Advanced Microwave Guided-Structures and Analysis
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Lecture 59
Cylindrical Wave Functions

So, till now we had dealt with essentially the rectangular coordinate system, the Eigen functions in the rectangular waveguide. We had a detailed study of the rectangular waveguide. So now, we are going to investigate the cylindrical waveguide or the circular waveguide that is more popularly called and the circular cavity but before that we have to have a general idea of the cylindrical wave functions that are used to describe the wave behavior in the circular waveguide domain.

So, the characteristics of the cylindrical wave functions, the types of cylindrical wave functions that are used, their properties so, all of them needs to be carefully studied in order to choose the appropriate wave functions for the cylindrical coordinate system. So, let us go to the lecture on this.

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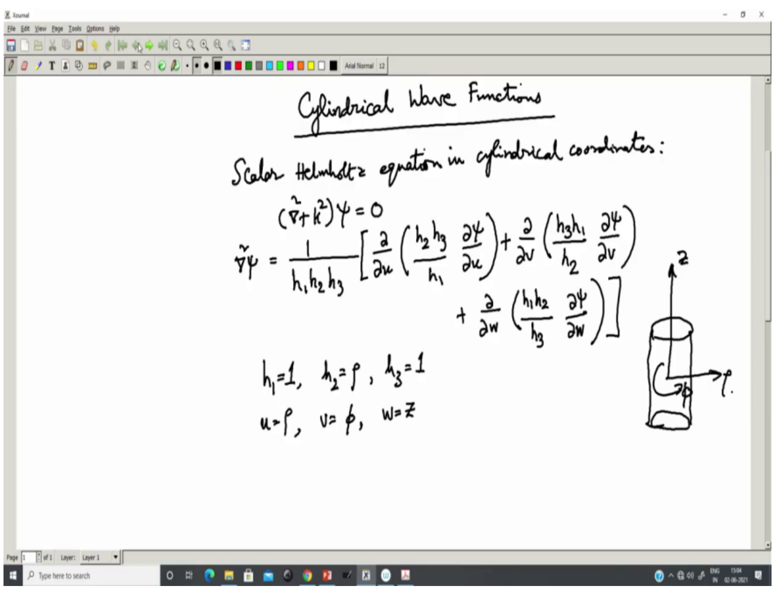

So, these are the cylindrical wave functions. So, just like in the planar domain or in the rectangular domain, we began with the Helmholtz equation. The solution to the Helmholtz equation or, appropriate to say the source free Helmholtz equation or the wave equation gives rise to the Eigen functions in the rectangular coordinate domain and then the Eigen values that are appropriate are chosen in order to satisfy the given boundary conditions in the rectangular coordinate system. In a similar way, we will study with the scalar Helmholtz equation in the cylindrical coordinate system.

So, the scalar Helmholtz equation in the cylindrical coordinates reads as follows $\left(\nabla^2 + k^2\right)\psi = 0$. And then we have to express $\nabla^2\psi$ in the cylindrical coordinate system. So, that is what exactly we proceed to do. So, $\nabla^2\psi$ is given by $\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_1 h_3}{h_2}\frac{\partial \psi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial w}\right)\right]$.

So, where h1 is 1, h2 is rho of the cylindrical coordinate system h3 is equal to 1 and u equal to rho v equal to phi and w is z.

So, ρ, ϕ, z being the coordinates of the cylindrical system. So, we are all familiar with the cylindrical system, this is a cylinder. So, this is the z -axis, this is the rho direction and this is the azimuthal direction phi. So, we are all familiar with this cylindrical coordinate system.

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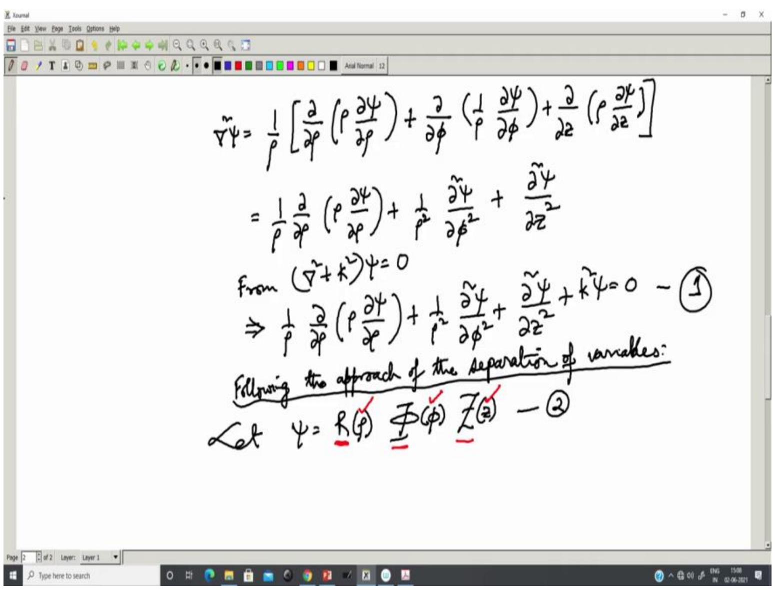

So, now, what we do is that substituting these values we get $\nabla^2\psi$ equal to $\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \psi}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial \psi}{\partial z}\right)\right]$. That is pretty much simple to understand.

So now, from the equation $\left(\nabla^2 + k^2\right)\psi = 0$ or the source free Helmholtz equation we can write this as that will imply $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial z}\right)+k^2\psi = 0$. So, this is the Helmholtz equation in the cylindrical coordinate system which we will try to solve using the separation of variables which is the same technique applied for the finding the relationship between the wave numbers and solving the rectangular Helmholtz equation in the rectangular coordinate system.

So, following the method of the separation of variables, we let $\psi = R(\rho)\Phi(\phi)Z(z)$. So, we break up ψ into three distinct functions one is $R(\rho)$ which is a function of only ρ, big phi a function of phi only and capitals Z which is a function of z only. So, this is the distinct decomposition of ψ. So, we call this equation 2. All we have to do is substitute 2 into 1. Because that was a purpose we broke up psi into distinct functions of R, capital phi or big phi and capital Z.

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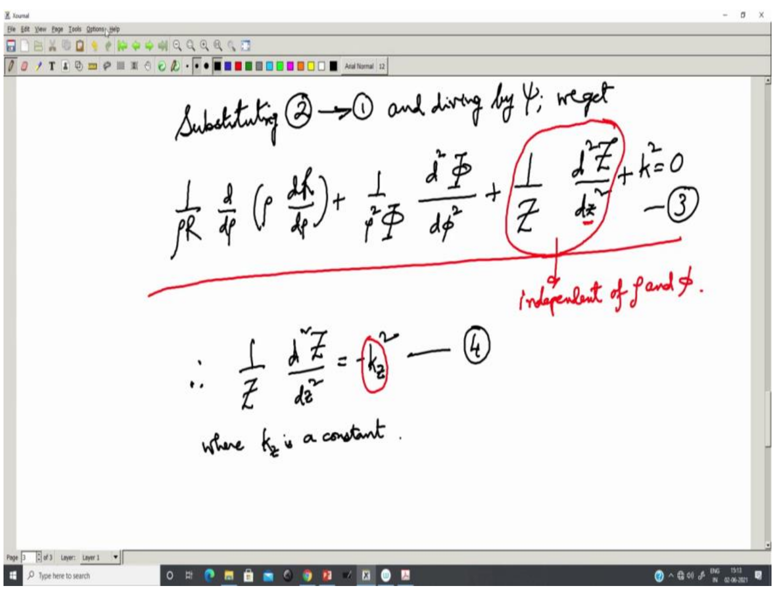

So, substituting 2 into 1 and dividing by ψ, finally get $\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}+k^2 = 0$. So, again this is capital Z which is a function exclusively of this small z and that is capitals Z double differentiation of capitals Z with respect to small z.

So, now so, we call this equation 3. So, we note that the third term which is this, it is independent of rho and phi. So, this is independent of and is a function only of small z because capital Z is a function of a small z only. So, therefore it must also be independent of z if this equation has to sum up to 0 for all rho, phi and z. I repeat that once again because this term is independent of rho and phi.

So, this must also be independent of z. If equation number 3 is to hold good and the left hand side is going to or sum up to 0, the left hand side sums up to 0 for all rho phi and z. So, therefore we can write. Therefore, $\frac{1}{Z}\frac{d^2Z}{dz^2}$ can be written as $-k_z^2$. This is 4 where this is our constant.

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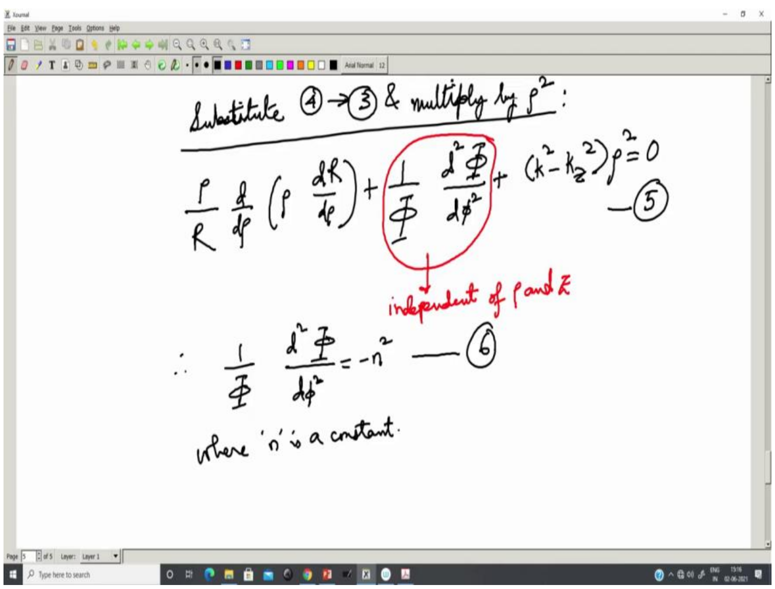

Now, we substitute 4 into 3 and multiply by rho squared. If we do that, we will obtain this equation $\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+\left(k^2-k_z^2\right)\rho^2=0$. We call this equation 5. Now, we note that the second term is independent of rho and z and the other terms are independent of phi.

So, this is independent of rho and z and other terms are independent of phi and therefore, $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$ must be equal to minus n square again where n is a constant because otherwise the left hand side cannot add up to 0. So, we call this equation 6.

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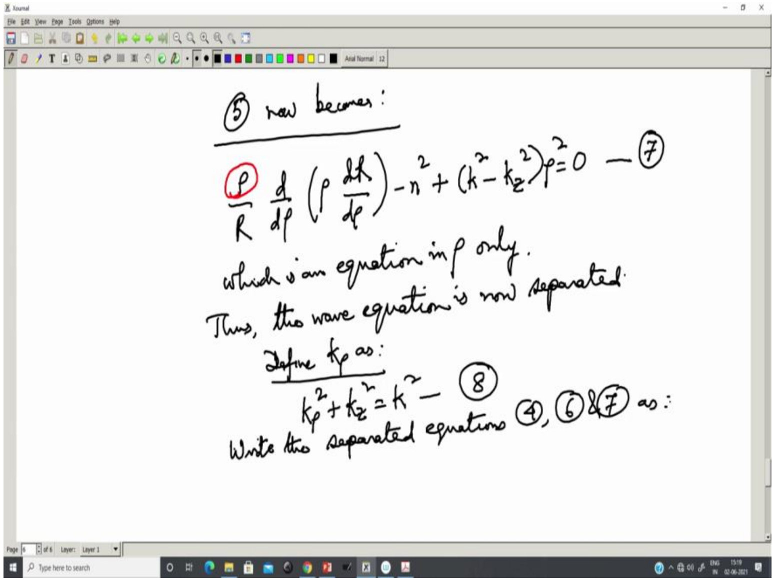

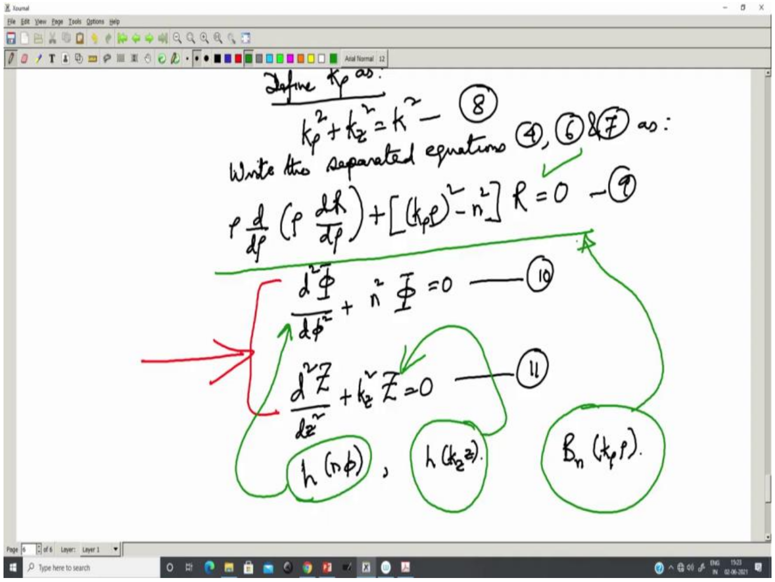

Using equation 6, 5 can be written as $\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)-n^2+\left(k^2-k_z^2\right)\rho^2=0$, plain and simple algebra. Call this equation 7 which is an equation in rho only. So, this is the equation in rho only there is no phi, there is no z. So, we write this is an equation in rho only.

So, thus the wave equation is separated. So, thus the wave equation or the Helmholtz equation, a sourcefree Helmholtz equation is now separated and we can define $k_\rho^2 + k_z^2 = k^2$. We call this equation number 8. After we do this we write the separated equations 4, 6 and 7 as $\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \left[\left(k_\rho \rho\right)^2 - n^2\right] R = 0$, rewritten we call that equation 9 and $\frac{d^2\Phi}{d\phi^2} + n^2\Phi = 0$ that comes from equation 6, we call this equation number 10.

And we also write $\frac{1}{Z}\frac{d^2 Z}{dz^2} + k_z^2 = 0$ that comes from equation number 4; we call this equation number 11. Now, we note that equation number 10 and 11 dealing with capital phi and capital Z, they are harmonic equations and therefore, they would be satisfied by harmonic functions. These harmonic functions we generally denote by $h(n\phi)$ and $h(k_z z)$. So, $h(n\phi)$ would satisfy equation number 10 and $h(k_z z)$ is going to satisfy equation number 11.

The R equation i.e. this equation is Bessel equations of order n solutions of which can be denoted by the term $B_n\left(k_\rho \rho\right)$. So, the R equation is Bessel equation of order n. So, this is the R equation which is Bessel equation of order n and the solutions will be denoted in general by $B_n\left(k_\rho \rho\right)$. So, we will denote the solutions to equation 9 by $B_n\left(k_\rho \rho\right)$. So, these will be generic solutions to equation number 9. Now, commonly used solution to Bessel's equation R.

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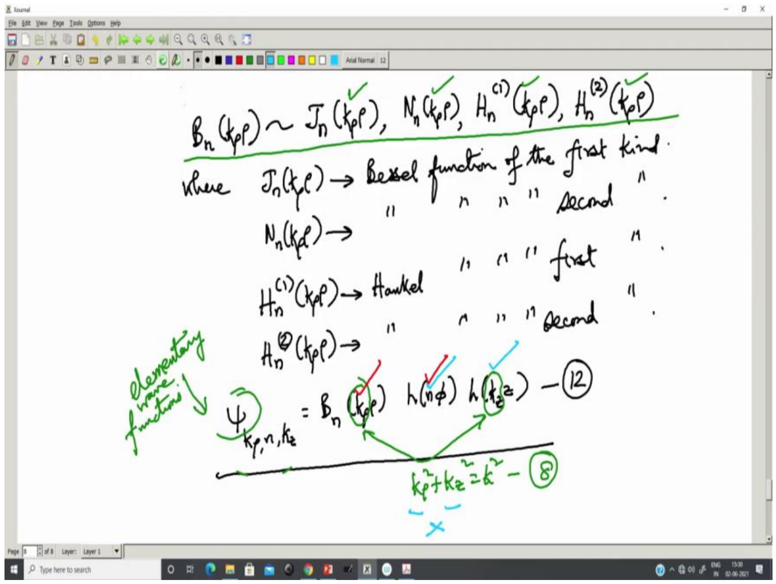

Let us write down the common solutions to the Bessel's equation $B_n\left(k_\rho \rho\right)$. It can take the form of $J_n\left(k_\rho \rho\right)$ which is the Bessel function of the first kind or $N_n\left(k_\rho \rho\right)$ which is the Bessel function of the n^th^ order of the second kind. It can take the form of $H_n^1\left(k_\rho \rho\right)$ which is the Henkel function of the n^th^ order of the first kind and $H_n^2\left(k_\rho \rho\right)$ which is the Henkel function of the second kind.

So, according to the equation 2 where we wrote ψ as a combination of three distinct functions of $R(\rho)\Phi(\phi)Z(z)$.

We can now write the same equation or the same solution in this form which is ψ_{k_ρ,n,k_z} equal to $B_n\left(k_\rho \rho\right)h(n\phi)h(k_z z)$ which we call equation number 12. So, these form the solutions to the Helmholtz equation in the cylindrical coordinate system. So, as a product of the two harmonic functions $h(n\phi)$ and $h(k_z z)$ in addition with this function $B_n\left(k_\rho \rho\right)$ which can take all these different forms.

So, which form it will take which form the function $B_n\left(k_\rho \rho\right)$ will take will depend on which region we are interested in to describe the fields in the cylindrical coordinate system. Does the region include the point rho is equal to infinity? Does it include the point rho equal to 0? Does it refer to any other kind of boundaries which are characteristic of any cylindrical region like a dielectric boundary or a metallic boundary?

So, based on these factors the form of $B_n\left(k_\rho \rho\right)$ is going to be decided whether it is going to be either of these four forms. In addition we must remember that k_ρ, n, kz, this k_ρ and kz they interrelated by equation number 8. So, they are interrelated by this equation which is $k_\rho^2 + k_z^2 = k^2$ which we had called equation number 8 previously.

So, we call this ψ_{k_ρ,n,k_z} elementary wave functions so, these are called elementary wave functions. Now, we must also remember that linear combinations of the elementary wave function are also solutions to Helmholtz equation. So, we can sum over possible values of n and k_ρ just like in the rectangular coordinate system we sum up over the values of kx, ky like that. Like in the rectangular waveguide we saw that we sum up over the values of m and n from 0 to infinity.

So, here also we can sum up over the possible values or eigen values of n and k_ρ or n and kz. But we cannot sum up over k_ρ and kz because they are interrelated. So, we can sum up over these two eigen numbers or eigen values or we can sum up over these two eigen values but we cannot sum up over k_ρ and kz because they are interrelated.

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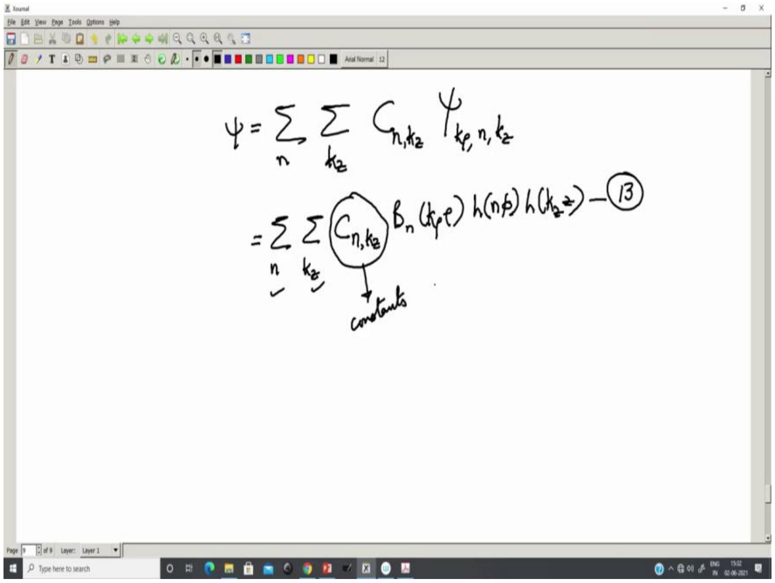

For example, we can write ψ as $\psi = \sum_{n}\sum_{k_z} C_{n,k_z}\psi_{k_\rho,n,k_z} = \sum_{n}\sum_{k_z} C_{n,k_z} B_n\left(k_\rho \rho\right) h\left(n\phi\right) h\left(k_z z\right)$ which is equation 13. So, let us stop here we will continue from here.