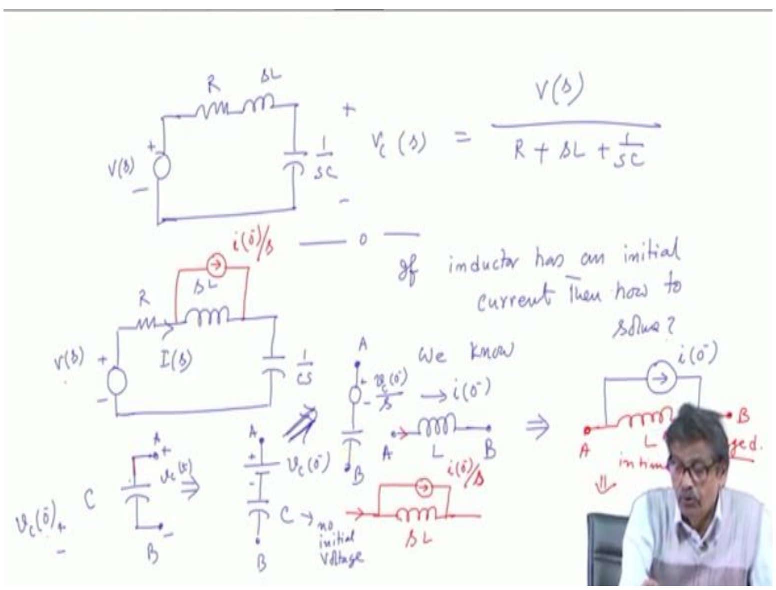

Because I already know that an RLC circuit could be excited sL by 1 over sC by any voltage you like. It need not be sC. Your Laplace transform domain analysis, suppose it is Vt. Vt could be anything sine omega t sort of thing, then also the voltage across the plate of the capacitor Vcs will be only thing, then you do not write 1 by s, but write the Laplace transform of that excitation voltage divided by same thing you do, maybe steps will be more and so on.

Not that very innocent term 1 by s will always appear here, but anyway that can be expanded in partial fraction in proper for, you try to bring those partial fraction terms. So that you can immediately see what should be the inverse and the last thing, I will tell in this lecture is about okay, what happens this sort of analysis you can always do, provided the initial conditions are 0. I told you. Now I will tell you what is going to happen.

Suppose I say, I have an RLC circuit in s domain. I need not draw it further okay. Let me draw it RL sL, s domain I will go 1 by sC, some voltage Vt Vs, I should write in s domain Vs, no Vt. This is the thing and this is is. Laplace transform gives you both the steady-state and transient part of this solution that is fine, but if I now say that this inductor, when you applied this Vs at t equal to 0 onwards I am applying, if inductor as an initial current, then what?

Then how to solve, how to solve means answer is known okay then write down the differential equation, take the Laplace transform. Initial condition terms will be taken care of, you know, Laplace transform of dx dt is s into xs minus x0 and so on, but I want to exploit this circuit property okay. I will not write down the differential equation go to s domain and take advantage of this particular method, what should I do?

Now you know that we know an inductor, we discussed at length in my initial lectures. If you have an inductor, we can initial current i0 minus across the point, terminal points of Ab it is equivalent to what? It is equivalent to an inductor, which is uncharged, but across it consider a current source is connected. In time domain, I am telling. So this inductor is uncharged and these are the points A and B.

So you can replace an inductor which is having some initial current in time domain. This is in time domain and it can be represented as an inductor without any initial current in parallel with a current source. Direction of that current source is what? The direction in which initial current was flowing. Magnitude of that is this. Therefore in a s domain, it will be this inductor is not having initial current. So I have every right to replace it by sL and this is a constant current.

Its Laplace transform will be i0 minus by s, got the point. Suppose in this example, if this inductor has got some initial current, you draw the circuit sL like this, but then put another parallel branch here with a current source i0 minus by s like this. Of course then, there are two sources coming in. That is different issue. I am not writing differential equation okay. This inductor has got an initial current.

So this inductor, I will presume which is written sL as having no initial current and mind you, the current flowing through the inductor is not the current in this uncharged inductor. If you are asked to calculate this current, you have to calculate this current plus this current will be the current here in this actual inductor understood this point, so i0 minus s. Similarly we know from our earlier discussion, which we did at the initial phase, that if a capacitor has got a capacitor which has got some Vc0 minus.

Then in time domain, how to represent this capacitor? This capacitor will be represented by, suppose I want to find out this is the voltage Vct across the capacitor, then between points A and B, I will say look here, there is this capacitor, which is uncharged, but in series with a battery fixed voltage, whose value is this Vc0 minus and what is the polarity of this voltage, depending upon how Vc0 minus was specified.

Suppose it was specified upper plate plus, lower plate minus, I will write it like this and this will be the description of the capacitor between points A and B. Only one capacitor is connected, but it had got some initial voltage, then between points A B, I will say this capacitor with no initial voltage in series with a constant voltage Vc0 minus in time domain. If that be the case, if this capacitor has also got some initial voltage in this example, then what should I do is this.

I should, in s domain, how this will look like? It will look like between points, from this to this I am drawing. A, I will say there is a voltage source plus minus whose Laplace transform is Vc0 minus by s and this uncharged capacitor, that is the point B. So voltage across the capacitor at any instant you try to find out you calculate VABs, that is okay, but then it will allow me to solve the circuit. So I will finally draw one circuit.

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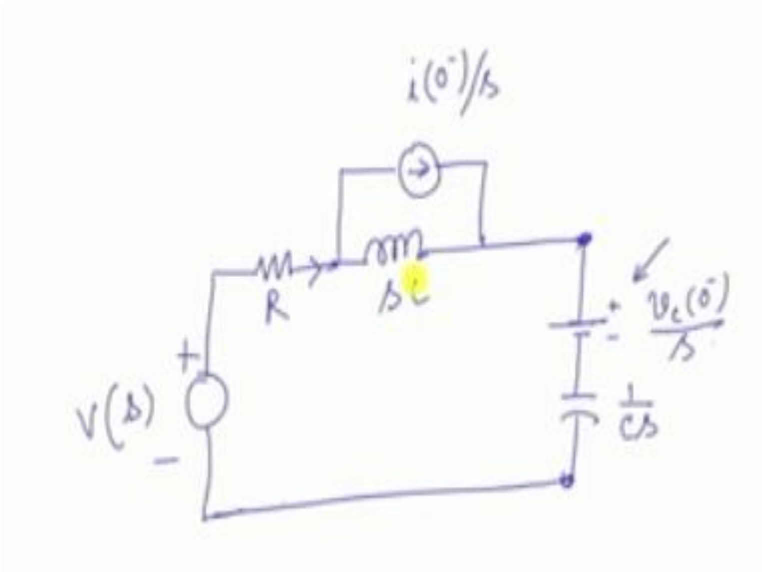

And next class, I will try to solve a problem with initial current, what I am telling, then in s domain, the circuit in general will look like an inductor sL parallel with a current source, then a capacitor with uncharged capacitor. If it is uncharged capacitor, then only you can write it as 1 by Cs and the initial voltage across the capacitor will be taken into account like this, which is Vc0 minus, this is a number that divided by s.

Similarly, this will be i0 minus divided by s and then here is your supply voltage, Laplace transform plus Vs. Then this circuit will have three sources, one source, two source, three source. Okay, there are various ways of handling this situation. Apply say superposition theorem or some other things nodal analysis and so on, but it can be also done in this way. It is a very nice way of looking at a circuit, inductors having initial current or capacitor.

Mind you, these are the two terminals of the capacitor. This is our thought process, which brought in this initial voltage in series with this uncharged capacitor. Similarly, this is the current flowing through the actual inductor in this circuit. In this circuit, there is no current source connected. Anyway, we will continue with this in the next class. Thank you.