Advanced Microwave Guided-Structure and Analysis
Professor Bratin Ghosh
Department of Electrical & Electronics Communication Engineering
Indian Institute of Technology, Kharagpur
Lecture 41
The Reciprocity Theorem, Computation of Amplitudes of Forward and Backward Propagating Waves for Electric and Magnetic Current Sources in the Waveguide

Welcome to this session on the reciprocity theorem, we will be using the reciprocity theorem in the computation of the amplitudes of the forward propagating and backward propagating waves due to electric current source and a magnetic current source inside a rectangular waveguide.

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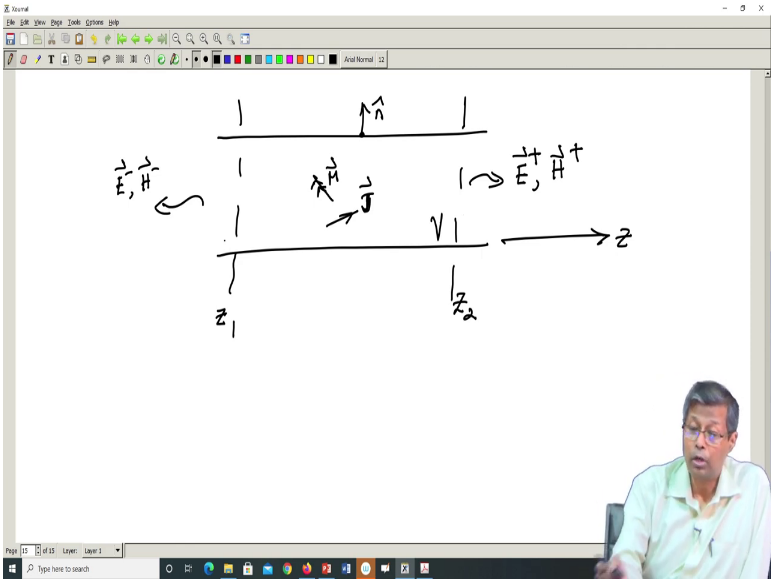

Let us go to the lecture and we have either a magnetic current source **M** or an electric current source **J** inside the waveguide. Let us consider 2 planes in the waveguide Z1 and Z2 and this is the Z axis. So, the forward propagating waves are given by **E** plus and **H** plus and the backward propagating waves are given by **E** minus and **H** minus and this is the surface normal $\hat{n}$. So, this is the volume V under consideration bounded by the waveguide metallic walls and the 2 imaginary surfaces.

So, and this is the volume V consisting of the waveguide metallic walls and the 2 imaginary surfaces at Z equal to Z1 and Z equal Z2. So, the problem is to find out the amplitudes of the forward and backward propagating waves radiated by the electric currents source **J** or the magnetic current source **M**.

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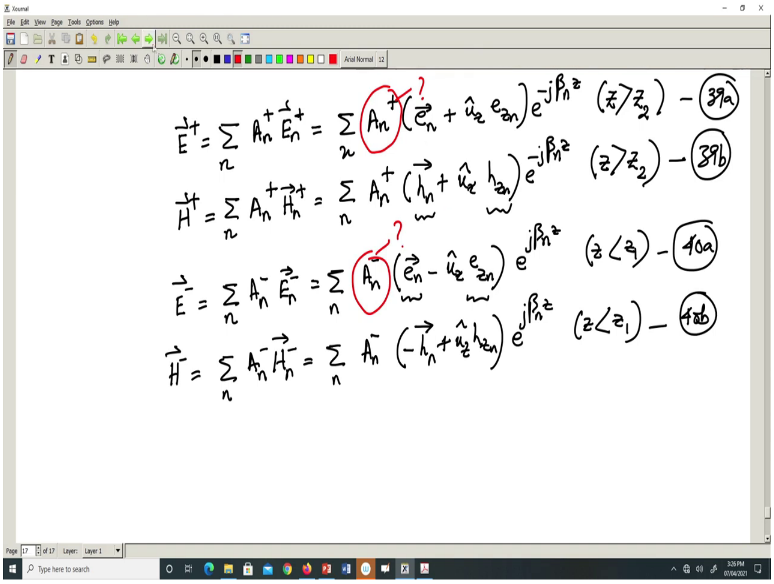

So, first let us write down the generic expressions of the electric and the magnetic fields propagating in the forward direction and the backward direction. So, then if I consider the electric field moving in the forward direction I can write it as a summation over n harmonics. An plus is the amplitude of the nth forward moving harmonic with **En** plus the distribution of the field of the nth harmonic. So, that can be broken up into the transverse component and the longitudinal component. The transverse component along the transverse walls of the waveguide and the longitudinal component along the Z direction.

So, it is broken up in to summation over n, An plus **en** plus **uz** ezn times $e^{-j\beta_n z}$ that is for Z greater than Z2 because the wave is moving in the forward direction. For the magnetic field moving in the forward Z direction we can write it as a summation over nth harmonics. So, An plus **Hn** plus, An plus **hn** plus **uz** hzn $e^{-j\beta_n z}$ for Z greater than Z2 and this is equation 39a that can be called equation 39b.

So, similarly, Hn denotes the transverse magnetic field component along the X and Y directions and hzn denotes the longitudinal magnetic field component along the Z direction. So, for the backward moving waves E minus can be written as summation over An minus **En** minus and that is summation over An minus **En** minus **uz** ezn $e^{-j\beta_n z}$ for Z less than Z1.

So, An minus is the amplitude of the nth harmonic propagating in the negative Z direction En minus is the field distribution of the electric field. So, it is composed of the transverse component En along the X and Y directions and the longitudinal component ezn along the Z direction and **H** minus is sigma n An minus **Hn** minus so, that is given by sigma over n An minus.

So, because the field is moving along the minus Z direction so, we will have the corresponding magnetic field component with a minus sign. So, En here it was En here it is minus Hn and here it was minus **uz** ezn and here there and for the magnetic field it will be plus **uz** hzn $e^{-j\beta_n z}$ for Z less than Z1. So, this is 40a. That is 40b. So, these are the generic expressions for the fields. So, our job is to find out An plus and An minus what is this and what is this.

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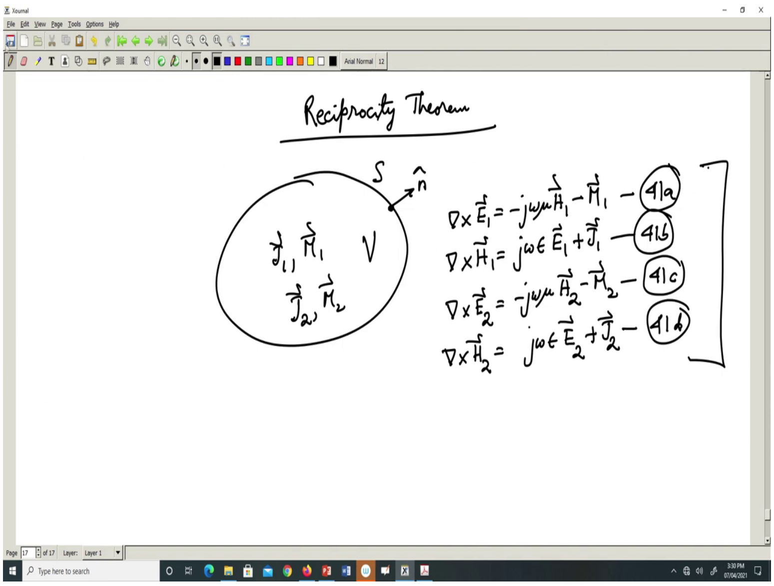

So, for that we invoke the reciprocity theorem. Let us, understand what the reciprocity theorem states we have an enclosed volume where sources **J1**, **M1** are present along with sources **J2**, **M2**. So, **J1** is the electric current source, **M1** is the magnetic source. This is the first pair of sources and then is the other pair of sources **J2** is the electric current source, **M2** is the magnetic current source, this is the volume V this is the surface n this is the surface normal $\hat{n}$.

So, now, we will simply write Maxwell's equations involving the electric and magnetic fields radiated by **J1, M1** and the electric and magnetic fields radiated by **J2, M2**. So, for that we have curl of **E1** equal to minus j omega mu **H1** minus **M1** let us call this 41a. So, this is due to the source **M1** again do to the source **J1** we have curl of **H1** as j omega epsilon **E1** plus **J1**.

So, let us call this 41b. For the second sources **J2, M2** similarly, we have curl of **E2** as minus j omega mu **H2** minus **M2**. So, this becomes 41c and curl of **H2** equal to j omega epsilon **E2** plus **J2**. Let us call this 41d. So, these four equations involve the fields radiated by the pair of sources of **J1, M1** and **J2, M2**.

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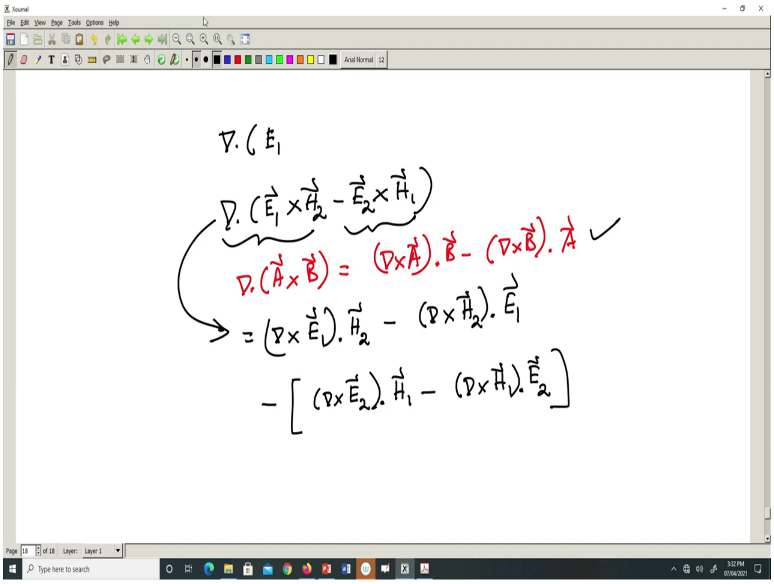

So, now consider this vector operation. Divergence of **E1** cross **H2** minus **E2** cross **H1.** we know that divergence of **A** cross **B** for any 2 general vectors **A** and **B** is given by curl of **A** dot **B** minus curl of **B** dot **A** and therefore using this can be written as curl of **E1** dot **H2** minus curl of **H2** dot **E1** that is for the first term this and for the second term we have curl of **E2** dot **H1** minus curl of **H1** dot **E2**. Now, using equations 41 a, b, c and d we can expand this equation.

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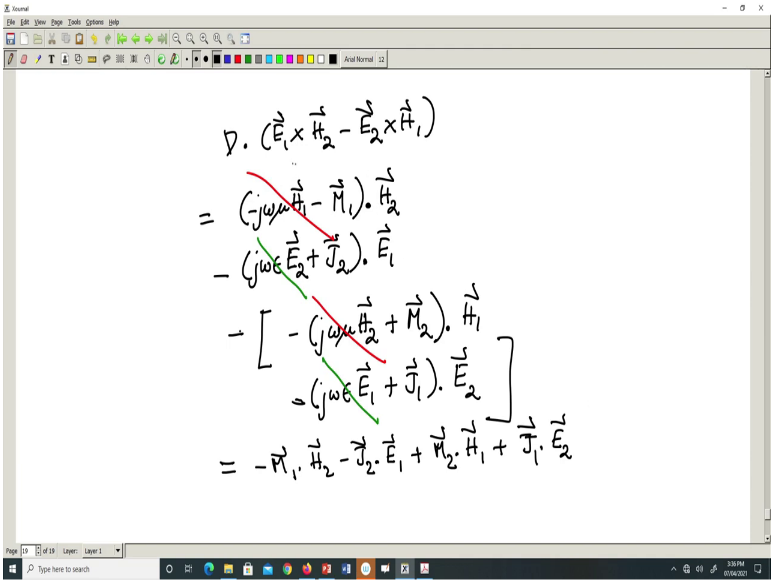

So, this can be further written as divergence of **E1** cross **H2** minus **E2** cross **H1** minus j omega mu **H1** minus **M1** dot **H2** minus j omega epsilon **E2** plus **J2** dot **E1** minus j omega mu **H2** plus **M2** dot **H1** minus j omega epsilon **E1** plus **J1** dot **E2.** So, now, we see that this term cancels with this term and this term cancels with that term. So, we are ultimately left with minus **M1** dot **H2** minus **J2** dot **E1** plus **M2** dot **H1** plus **J1** dot **E2**. So, this is what we are left with.

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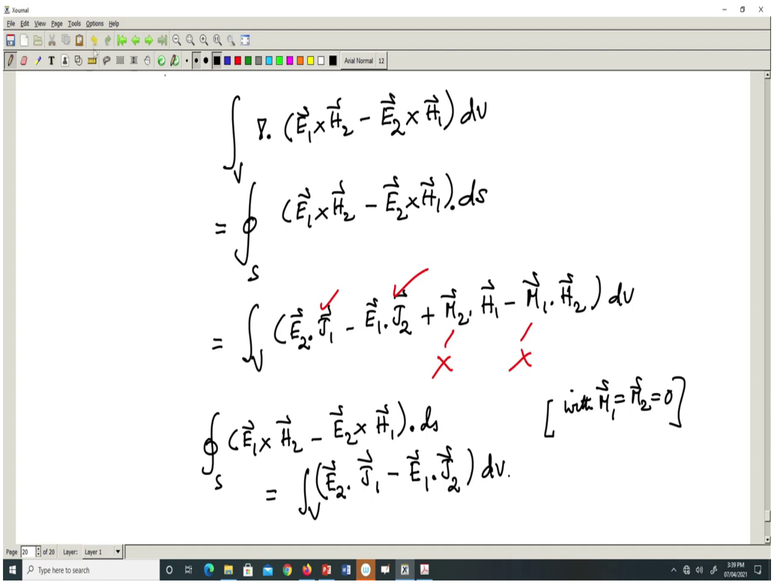

Now, we integrate with the volume V and apply divergence theorem. So, once we do that integral over the volume V divergence of **E1** cross **H2** minus **E2** cross **H1** dv, that is the close surface integral by divergence theorem **E1** cross **H2** minus **E2** cross **H1** dot ds and that is integration over volume V **E2** dot **J1** minus **E1** dot **J2** plus **M2** dot **H1** minus **M1** dot **H2** dv.

So, now, if there is only the presence of the electric current source, in other words, if only **J1** and **J2** are present, but **M1** and **M2** they are not present then the above equation will read closed surface integral **E1** cross **H2** minus **E2** cross **H1** dot ds is the volume integral **E2** dot **J1** minus **E1** dot **J2** dv. So, that is with **M1** equal to **M2** equal to 0. So, S is the surface including the volume V.

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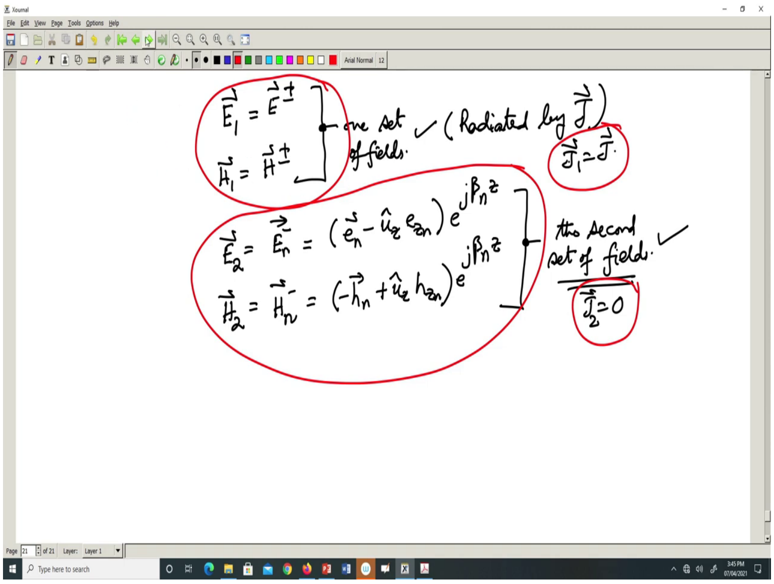

Now, we go back to the previous problem to find out the amplitude of nth forward moving harmonic An plus. What is the amplitude of the nth forward moving or what is the amplitude of the nth forward going harmonic radiated by a current source J inside the waveguide.

So, there is no magnetic current source only there is a single current source **J** inside the waveguide and what will be the amplitude of the nth going forward propagating wave. What is the amplitude of the nth forward propagating wave or what is the amplitude of the nth harmonic for the forward propagating wave.

So, for that we let the 2 sets of fields be defined by the electric field **E1** as **E** plus minus or the scattered field inside the waveguide radiated by the current source **J** and the corresponding **H1** being the scattered field **H** due to the current source **J** placed inside the waveguide. So, this is 1 set of fields.

The other set of fields are considered to be test modes. So, if I am interested to find out the amplitude of the nth harmonic moving in the forward direction we consider the test mode to be **E2** equal to **En** minus the nth moving wave moving in the minus Z direction. So, **En** minus that is given by **En** minus **uz** ezn $e^{-j\beta_n z}$.

We wrote the expression of this before note that this is composed of only 1 harmonic n it is moving in the backward direction or the negative Z direction and is composed of 1 harmonic n. Similarly, for **H2** the corresponding **H2** we have **Hn** minus and that is minus **Hn** plus **uz** hzn $e^{-j\beta_n z}$.

The expression for this also we have written before. So, this is the nth going harmonic for the magnetic field which is moving in the backward Z direction then we substitute these 2 sets of fields. So, this is the other set of fields the second set of fields. So, these 2 fields are substituted into the reciprocity theorem.

So, this set of fields are radiated by see this is radiated by the current source **J**. So, **J1** equal to **J** and this set of fields is coming from infinity. So, there are no sources for these set of fields is the second set of fields. So, this is just traveling along the minus Z direction and these are source free. So, then for this set we have **J2** equal to 0. So, these are just negative going nth harmonic modes without any source. So, therefore, if I substitute these conditions which are this condition this condition this condition and this condition into the reciprocity theorem.

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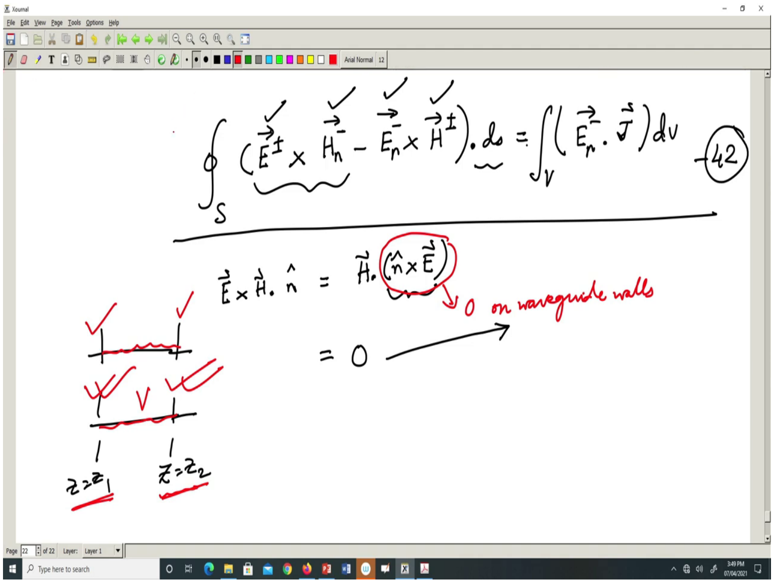

We have closed integrals is S **E** plus minus cross **Hn** minus **En** minus cross **H** plus minus dot ds is closed follow integral **En** minus dot **J** dv. Because this is the electric field radiated by the source **J1** which is the scattered field or the scattered electric field. This is the field **H2** which is the nth moving backward nth which is the nth backward moving harmonic.

This is **E2** which is the nth backward moving harmonic for the electric field and this is the magnetic field radiated by the source **J** and this is the electric field again **E2**. So, which is the nth backward moving harmonic. This is the current source **J** and then because there is no second current source because **J2** is equal to 0. Therefore, we do not have any second term. So, this is the now the statement of the reciprocity theorem for this case. So, we call this equation 42.

So, we considered this term now equals **E** cross **H** dot ds. If we consider this term **E** cross **H** dot n that can be written according to the rules of the vector triple product as **H** dot n cross **E** and n cross E is the tangential component of E. So, this component is 0 on the waveguide walls on the 4 waveguide walls the 2 narrow walls and the 2 broad walls this part is 0 and therefore, this is 0 on the waveguide walls.

So, we are now left with integration over the guide cross section which is we had the waveguide walls as this and then we had the 2 walls at Z equal to Z1 and Z equal to Z2. So, because the integration over these 2 parts they vanish we are left with only integration over these 2 surfaces because these is the entire volume under consideration V. It is bounded by the 4 metallic also the waveguide and these 2 surfaces at Z equal to Z1 and Z equal to Z2. So, we are left with integration only at these 2 planes at Z equal to Z1 and Z equal to Z2 in addition, we note that the waveguide modes are orthogonal. What does it mean?

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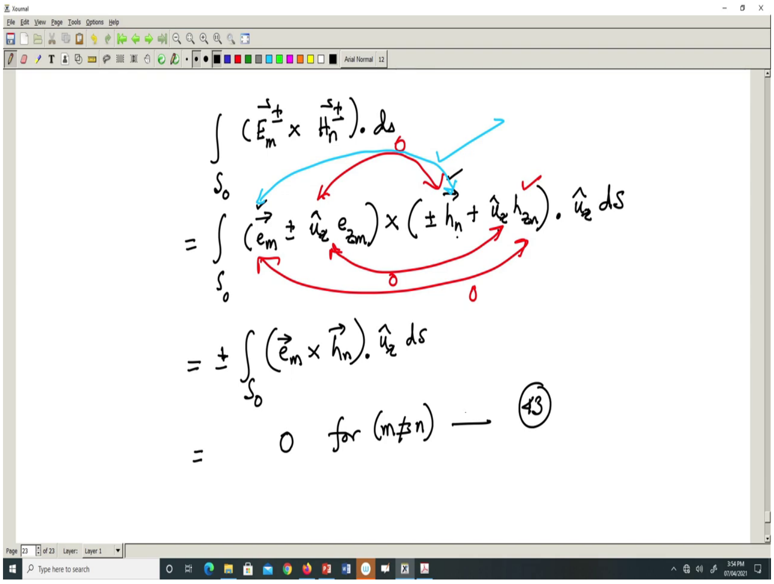

It means consider any transverse cross section of the waveguide if I integrate **Em** plus minus and **Hn** plus minus over ds. So, this can be written as So again in terms of the transverse component and the longitudinal component. So, **em** plus minus **uz** ezm cross plus minus **hn** plus **uz** hzn dot **uz** ds. Now, out of this **em** cross **hn** this term has so, **em** cross **hm** is pointing along the **uz** direction it is pointing along **uz** direction so, when it is so, **em** cross **hn** is pointing along the **uz** direction.

So, when we perform the dot product with respect to **uz** this term which is **em** cross **hn** will survive but **em** cross **uz hzn** will point either along the X or Y direction and when we perform the dot product with respect to **uz** this term is going to yield 0. Because em cross hn is pointing along either X or Y. Similarly, ezm cross hn is either pointing along the X or Y. So when I perform the dot product with **uz** this term is going to vanish. So, the product of this and this the product of this and this is going to vanish the product of this and this is going to vanish and **uz** cross **uz** is also going to vanish because we are pointing along the same direction.

So, two vectors pointing along the same direction if I form the cross product, it is going to vanish by the rules of cross product. So, then this product will also vanish. So, therefore, we are left with only this product. This is going to stay intact. However, if m is not equal to n. So, this can written as plus minus So **em** cross **hn** dot **uz** ds. So, this is the only surviving term. However, if m is not equal to n then this cross product will always yield 0 for m not equal to n, we call this equation 43.

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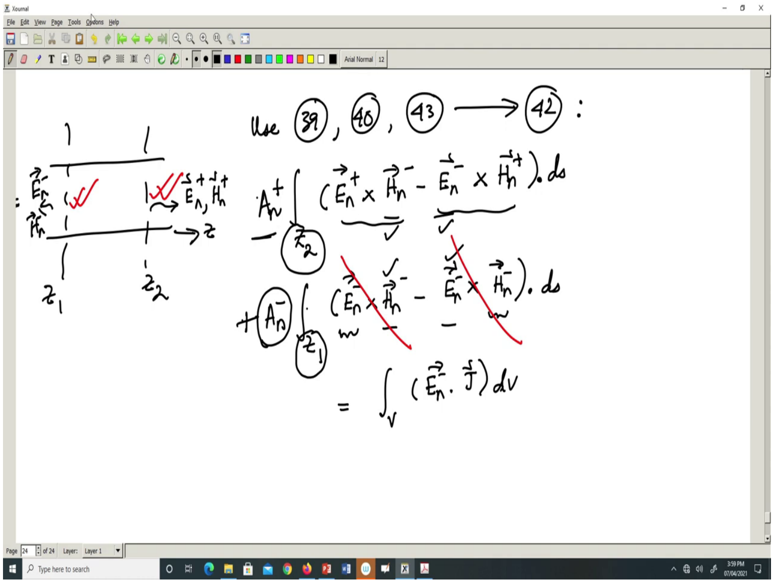

Now, we use 39, 40 which are the equations for the electric and magnetic field the scattered electric and the magnetic field moving along the plus Z and the minus Z directions respectively and we use 43 and we use them in equation 42. So, what happens in the surface Z2 again I draw this. So, this is my surface Z2 this is my surface Z1 this is my surface Z we have already shown that the integration is only restricted to this surface and this surface over Z1 and Z2. So, therefore, over the surface Z2 I have **An** plus Z2 **En** plus cross **Hn** minus minus **En** minus cross **Hn** plus dot ds.

So, if we look at equation 42 this was **E** plus minus. So, because I am forming the cross product with each **Hn** minus so, only **En** plus is going to be surviving because it is over the surface Z2. So, the relevant scattered mode is the forward propagating wave in this surface. So, that is **En** plus here. So, that becomes **En** plus cross **Hn** minus it was already there cross minus **En** minus **En** minus was already there cross **Hn** plus minus in equation 42.

So, **Hn** plus minus becomes **Hn** plus because we are considering the surface at Z2 in which I have **En** plus **Hn** plus moving around the positive opposite direction. Then, we have **An** minus we consider the surface Z1. So, we have **En** minus cross **Hn** minus minus **En** minus cross **Hn** minus dot ds. This **Hn** minus and this **En** minus are as before like this one this one this one and this one that the same.

Now, because **E** plus minus is the scattered field at Z equal to Z1. We have the backward wave **E** we have the backward wave **En** minus as the scattered field. So, that is how the **E** plus minus in equation 42 becomes **En** minus. Similarly, the **H** plus minus in equation 42 because we are considering the surface Z1 becomes **Hn** minus because the relevant field amplitudes are **En** minus and **Hn** minus here.

Now, we see that these two terms neatly cancel each other. So, that must be equal to the right hand side of equation 42 which is **En** minus dot **J** dv. So, now we expand these terms the remaining terms involving **An** plus. So, you see that the backward wave amplitude **An** minus the terms involving **An** minus totally vanishes out. So, we are left only with **An** plus.

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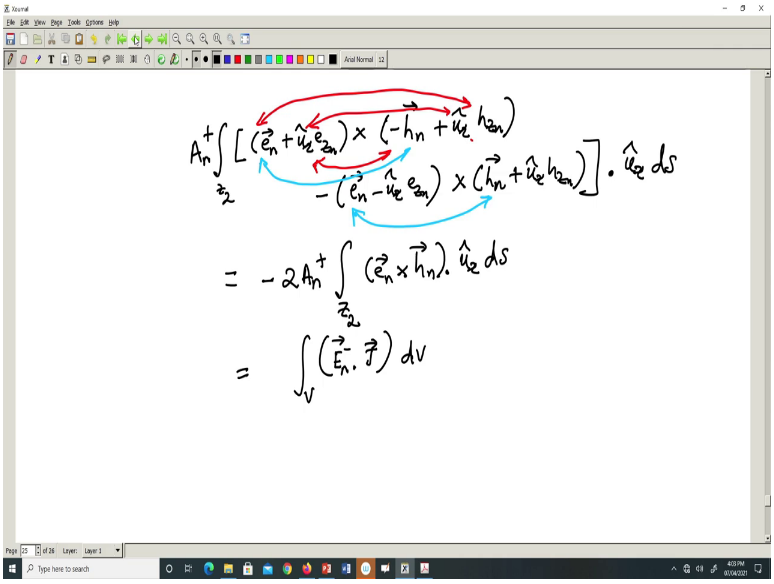

So, therefore, we have An plus integration over Z2 **en** plus **uz** ezn this is for **en** plus **en** plus **uz** ezn then we have **hn** minus we just substitute for what we wrote previously. So, that is **hn** minus so, that is minus **hn** plus **uz** hzn then we have minus **en** minus so, that becomes **en** minus **uz** ezn cross **hn** plus **hn** plus becomes **hn** plus **uz** hzn dot **uz** ds.

So, that becomes equal to again when we perform this times this it will vanish because these times **uz** hzn will yield a component along the X or Y direction and then when we are performing the dot product with **uz** that term will vanish. Similarly, this will lead to a product along the X or Y direction and then that when we perform the dot product with **uz** will finish.

Similarly, **uz** cross **uz** term or these two terms cross product between these two terms will also vanished because they are pointing along the same direction. So, the only surviving term will be these two terms **en** cross **hn** or minus **en** cross **hn.** Similarly for here the only surviving term will be minus **en** cross **hn**. So, therefore, this will be equal to minus 2 **An** plus over Z2 **en** cross **hn** dot **uz** ds and then that is equal to the right hand side volume integral V **En** minus dot **J** dv. So, from here An the and from here An plus the amplitude of the forward going harmonic can be easily evaluated.

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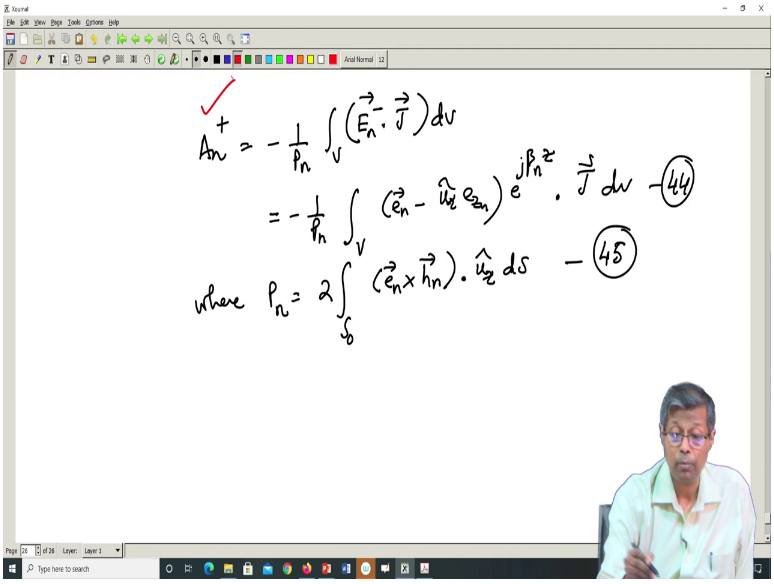

So, An plus becomes equal to minus 1 by Pn integral over the volume V **En** minus dot **J** dv. So, that is equal to minus 1 by Pn integral over the volume V **en** minus **uz** ezn which is **en** minus times $e^{-j\beta_n z}$ dot **J** dv. So, let us call this equation 44 where Pn is twice integral over the cross sectional surface is So **en** cross **hn** dot **uz** ds. So, let us call this 45.

So, this compares the derivation of the forward moving n^{th} harmonic radiated by the current source **J** inside the waveguide. So, we consider test mode which is going along the riverse Z direction as the second set of fields with the first set of fields being the fields radiated by the current source inside the waveguide considering the negative going second set of fields or the test mode, we found the amplitude of the forward going wave An plus the terms involving the backward moving harmonics An minus they cancel out.

So, next we are going to see how we find out the amplitudes of the negative moving harmonics. So, the negative going harmonics An minus and also the fields the and also the positive and negative going modes radiated by a magnetic current source. We will continue further. Let us, stop here.