Advanced Microwave Guided-Structure and Analysis

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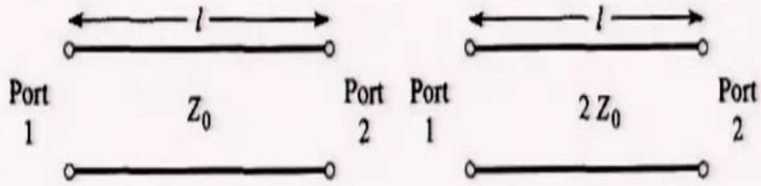
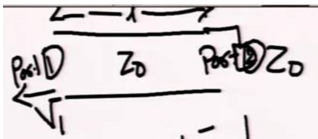
Lecture No. 06

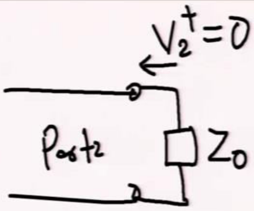
Scattering Matrix Concepts Tutorials (Contd.)

3. Derive the scattering matrix for each of the lossless transmission lines shown below, relative to a system impedance of Z_0. Verify that each matrix is unitary.





$$S_{11} = \left. \frac{V_1^-}{V_1^+} \right|_{V_2^+=0}$$



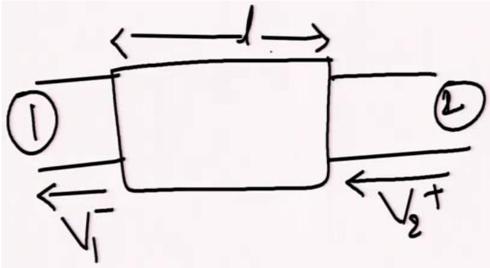
$$S_{11} = 0$$

$$S_{22} = \left. \frac{V_2^-}{V_2^+} \right|_{V_1^+=0}$$

$V_2^- = 0$

$$S_{22} = 0$$

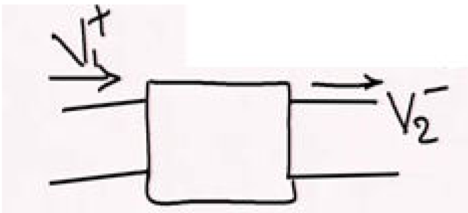
$$S_{12} = \left. \frac{V_1^-}{V_2^+} \right|_{V_1^+=0}$$



$$S_{12} = e^{-j\beta l}$$

$$S_{21} = \left. \frac{V_2^-}{V_1^+} \right|_{V_2^+=0}$$

$$= e^{-j\beta l}$$

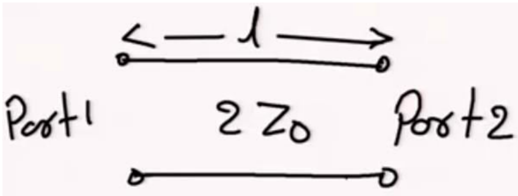


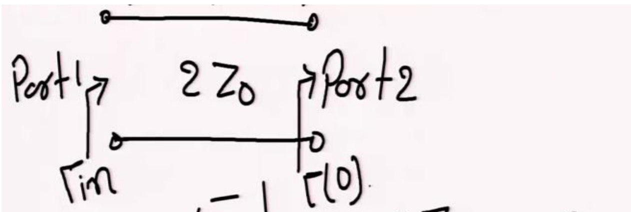
So, we have got $S_{11} = S_{22} = 0$ and $S_{12} = S_{21} = e^{-j\beta l}$.

Scattering Matrix can be written as,

$$[S] = \begin{bmatrix} 0 & e^{-j\beta l} \\ e^{-j\beta l} & 0 \end{bmatrix}$$

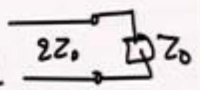
After this we can go to the second configuration





$$S_{11} = \frac{V_1^-}{V_1^+}\bigg|_{V_2^+=0} = \Gamma_{in} = S_{22}$$

$$\Gamma(0) = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_0 - 2Z_0}{Z_0 + 2Z_0} = \frac{-Z_0}{3Z_0} = \frac{-1}{3}$$



$$Z_{in} = Z_0 \frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-2j\beta l}} = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}$$

$$Z_{in} = 2Z_o \frac{1 + \Gamma(0)\, e^{-2j\beta l}}{1 - \Gamma(0) e^{-2j\beta l}}$$

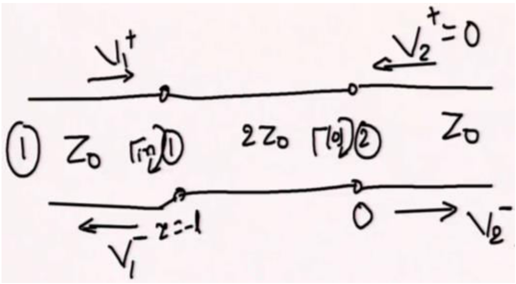
$$= 2Z_o \frac{1 - \frac{1}{3}\, e^{-2j\beta l}}{1 + \frac{1}{3} e^{-2j\beta l}}$$

And from this Z_{in} we can find out Γ_{in}

$$\Gamma_{in} = \frac{Z_{in} - Z_o}{Z_{in} + Z_o}$$

$$= \frac{2\left(1 - \frac{1}{3} e^{-2j\beta l}\right) - \left(1 + \frac{1}{3} e^{-2j\beta l}\right)}{2\left(1 - \frac{1}{3} e^{-2j\beta l}\right) + \left(1 + \frac{1}{3} e^{-2j\beta l}\right)}$$

$$\Gamma_{in} = \frac{1 - e^{-2j\beta l}}{3 - \frac{1}{3} e^{-2j\beta l}} = S_{11} = S_{22}$$



Now, we have to find out S_{21} and S_{12}.

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+ = 0}$$

$$V_1^+ + V_1^- = V_1^+(1 + \Gamma_{in}) = V^+ e^{j\beta l} - \frac{1}{3} V^+ e^{-j\beta l}$$

$$V_2 = V_2^+ + V_2^- = V_2^- = V^+\left(1 - \frac{1}{3}\right)$$

we are finding out for S_{21},

$S_{21} = \frac{V_2^-}{V_1^+}$ when $V_2^+ = 0$

$$V_2^- = \frac{2}{3}V^+ = \frac{2}{3}\frac{V_1^+\left(1+\Gamma_{in}\right)}{e^{j\beta l} - \frac{1}{3}e^{-j\beta l}}$$

We have

$$\Gamma_{in} = \frac{1 - e^{-2j\beta l}}{3 - \frac{1}{3}e^{-2j\beta l}}$$

$$\frac{V_2^-}{V_1^+} = \frac{\frac{2}{3}\left(1 + \frac{1 - e^{-2j\beta l}}{3 - \frac{1}{3}e^{-2j\beta l}}\right)}{e^{j\beta l} - \frac{1}{3}e^{-j\beta l}}$$

$$= \frac{8}{3}\frac{e^{-j\beta l}}{3 - \frac{1}{3}e^{-2j\beta l}} = S_{21} = S_{12}$$

Now, to satisfy the unitary condition,

$$|S_{11}|^2 + |S_{21}|^2$$
$$= \left|\frac{1 - e^{-2j\beta l}}{3 - \frac{1}{3}e^{-2j\beta l}}\right|^2 + \frac{64}{9}\cdot\frac{1}{\left|3 - \frac{1}{3}e^{-2j\beta l}\right|^2}$$

$$= \frac{\left|1 - e^{-2j\beta l}\right|^2 + \frac{64}{9}}{\left|3 - \frac{1}{3}e^{-2j\beta l}\right|^2}$$

$$= \frac{\left(1 - e^{-2j\beta l}\right)\left(1 - e^{2j\beta l}\right) + \frac{64}{9}}{\left(3 - \frac{1}{3} e^{-2j\beta l}\right)\left(3 - \frac{1}{3} e^{2j\beta l}\right)}$$

$$= \frac{1 - e^{-2j\beta l} - e^{2j\beta l} + 1 + \frac{64}{9}}{9 - e^{2j\beta l} - e^{-2j\beta l} + \frac{1}{9}}$$

$= 1$ Unitary