Advanced Microwave Guided-Structures and Analysis
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Lecture 12
Instantaneous Form of Maxwell's Equations (Contd.)

So, welcome to this session. We will be continuing with the instantaneous form of Maxwell's equations. So, let us go to the lectures. We were seeing in the previous lecture, the different terms which are relevant in the vector identity, which will be remarked, that will ultimately take the shape of power conservation theorem. So, this time we are going to see how exactly the terms fit together to manifest itself, so that the equation manifests itself in the form of the power conservation theorem. So, let me write down the equation, so this is the equation:

$$p_f + \frac{\partial}{\partial t}(W_m) + \vec{H}_{in} \cdot \vec{M}_{in}^{i} + \frac{\partial}{\partial t}(W_e) + p_d + \vec{E}_{in} \cdot \vec{J}_{in}^{i} = 0$$

So, we saw the expressions of the time rate of change of the electric stored energy, which is $\frac{\partial W_e}{\partial t}$, the time rate of change of the magnetic stored energy, the power flow and the power dissipated terms. So, we now say that terms

$$\vec{E}_{in} \cdot \vec{J}_{in}^{i} = 0 , \quad \vec{H}_{in} \cdot \vec{M}_{in}^{i}$$

should be the power provided by the sources $\vec{J}_{in}^{i}$ and $\vec{M}_{in}^{i}$. Let me rearrange the terms once again.

$$p_f + \frac{\partial}{\partial t}(W_e + W_m) + p_d = -\left[\vec{E}_{in} \cdot \vec{J}_{in}^{i} + \vec{H}_{in} \cdot \vec{M}_{in}^{i}\right]$$

So the left hand side contains the power flow term, the energy densities of the electric and magnetic fields, the time rate of change and it contains the power distribution terms pd.

If I now shift as we said that right hand side terms contain the term involving the power impressed by the sources, so this term $\vec{E}_{in} \cdot \vec{J}^{i}_{in}$ or $\vec{H}_{in} \cdot \vec{M}^{i}_{in}$, contains the power supplied by the sources. So, then what we can do is that we can put that term for the power supplied by the sources, so this is the power supplied by $\vec{J}^{i}_{in}$, $\vec{E}_{in} \cdot \vec{J}^{i}_{in}$, the electric source the power supplied by the magnetic current source will be $\vec{H}_{in} \cdot \vec{M}^{i}_{in}$.

So, therefore, so the equation is now arranged such that the power flow term pf, the time rate of change of the stored energy, the electric stored energy and the magnetic stored energy and the power dissipation terms are to the left hand side of the equation, the right hand side of the equation contains the power supplied by the sources together with the minus sign.

So, this is the statement of the power conservation theorem, it says that the power supplied by the sources must be spent into the power flow, the time rate of change of the stored electric and magnetic energy densities and power dissipated term.

So, therefore we write ps, so let us call this

$$p_f + \frac{\partial}{\partial t}(w_e + w_m) + p_d = -\left[\vec{E}_{in} \cdot \vec{J}^{i}_{in} + \vec{H}_{in} \cdot \vec{M}^{i}_{in} \right]$$

as equation 22, and we call this

$$p_s = p_f + \frac{\partial}{\partial t}(w_e + w_m) + p_d \quad - (23)$$

as equation 23, so equation 23 basically states that the density of power supplied by the sources must equal that leaving the point, plus that dissipated plus the rate of increase of stored electric and magnetic energy densities..

So, now we will integrate for the whole region, so this is a point relationship, for the entire region of space or the for the regional space under consideration this

$$W_e = \frac{1}{2}\iiint \epsilon \left(\vec{E}_{in} \cdot \vec{E}_{in}\right) d\tau$$

is the total energy We in the entire region under consideration, so we have formed the triple integral because it was the energy density.

Similarly, Wm becomes the stored magnetic energy and we have derived that from by triple integrating the energy density. So, this

$$W_m = \frac{1}{2}\iiint \mu \left(\vec{H}_{in} \cdot \vec{H}_{in}\right) d\tau$$

is the total magnetic energy. And the net power converted to heat energy will similarly be found big Pd:

$$P_d = \iiint \sigma \left|\vec{E}_{in}\right|^2 d\tau \quad \left[\text{net power converted to heat energy}\right]$$

Note that this is this term is different from this stuff, this is small we, this is capital We, this is small wm this is capital Wm, this is small pd, this is capital Pd.

So, this is found by performing a triple integration over $\sigma\left|\vec{E}_{in}\right|^2$ which was small pd. So, this is the net power that is converted to heat energy, or the ohmic loss the total ohmic loss in the region.

Similarly, we can find the net power supplied by the sources in the region and that will be Ps it is different from small ps. So, this

$$P_s = -\iiint \left(\vec{E}_{in} \cdot \vec{J}_{in}^{\,i} + \vec{H}_{in} \cdot \vec{M}_{in}^{\,i}\right) d\tau$$

is the net power supplied by the sources in the region, the net power supplied by sources within the region, it is found with a triple integration of the power density. So, it is a fun way to build integration of the power density torque.

So, if we now use these terms into equation 19, which we derived before, so using these equation 19 can now be written as, Ps is Pf plus $\frac{d}{dt}(W_e + W_m)$ plus Pd.

$$P_s = P_f + \frac{d}{dt}\left(W_e + W_m\right) + P_d$$

Hence, the power supplied by sources within a region must equal to that leaving the region plus that dissipated plus the rate of increase in electric and magnetic energies stored within that region. So, this is the statement of the power conservation in the instantaneous domain, you see the explicit appearance of the time term.

So this completes the statement of the power conservation in the instantaneous domain. So, we have the power flow terms, the rate of increase of the stored energy term, the power dissipated term and the power supply term. Now, let us go to the time harmonic form of these same equations; the point form, the integral form and then graduating to the pointing vector, the relationship between the instantaneous form and the time harmonic form and finally the statement of the power conservation in the time harmonic form.

So, we are all aware of the time harmonic notation, but we need to state that here because we will be using this kind of mathematical language, we use this mathematical language everywhere electromagnetics, in the time harmonic domain, so we need to first of all clarify what exactly we mean by the term harmonic domain and what are the fundamental mathematical operations in the time harmonic domain.

So, we begin with a brief review of complex quantities, we have a voltage v which is expressed as

$$\begin{aligned} v &= \sqrt{2}\,|V| \cos(\omega t + \alpha) \\ &= \sqrt{2}\, \mathrm{Re}\left(V e^{j\omega t}\right) \quad \text{where} \quad V = |V| e^{j\alpha} \end{aligned}$$

So, in this notation the instantaneous electric field Ein can be written as

$$\begin{aligned} \vec{E}_{in} &= \sqrt{2}\, |\vec{E}| \cos(\omega t + \alpha) \\ &= \sqrt{2}\, \mathrm{Re}\left(\vec{E} e^{j\omega t}\right) \\ \text{where} \quad \vec{E} &= |\vec{E}| e^{j\alpha} \end{aligned}$$

So, we also will write some fundamental operations which we use in the complex domain, which is:

$$\begin{aligned} \mathrm{Re}(A) + \mathrm{Re}(B) &= \mathrm{Re}(A + B) \\ \mathrm{Re}(\alpha A) &= \alpha\, \mathrm{Re}(A) \\ \frac{\partial}{\partial x} \mathrm{Re}(A) &= \mathrm{Re}\left(\frac{\partial A}{\partial x}\right) \end{aligned}$$

real part of A plus real part of B is real part of A plus B. Real part of alpha A is alpha real part of A, where alpha is a constant del del x of real part of A, real is real part of del A del x.

$$\int Re(A)\, dx = Re\int (A\, dx)$$

Integral of real part of A dx is real part of A dx. So, as an example we can write:

$$v = \int \vec{E}_{in} \cdot dl$$

$$\sqrt{2}\, Re\,(V e^{j\omega t})$$

$$v = \int \vec{E}_{in} \cdot dl$$

$$v = \int \vec{E}_{in} \cdot dl$$

Ein dot dl, so we can write root 2 real part of V e to the power j omega t, we right v is integral Ein dot dl, we write v equal to integral Ein dot dl and therefore root 2 real part of V e to the power j omega t equal to root 2 real part of e to the power j omega t integral E dot dl.

$$\sqrt{2}\, Re\,(V e^{j\omega t}) = \sqrt{2}\, Re\left(e^{j\omega t}\int \vec{E} \cdot dl\right)$$

$$V = \int \vec{E} \cdot dl$$

So, we have cancelling all the e to the power j omega t variation, so cancelling this term with this term, this with this and this with that, we have V equal to integral E dot dl.

We can similarly write u equal to integral H dot dl.

$$U = \int \vec{H} \cdot dl$$

So, what this essentially means is that we are explicitly not mentioning the time variation, so the time variation because it is appearing on the left and right side of the equation that time variation is cancelled on the left and right side of the equation, so that we do not need to bother about the e to the power j omega t variation, it also happens in low frequency, it is independent of frequency.

So, the same thing will happen at any given frequency, so we use the time harmonic form and basically we do not need to use the e to the power j omega t term. So, if you look at any paper they will say that the e to the power j omega t term is suppressed, this term is suppressed. So, then we have by the same token I is equal to j dot ds and K as M dot ds.

$$I = \iint \vec{J} \cdot ds$$
$$K = \iint \vec{M} \cdot ds$$

We call these sets of equations including the previous one equation 26. Now, we can also find out in a similar manner by omitting the time variation the complex form of the Maxwell's equation.

So, the time harmonic form of the Maxwell's equations from equation number 16 and that is called of E equal to minus Mt and curl of H equal to Jt. So, we put a vector here and we call this question 27.

$$\left.\begin{aligned} \nabla \times \vec{E} &= -\vec{M}^t \\ \nabla \times \vec{H} &= \vec{J}^t \end{aligned}\right\} - (27)$$

However, consider the equation having are the explicit d dt term or del del t term in that case, for example, I have that equation curl of Ein, we have the equation curl of Ein is minus del Bin del t, so we can write this as curl root 2 real part E to power j omega t equal to minus del del t of root 2 real part of Bin e to the power j omega t, therefore root 2 real part curl of Ein e to the j omega t.

$$\nabla \times \vec{E}_{in} =$$

$$\nabla \times E_{in} = -\frac{\partial B_{in}}{\partial t}$$

$$\nabla \times \left\{ \sqrt{2}\, Re \left(\vec{E} e^{j\omega t} \right) \right\} = -\frac{\partial}{\partial t} \left\{ \sqrt{2}\, Re \left[\vec{B} e^{j\omega t} \right] \right\}.$$

$$\sqrt{2}\, Re \left(\nabla \times \vec{E} e^{j\omega t} \right\}$$

$$\sqrt{2}\, Re \left(\nabla \times \vec{E} e^{j\omega t} \right) = -\sqrt{2}\, Re \left(j\omega \vec{B} e^{j\omega t} \right)$$

$$\nabla \times \vec{E} = -j\omega \vec{B}$$

Therefore, root 2 real part curl of E e to the power j omega t becomes minus root 2 real part because of the del del t operation we pull the j omega term out, say j omega B e to the j omega t, therefore curl of E becomes equal to minus j omega B. So, it simply pulls the minus j omega term out, because of the minus sign, this stays minus sign and because of the del del t operation, we get the j omega term here. So, the other equations can be similarly written.

$$\left.\begin{aligned} \nabla \times \vec{E} &= -j\omega \vec{B} \\ \nabla \times \vec{H} &= j\omega \vec{D} + \vec{J} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \cdot \vec{D} &= Q_v \end{aligned}\right\} - (28)$$

For (13):

$$\left.\begin{aligned} \vec{D} &= \epsilon \vec{E} \\ \vec{B} &= \mu \vec{H} \\ \vec{J} &= \sigma \vec{E} \end{aligned}\right\} - (29)$$

So, we write curl E as minus j omega B, curl of H as j omega D plus J, divergence of B is 0 and divergence of D is Qv. So, from 13 also we can write the complex forms of 13, so D equal to epsilon E, B is mu H, J is sigma E, this correspond to 29, so let us stop here, we will continue from here.