Advanced Microwave Guided Structures and Analysis
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Lecture No. 19
Wave Equation and Solution

Hello, welcome to this session on Wave Equation and Solution. So, in this session we are going to learn the formulation of the wave equation, the source free wave equation and how it is solved. So, let us write down the wave equation in source free medium.

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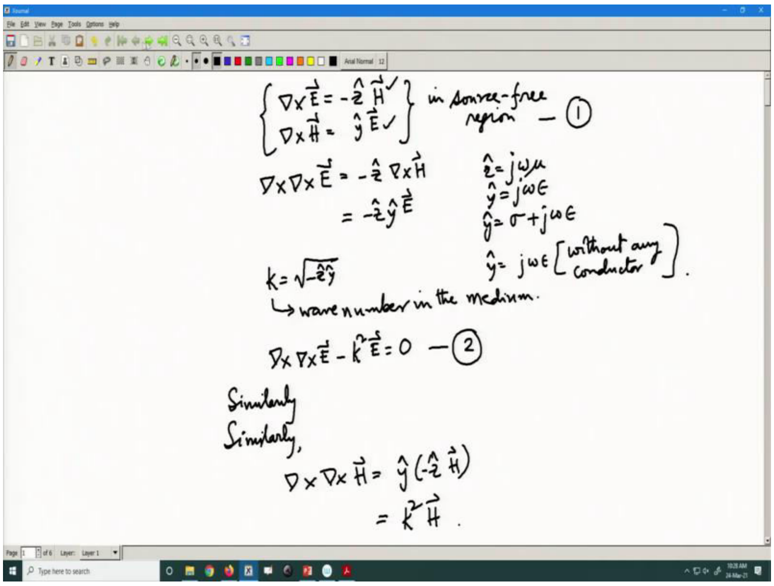

So, $\nabla\times\vec{E}=-\hat{z}\vec{H}$ and $\nabla\times\vec{H}=\hat{y}\vec{E}$. So, there are no sources in this equation. Let us call this equation 1. Now, in order to recast this equation in terms of either $\vec{E}$ or $\vec{H}$, we need to drive out $\vec{H}$ from this equation or we need to drive out from this equation.

So, we need to perform the curl of the first of equation and substitute from the second equation to the first equation, as simple as that. So, we perform $\nabla\times\nabla\times\vec{E}$ equal to $-\hat{z}\nabla\times\vec{H}$ and substituting from the second set of equation which is the second of the equation 1, which is $\nabla\times\vec{H}=\hat{y}\vec{E}$.

So, needless to say $\hat{z}$ is $j\omega\mu$ and $\hat{y}$ is $j\omega\varepsilon$ and $\hat{Y}$ is $\sigma+j\omega\varepsilon$. So, we can write $k=\sqrt{-\hat{z}\hat{y}}$ .It is the wave number in the medium. So, if there is no conductor, it is a source free case, homogeneous medium without any conductor, so $\hat{y}$ will be equal to $j\omega\varepsilon$.

So, K is called the wave number in the medium and therefore, we can write the previous equation as $\nabla\times\nabla\times\vec{E}-k^2\vec{E}=0$, this is equation number 2. In a similar fashion, we can find $\nabla\times\nabla\times\vec{H}-k^2\vec{H}=0$. So, we take the curl of the second equation and substitute for curl from the first equation to the second equation as simple as that.

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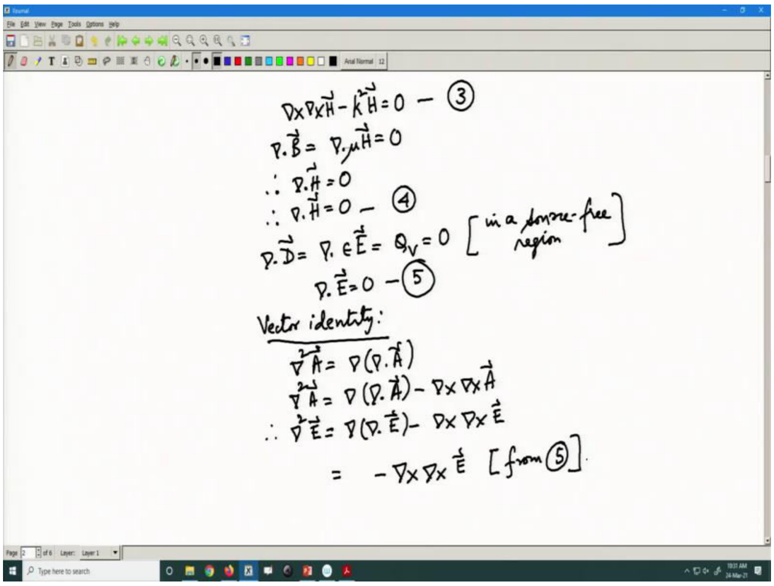

So, we can write this equation in terms of $\nabla\times\nabla\times\vec{H}-k^2\vec{H}=0$. So, this is equation number 3, which is similar to equation number 2, in terms of magnetic field. Now, we also know that $\nabla\cdot\vec{B}=\nabla\cdot\mu\vec{H}=0$, we already know that fact that the magnetic flux forms close loops, there are no magnetic sources. We call this equation number 4, also $\nabla\cdot\vec{D}=\nabla\cdot\varepsilon\vec{E}=Qv$, the charge enclosed and that is equal to 0, because we are in a source free region that is equation 5.

Now, we have from a vector identity $\nabla^2\vec{A}=\nabla(\nabla\cdot\vec{A})-\nabla\times\nabla\times\vec{A}$, true for any vector and therefore, if I substitute the electric field for $\vec{A}$, it is $\nabla^2\vec{E}=\nabla(\nabla\cdot\vec{E})-\nabla\times\nabla\times\vec{E}$ and because of 5 where $\nabla\cdot\vec{E}=0$, this can be written in terms of $-\nabla\times\nabla\times\vec{E}$, that is from 5.

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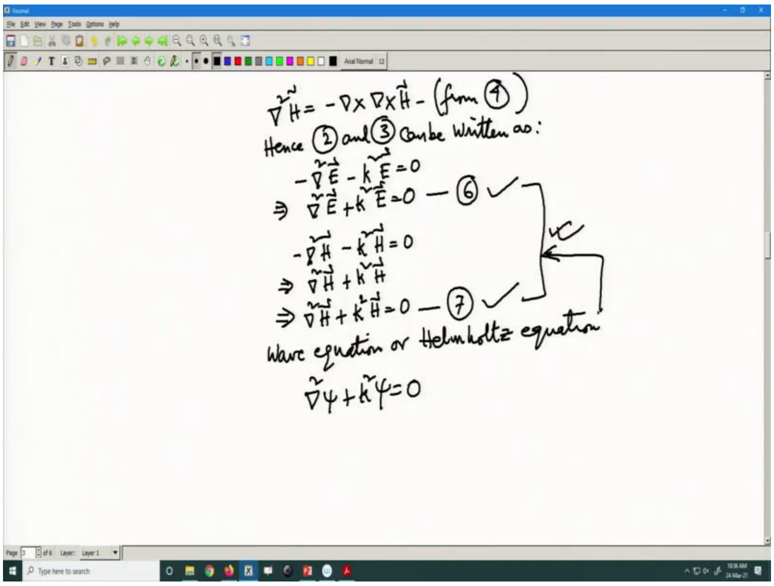

Similarly, for the magnetic field we have $\nabla^2 \vec{H} = -\nabla \times \nabla \times \vec{E}$ from 4, hence 2 and 3 can be written as $-\nabla^2 \vec{E} - k^2 \vec{E} = 0$, which will imply $\nabla^2 \vec{E} + k^2 \vec{E} = 0$ and minus $-\nabla^2 \vec{H} - k^2 \vec{H} = 0$, which will mean $\nabla^2 \vec{H} + k^2 \vec{H} = 0$.

We call this equation 6 and we call this equation 7. So, these are the wave equations satisfied by the electric field and the magnetic field, the same way the equation is satisfied by the electric field and the magnetic field, this wave equation is also called the Helmholtz equation. So, these are the two equations. So, both electric and magnetic fields satisfies the wave equation or the Helmholtz equation.

So, both these equations can be succinctly or shortly written as $\nabla^2 \psi + k^2 \psi = 0$, where ψ can be either $\vec{E}$ or $\vec{H}$. So, as we said if the medium is a perfect dielectric, $\hat{y}$ is $j\omega\varepsilon$ and $\hat{z}$ is $j\omega\mu$.

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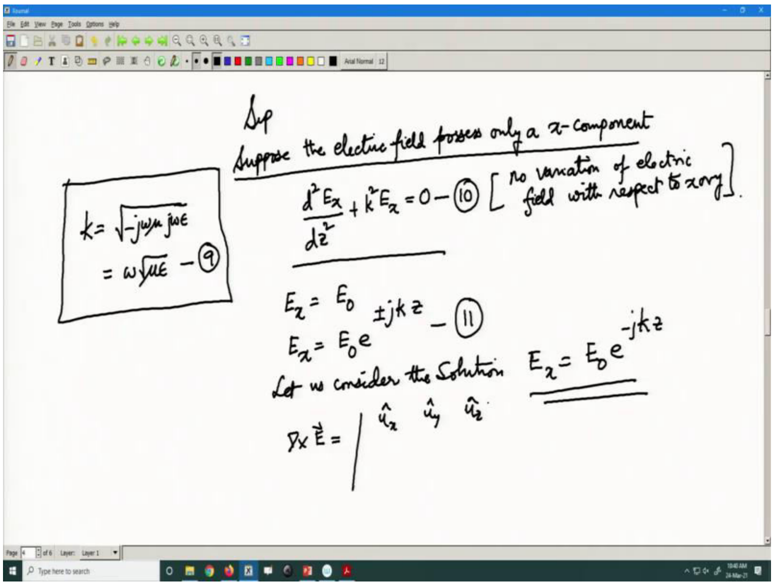

Now, suppose the electric field poses only a x component, that is independent of x or y, it is dependent only on z. So, the electric field is x polarized, which is independent of x or y and dependent only on z. Therefore, we have $\frac{dEx}{d^2z} + k^2Ex = 0$.

So, as I said two things need to be noted here that there is no variation with respect to x or y. No, variation of electric field with respect to x or y, and we have one polarization which is dependent only on z.

Now, the solution to this equation is of the form $Ex = Eoe^{\pm jkz}$, this becomes equation number 11.

So, now let us consider the solution, let us consider the solution, $Ex = Eoe^{-jkz}$. So, this is a forward going wave.

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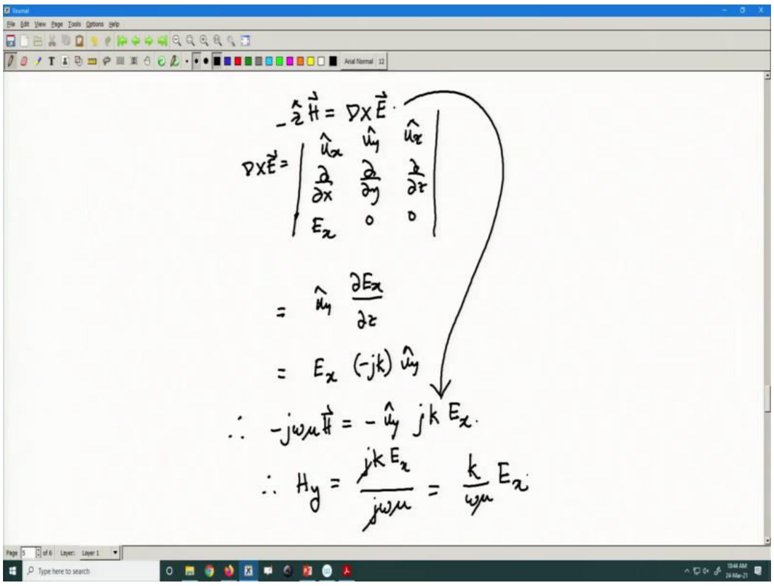

Then the source free Maxwell's equation can be written minus $-\hat{z}\vec{H} = \nabla \times \vec{E}$ with $\nabla \times \vec{E}$ given by $\begin{vmatrix} \hat{u}_x & \hat{u}_y & \hat{u}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ Ex & 0 & 0 \end{vmatrix}$ so, that will become $\hat{u}_y \dfrac{\partial Ex}{\partial x}$.

So, there will be no component along the z direction. So, this can be written as $Ex(-jk)\hat{u}_y$ because of this $\dfrac{\partial}{\partial z}$ differentiation. So, minus j K is going to come out and then we are going to have E to the power minus j kz. So, that will be the same as Ex. So, because of this differentiation the minus j K term is going to come out of the E to the power minus j Kz.

So, therefore, $-j\omega\mu\vec{H} = Ex(-jk)\hat{u}_y$. That is from here. Therefore, Hy has only a y component will be $Ex(-jk)\hat{u}_y / -j\omega\mu$. So, that will be equal to $(k / \omega\mu)Ex$. So, now we come to the concept of wave impedance.

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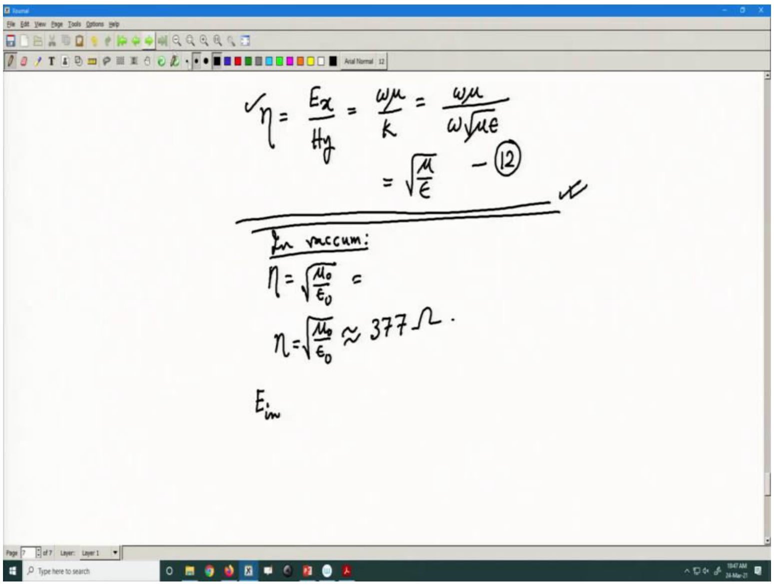

So, η the wave impedance is Ex by Hy and that will be equal to omega mu by K that is omega mu by omega root mu epsilon that is root of mu by epsilon, is 12. So, what is the significance of this wave impedance? The significance particularly is that once we can calculate Ex, once the Ex is calculated, we can find out Hy.

Now, you will see that this is free space, the wave impedance is uniquely defined, if the medium is a guided wave structure, the unique definition of wave impedance is going to slowly vanish, and we will see that in the waveguide, impedance cannot be uniquely defined or the characteristic impedance cannot be uniquely defined.

In hybrid guided structures, when you take the ratio between different transverse components you are going to get different results. So, that concept of wave impedance or the concept of characteristic impedance in hybrid guided structures where both the electric and magnetic field along the z direction exists simultaneously, there the concept of wave impedance totally vanishes or the characteristic impedance totally finishes.

So, in this case, this is called the wave impedance and in vacuum η is $\sqrt{\mu o / \varepsilon o}$ and if you substitute values in vacuum that is approximately equal to 377 Ohm.

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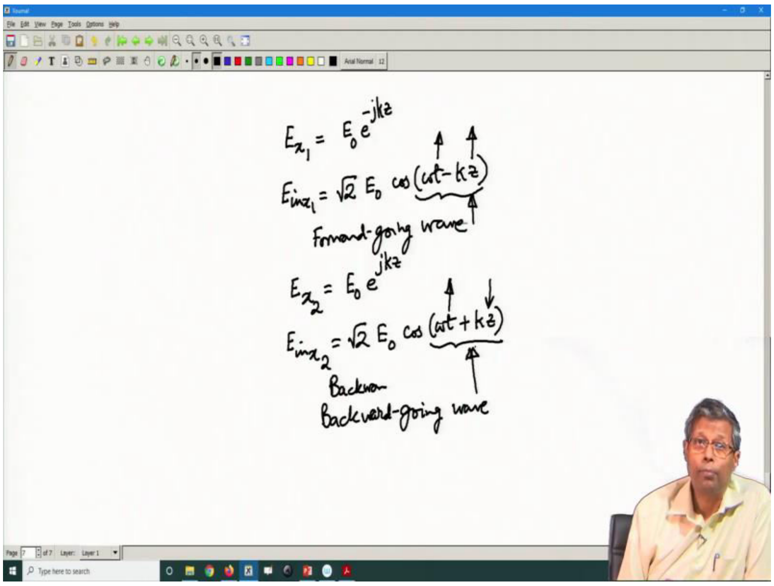

Now the concept of a forward going wave as we said, so let us suppose Ex1 is Eoe^{-jkz}. So, the corresponding instantaneous wave in Einx1 is $\sqrt{2}Eo\cos(\omega t - kz)$. So, you see for a constant phase point as time increases, z will also increase for a constant phase point. As time increases the z will also increase, in order to maintain constant phase. So, this is a forward going wave. So, the wave moves forward as time increases.

Similarly, we have another wave Ex2 as Eoe^{jkz}, the corresponding instantaneous wave is Einx2, which is $\sqrt{2}Eo\cos(\omega t + kz)$. So, you see now, in order to maintain constant phase, for a constant phase point as time increases, z has to decrease. So, as time keeps on increasing the wave moves backward, so as time keeps on increasing the wave moves backward.

So, this represents a backward going wave. So, this represents backward going wave. So, we discussed the wave equation and we took the representative example of a x polarized wave which is independent of x or y variation and we came also to the concept of characteristic impedance and the concept of the forward going wave and their representations. So, we will end the lecture here. We will continue with this.