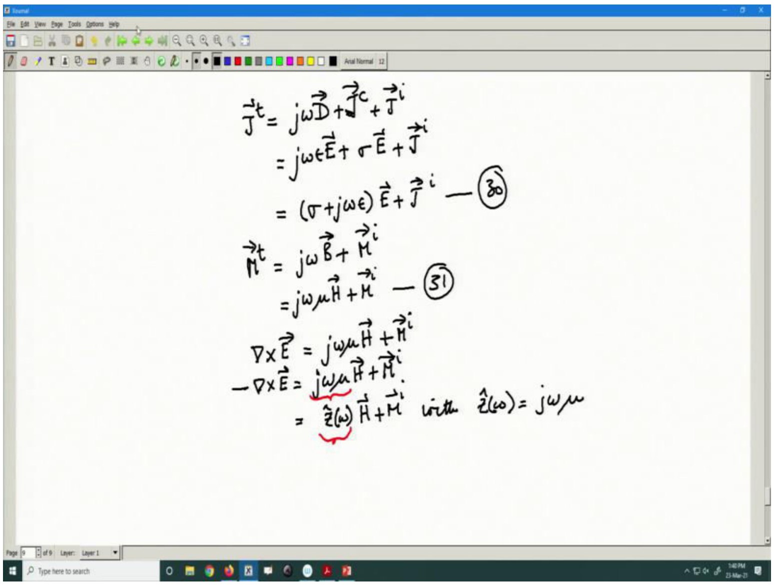
Advance Microwave Guided Structures and Analysis
Professor Bratin Ghosh
Department of E and ECE
Indian Institute of Technology, Kharagpur
Lecture 15
Time - Harmonic Form of Maxwell's Equations

So, let us start from where we left last time. So, after the constitutive relationships, we go to the generalised current concept, we wrote that in the instantaneous domain. So, let us write the generalised current in the time harmonic domain, it is a simple matter.

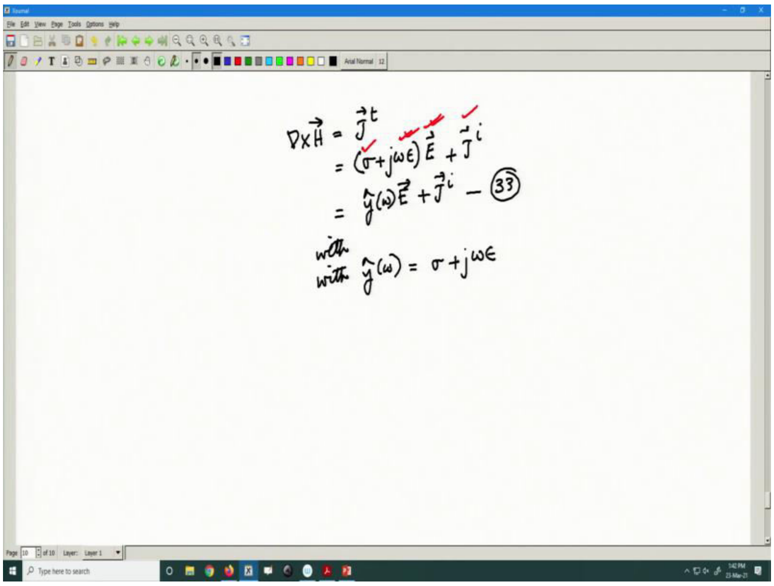
(Refer Slide Time: 00:36)



So, $\vec{J}^t = j\omega\vec{D} + \vec{J}^c + \vec{J}^i$ that is $\vec{J}^t = j\omega\varepsilon\vec{E} + \sigma\vec{E} + \vec{J}^i$, and therefore that can be written as $\vec{J}^t = (\sigma + j\omega\varepsilon)\vec{E} + \vec{J}^i$. So, we call this equation 30. So, also the equation for the total magnetic current $\vec{M}^t = j\omega\vec{B} + \vec{M}^i$ that is $\vec{M}^t = j\omega\mu\vec{H} + \vec{M}^i$. So, this we call this 31.

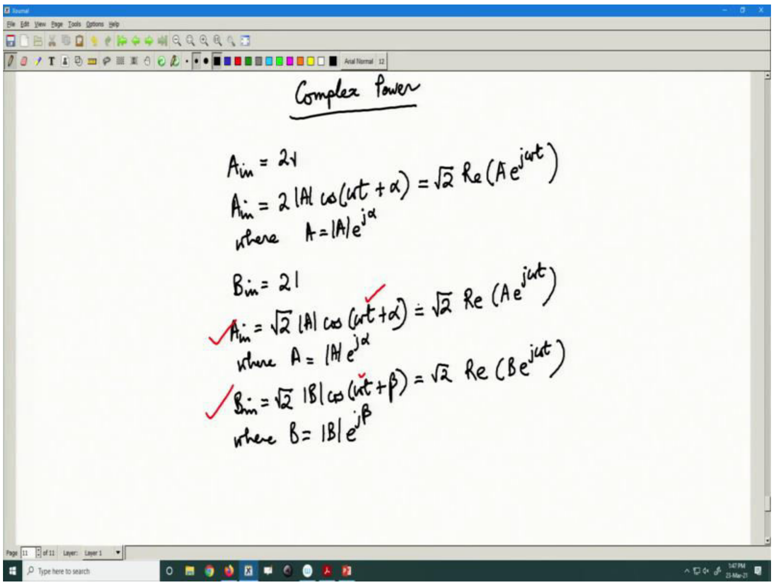
So, using this, the complex form of equation 16 can be written as $-\nabla\times\vec{E} = j\omega\mu\vec{H} + \vec{M}^i$ and that can be written as $-\nabla\times\vec{E} = \hat{z}(\omega)\vec{H} + \vec{M}^i$ where this $j\omega\mu$, it is captured within this $\hat{z}(\omega)$. So, this $\hat{z}(\omega) = j\omega\mu$.

(Refer Slide Time: 04:25)



Also $\nabla \times \vec{H} = (\sigma + j\omega\varepsilon)\vec{E} + \vec{J}^i$. So, $\sigma\vec{E}$ is the conduction current $j\omega\varepsilon\vec{E}$ is the displacement current and $\vec{J}^i$ is the impressed current. So, therefore, I can write $\nabla \times \vec{H}$ as= $\nabla \times \vec{H} = \hat{y}(\omega)\vec{E} + \vec{J}^i$, we call this equation 33 with $\hat{y}(\omega) = (\sigma + j\omega\varepsilon)$. So, in equations 32 and 33, $\hat{z}(\omega)$ and $\hat{y}(\omega)$ are the media parameters it specifies the characteristics of the media.

(Refer Slide Time: 06:24)

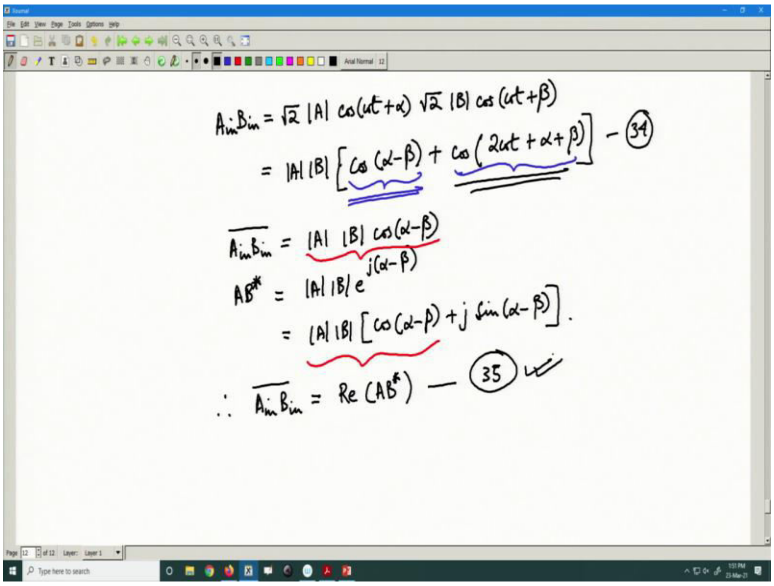


Now, we go to the concept of complex power. So, how the complex power is calculated in the time harmonic domain? So, we have considered the expressions for instantaneous power and energy in terms of instantaneous field vectors. We shall now show that similar expressions in terms of complex field vectors represent the time average power and energy in the AC fields.

So, we shall write this as let us begin with two the expressions for the two quantities from the instantaneous domain. So, $A_{in} = \sqrt{2}\left|A\right|\cos(\omega t + \alpha)$ that is $\sqrt{2}\,\mathrm{Re}(Ae^{j\omega t})$ where A is $\left|A\right|e^{i\alpha}$, similarly, $B_{in} = \sqrt{2}\left|B\right|\cos(\omega t + \beta)$ that is $\sqrt{2}\,\mathrm{Re}(Be^{j\omega t})$, where B equals to $\left|B\right|e^{i\beta}$.

Now, let us multiply these two quantities, because ultimately the power is the multiplication of voltage into current. So, if you multiply two quantities which are both varying with time and which has different phases, what happens in the time harmonic domain and how that is related to the instantaneous domain.

(Refer Slide Time: 11:16)

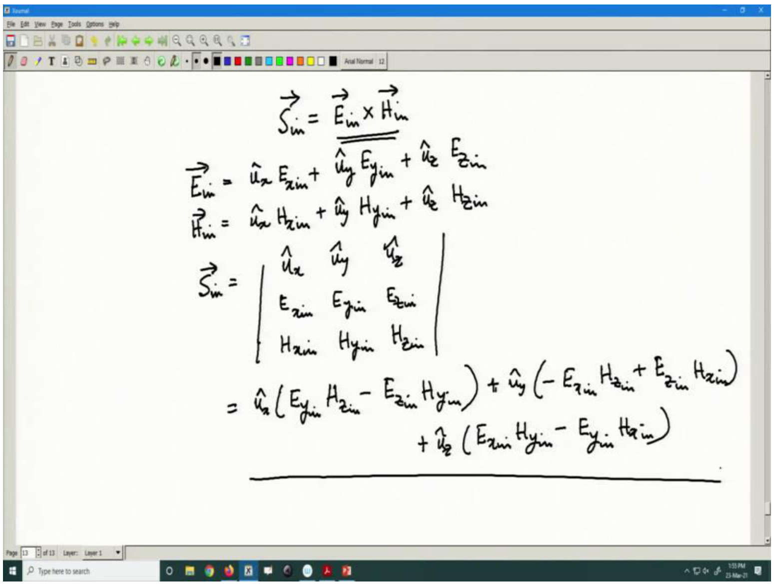


So, $A_{in}B_{in} = \sqrt{2}\left|A\right|\cos(\omega t + \alpha)\times\sqrt{2}\left|B\right|\cos(\omega t + \alpha)$. So, using the cosine rule simply we can write $A_{in}B_{in}$ as $\left|A\right|\left|B\right|[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)]$. So, we call this 34. Now, we compute that time average of this product, we see that there is a time dependent term and a time independent term. So, when you compute the time average obviously, $\cos(2\omega t + \alpha + \beta)$ is a cosine function, so the time average of this will be 0.

So, the time average will only be contributed by $\cos(\alpha - \beta)$. So, therefore, we find out the time average of $A_{in}B_{in}$, and that will be equal to $\left|A\right|\left|B\right|\cos(\alpha - \beta)$. So, the contribution of $\cos(2\omega t + \alpha + \beta)$ term is going to be 0. We also now note that if we perform this operation AB^*, then we are going to get $|A||B|e^{j(\alpha-\beta)}$ and that is expanding to get $\cos(\alpha-\beta)+j\sin(\alpha-\beta)$.

So, we find that the real part of this term AB^* is the same as the time average of the instantaneous product $A_{in}\bar{B}_{in}$, so we write therefore $A_{in}\bar{B}_{in}$ is equal to real part of $\mathrm{Re}(AB^*)$,we named this equation 35. So, this equation is the basis for our definition for the complex power, so we will see how.

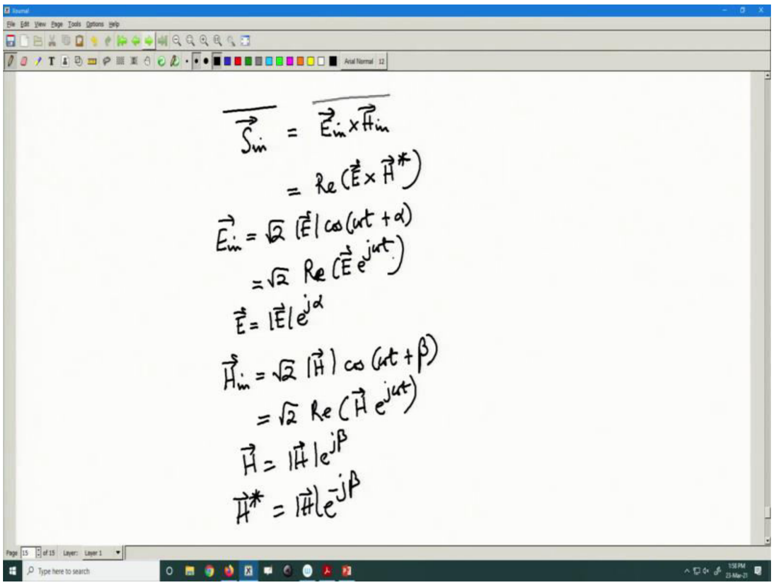
(Refer Slide Time: 15:47)



So, we know the instantaneous pointing vector is given by $\vec{E}_{in}\times\vec{H}_{in}$. So, $\vec{E}_{in}$ if I expand that in xyz coordinates will be $\vec{E}_{in}=\hat{u}_x E_{xin}+\hat{u}_y E_{yin}+\hat{u}_z E_{zin}$, similarly, $\vec{H}_{in}=\hat{u}_x H_{xin}+\hat{u}_y H_{yin}+\hat{u}_z H_{zin}$. So, now if I perform this operation $\vec{S}_{in}$ it will ultimately be a product of these terms. So, $\vec{S}_{in}$ will be $\begin{vmatrix}\hat{u}_x & \hat{u}_y & \hat{u}_z \\ E_{xin} & E_{yin} & E_{zin} \\ H_{xin} & H_{yin} & H_{zin}\end{vmatrix}$.

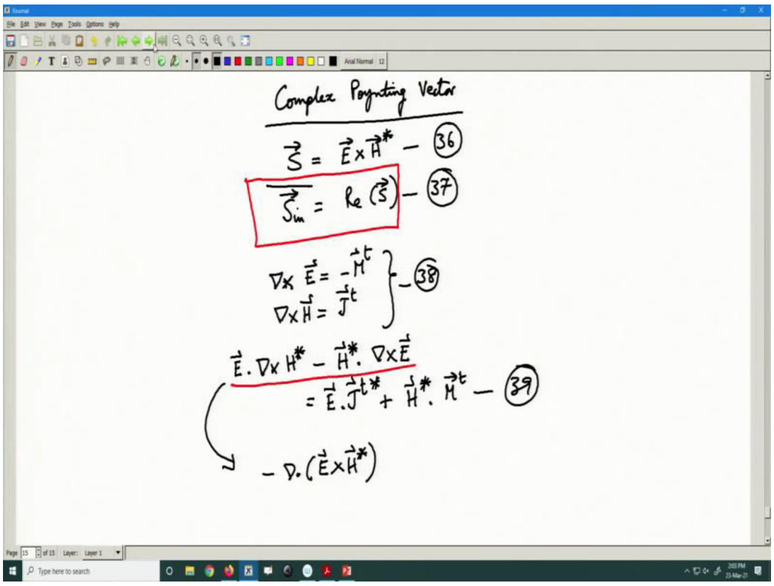
So, that will be equal to $\hat{u}_x(E_{yin}H_{zin}-E_{zin}H_{yin})+\hat{u}_y(-E_{xin}H_{zin}+E_{zin}H_{xin})+\hat{u}_z(E_{xin}H_{yin}-E_{yin}H_{xin})$. So, you see now this is a sum of terms, each of which is in the form of equation number 34.

(Refer Slide Time: 19:52)



It therefore follows that the time the average of the instantaneous pointing vector is equal to $\overline{\vec{E}_{in} \times \vec{H}_{in}}$ and that will be equal to real part of $\vec{E} \times \vec{H}^*$ as we saw previously. So, if $\vec{E}_{in}$ is $\vec{E}_{in} = \sqrt{2}\left|\vec{E}\right|\cos(\omega t + \alpha)$ i.e. $\sqrt{2}\,\mathrm{Re}(\vec{E}e^{j\omega t})$. $\vec{H}_{in}$ is $\sqrt{2}\left|\vec{H}\right|\cos(\omega t + \beta)$, that is $\sqrt{2}\,\mathrm{Re}(\vec{H}e^{j\omega t})$. So, $\vec{H}$ is equal to $\left|\vec{H}\right|e^{j\beta}$ and $\vec{H}^*$ is $\left|\vec{H}\right|e^{-j\beta}$.

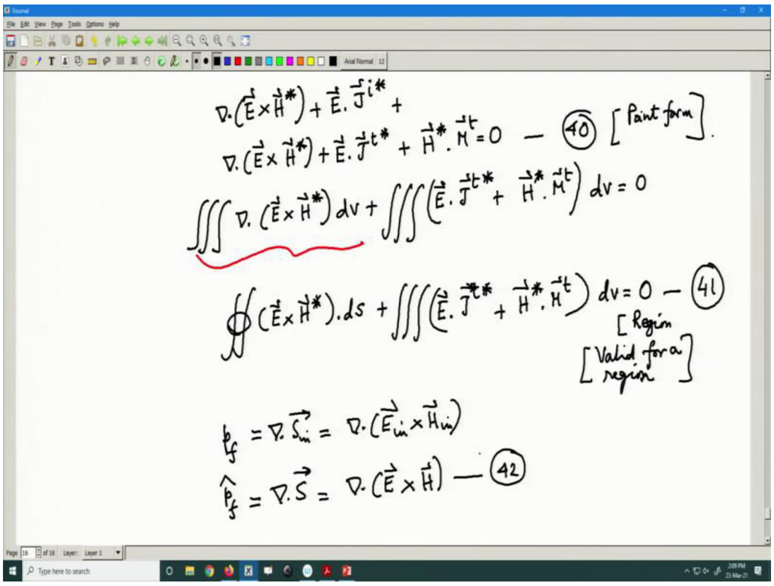
(Refer Slide Time: 22:41)



So, in view of this we can define a Complex Pointing Vector. $\vec{S}$ is $\vec{E} \times \vec{H}^*$ whose real part, is the time average of the instantaneous pointing vector for $\vec{S}_{in}$. So, this assumes a very important relationship, linking up the time harmonic form with the instantaneous form of the complex pointing vector. So, after we have found out and discussed regarding the complex pointing vector, let us now rephrase the power conservation theorem as we learnt in the instantaneous domain. So, if we rephrase the power conservation theorem, as we learned is in the instantaneous domain, we will again begin with different vector identity and in this case, we will write.

First of all starting from the Maxwell's equation in the time harmonic domain. So, this is Maxwell's equations time harmonic domain, we will now compute this one. So, it will involve the complex conjugate while in the instantaneous domain we had only the instantaneous quantities. So, that is equal to $\vec{E}\cdot\vec{J}^{t*}+\vec{H}\cdot\vec{M}^{t*}$, call this 39. So, we identify the left-hand side as $-\nabla\cdot(\vec{E}\times\vec{H}^{*})$.

(Refer Slide Time: 27:32)

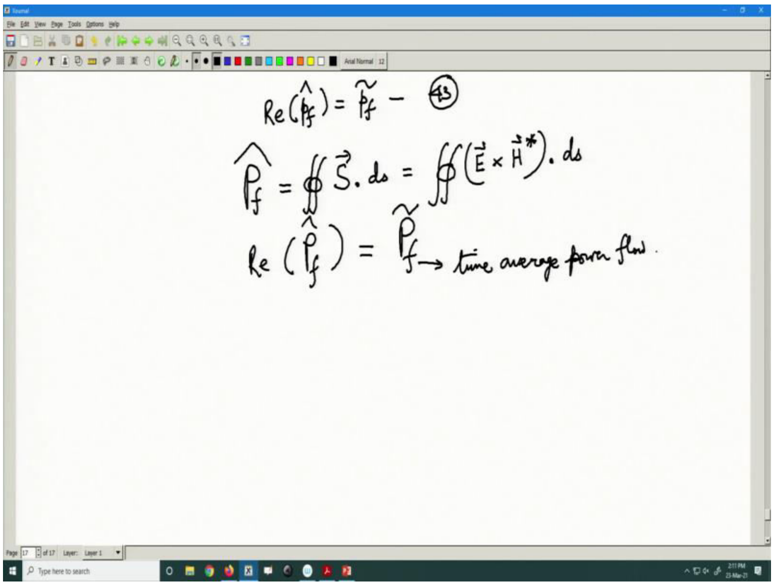


And therefore, the equation will read $\nabla\cdot(\vec{E}\times\vec{H}^{*})$ plus $\vec{E}\cdot\vec{J}^{t*}+\vec{H}\cdot\vec{M}^{t*}$ is 0, we call this equation number 40. So, similar to this equation now, as we did previously to this equation, we will now find the region form this is the point form and the equation applied to a region of space.

So, this is the point form of the power conservation. So, if we apply this to a region of space we will perform the triple integration on the left and the right. So, we will say $\iiint\nabla\cdot(\vec{E}\times\vec{H}^{*})dv+\iiint(\vec{E}\cdot\vec{J}^{t*}+\vec{H}\cdot\vec{M}^{t*})dv=0$. And by using divergence theorem to $\iiint\nabla\cdot(\vec{E}\times\vec{H}^{*})dv$ term, we will get $\iiint(\vec{E}\times\vec{H}^{*})\cdot\vec{ds}+\iiint(\vec{E}\cdot\vec{J}^{t*}+\vec{H}\cdot\vec{M}^{t*})dv=0$. So, we call this equation number 41.

So, we compare 40 and 41 with 18 and 19 for the corresponding instantaneous domain. So, we call 40 and 41 expressions for conservation of complex power for a point and for a region. So, as suggested by the point relationship we had the point relationship $p_f = \nabla \cdot \vec{S}_{in}$. We noted this in the instantaneous domain that is equal to $\nabla \cdot (\vec{E}_{in} \times \vec{H}_{in})$. We define a complex volume density of power leaving a point as $\hat{p}_f$. So, this is the complex volume density of power leaving the point as $\nabla \cdot \vec{S}$ that is $\nabla \cdot (\vec{E} \times \vec{H})$, forty two.

(Refer Slide Time: 33:31)



Now, the real part of this is a time average volume density of power leaving a point. So, the real part of $\hat{p}_f$ is equal to $\tilde{p}_f$ and it is the time average volume density of power leaving a point. Similarly, we can define a complex power leaving a region as $\hat{P}_f$.

So, closed integral the pointing vector dot ds and that is equal to close surface integral $\vec{E} \times \vec{H}^*$ dot ds, the real part of this is the time average power flow. So, we write real part of $\hat{P}_f$ is $\tilde{P}_f$ or the time average power flow. So, let us stop here we will continue with the power conservation theorem.