Advanced Microwave Guided-Structures and Analysis
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Lecture 60
Cylindrical Wave Functions (cont.)

So, we have been discussing regarding the cylindrical wave function and solution to Helmholtz equation, the source free Helmholtz equation in the cylindrical coordinate system, the solution in terms of ρ , ϕ and z and what are the characteristics of the so called solutions which satisfy the Helmholtz equation and we came to the point where the ψ can be expressed as a summation over the azimuthal wave number n and k_z and summed up over the elementary wave functions. So, let us continue from there.

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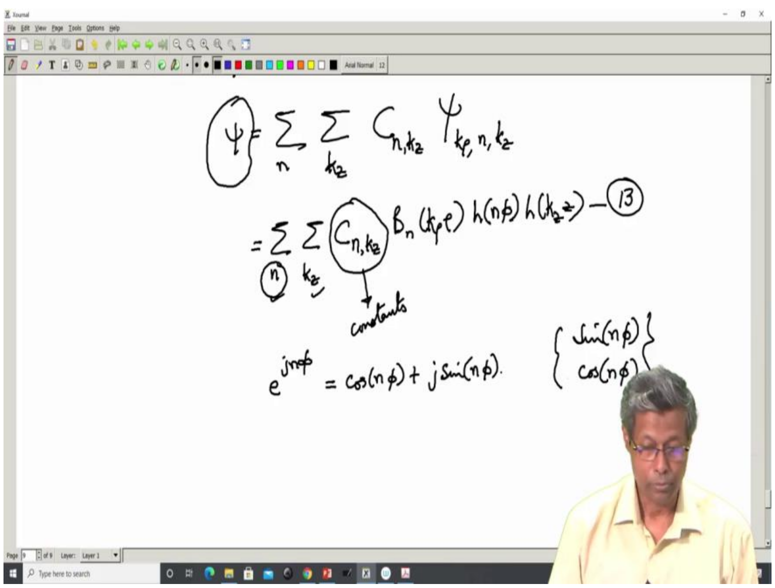

We can also integrate over the separation constants though the value of n is usually discrete, this n is usually discrete. So, we will come to that later when we will find that equation number 10 will be satisfied by functions which are in the form of $e^{jn\phi}$ which is $\cos(n\phi)+j\sin(n\phi)$. So, it will be usually expressed in terms of the cosine and sine terms or a combination of $\sin(n\phi)$ and $\cos(n\phi)$ terms. So, the value of n will usually be discrete.

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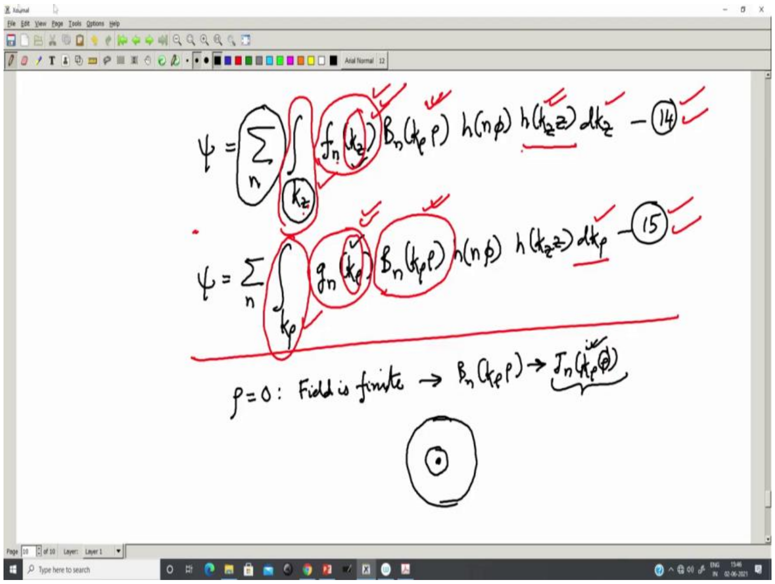

So, now, we can write the possible solution to Helmholtz equation as $\sum_{n}\int_{k_z} f_n(k_z)B_n(k_\rho\rho)h(n\phi)h(k_z z)dk_z$ which is equation 14. So, these can also be possible solutions to Helmholtz equations and here we integrate over the separation constants because n is discrete we express the summation over n and express the summation over kz it in terms of the integral over kz.

So, we can also express ψ as $\sum_{n}\int_{k_\rho} g_n(k_\rho)B_n(k_\rho\rho)h(n\phi)h(k_z z)dk_\rho$. So, these kinds of expressions are also possible because we appreciate the fact that k_ρ and kz these two terms are ultimately inter related through the equation $k_\rho^2 + k_z^2 = k^2$ which was equation number 8.

So, in many cases these kinds of alternate interpretations are very-very important because in many cases of the cylindrical coordinates, we will find the integration over kz or integration over k_ρ poses different features. In many cases, one formulation can be referred to as the Sommerfeld formulation and another formulation can be referred to as the Schelkunoff formulation and these Sommerfeld and Schelkunoff formulation corresponding to integration over kz or k_ρ can have distinct different features when we plot the integrand over kz or k_ρ or when we compute the integral due to different generic forms of $f_n(k_z)h(k_z z)$ and $g_n(k_\rho)h(k_z z)$ in addition to $B_n(k_\rho \rho)$.

So, these functions are combinations of the function $f_n(k_z)h(k_z z)$ which in this integration kernel is a function of kz. Integration of this function over kz can have distinctly different features then integration over this kernel which is $g_n(k_\rho)h(k_z z)$ integrated over k_ρ that is why these alternate formulations are very important in the cylindrical domain. At this point we all realize their importance.

So, these integrations can be over any contour in the complex plane and $f_n(k_z)$ and $g_n(k_\rho)$ are functions to be determined from the boundary conditions. Now, you might ask me suddenly that where these contour integrations come because it appears that dk_z, dk_ρ are real numbers.

The concept of contour integration comes because for this integration path which is $\int_{k_\rho} f_n(k_z)B_n(k_\rho \rho)h(n\phi)h(k_z z)dk_z$ there can be singularities of these functions in terms of poles or in terms of more complex singularities in that form of branch points which we are not going to discuss at this level.

So, because in the path of integration of equation 14, when we are integrating over k_z we will encounter poles of $f_n(k_z)h(k_z z)$. We have to deform the path of integration and generalize this integration real axis integration of kz into the complex plane and thereby the concept of contour integration comes. So, we are going to go to the complex plane and integrate over 14 and 15. So, we would not go further in that direction because it is going to lead us to more advanced treatment of cylindrical wave functions which are not really part of this course.

Now, on the other hand, we are going to focus on the nature of $B_n(k_\rho \rho)$ and when to use different forms of $B_n(k_\rho \rho)$. We already said that $B_n(k_\rho \rho)$ can have four distinct forms $J_n(k_\rho \rho)$ and $H_n^2(k_\rho \rho)$ if the field is finite. At ρ equal to 0 if the field is finite $B_n(k_\rho \rho)$ must be $J_n(k_\rho \rho)$. Because all other functions lead to non computable values.

It can be extremely large values in the negative sense. But, they have a singular behavior that is $N_n\left(k_\rho\rho\right), H_n^1\left(k_\rho\rho\right), H_n^2\left(k_\rho\rho\right)$ and will have non-computable singular values as their argument as the as their argument $k_\rho\rho$ becomes 0. Because we are dealing with the spatial location we are not concerned with the behavior of k_ρ here we our concern is over the rho variable.

So, what happens to these functions as rho changes? If at rho equal to 0, on the centre of the cylindrical coordinate system the field is finite, in order to describe this finite behavior the appropriate wave function along the radial direction must have to be $J_n\left(k_\rho\rho\right)$ and not otherwise. This has to be very-very carefully born in mind.

Because inappropriate choice of functions is absolutely not acceptable anywhere. So, we must absolutely know the proper recipe to use or the proper function to use to describe the wave behavior under the different circumstances or in the different spatial locations.

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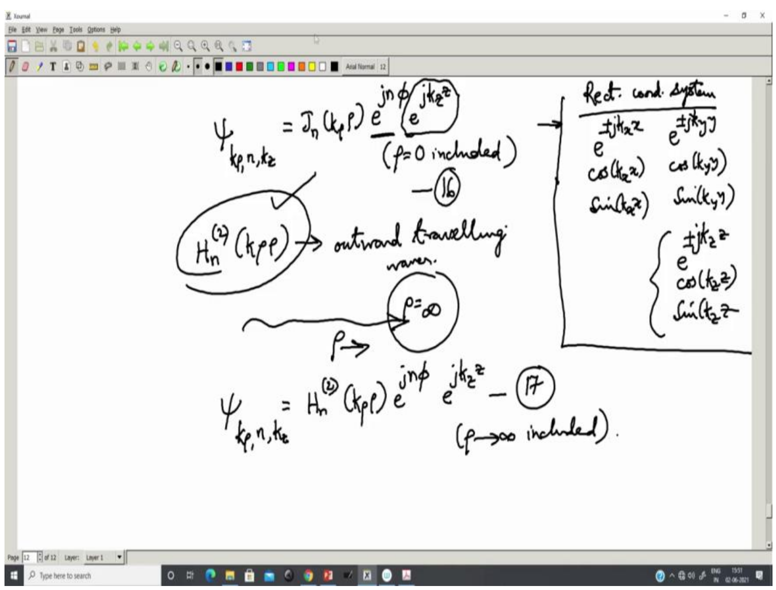

So, in this case, the elementary wave functions will be of the form ψ_{k_ρ,n,k_z} equal to $J_n\left(k_\rho\rho\right)e^{jn\phi}e^{jk_z z}$ with rho equal to 0 included. We have already encountered this function before as a solution to equation number 10 and we note that equation number 11 is pretty much of the same form as equation number 10.

So, solution is going to be $e^{jk_z z}$. This equation number 11 is also of the similar type which we meet in the rectangular wave decomposition of the psi function when we apply the separation of variable to the psi function or which describes the Helmholtz equation in the rectangular coordinate system. We come across exactly the same equation in the z-direction and the x-direction and the y-direction as equation number 11.

And solution is of these form. We are always aware of this because the solution in the rectangular wave coordinates is either $\sin(k_x x), \cos(k_x x)$ or $e^{jk_x x}$. So, we just recall or compare with the rectangular coordinate system where the solutions could be $e^{\pm jk_x x}, \sin(k_x x), \cos(k_x x)$ in the x direction. It can be $e^{\pm jk_y y}, \sin(k_y y), \cos(k_y y)$ in the y-direction or it can be $e^{\pm jk_z z}, \sin(k_z z), \cos(k_z z)$ in the z-direction.

So, this is the description of the rectangular coordinate system. So, we already know the origin of this function from the equation which is exactly similar to equation number 11 which we discussed here. So, we call this equation number 16. So, note that this function on the other hand $H_n(k_\rho \rho)$ are the only solutions, which will vanish for large rho if k_ρ is complex and therefore, they will represent outward travelling waves that are going in this direction in increasing rho and include the region rho equal to infinity.

If our point of interest includes rho equal to infinity and has to describe outward travelling waves, then this is the only function that can be used to describe such behavior. So, they will represent outward travelling waves if k_ρ is real therefore, if there are no sources at infinity $B_n(k_\rho \rho)$ must be $H_n^2(k_\rho \rho)$, if rho equal to infinity or rho tends to infinity if this region is to be included.

Hence, in this case the ψ function will become ψ_{k_ρ,n,k_z} equal to $H_n^2(k_\rho \rho) e^{jn\phi} e^{jnk_z z}$ which is equation number 17. So, here rho tends to infinity. The region rho tends to infinity is included.

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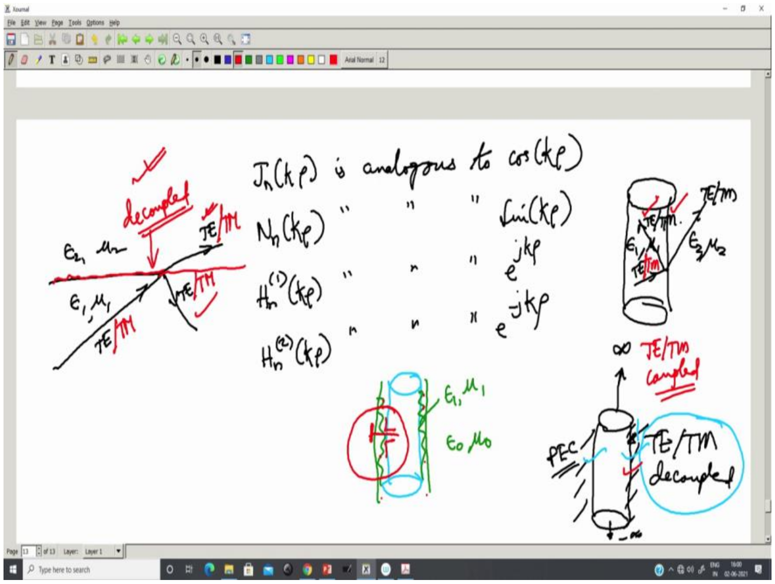

We can also make some qualitative analogies or qualitative comparisons for the cylindrical wave functions. For instance, $J_n\left(k_\rho \rho\right)$ is analogous to $\cos\left(k_\rho \rho\right)$, $N_n\left(k_\rho \rho\right)$ is analogous to $\sin\left(k_\rho \rho\right)$, $H_n^1\left(k_\rho \rho\right)$ is analogous to $e^{jk_\rho \rho}$ or incoming waves from infinity and $H_n^2\left(k_\rho \rho\right)$ is analogous to $e^{-jk_\rho \rho}$ outgoing waves.

So, now, we are armed with finding out the solution to the cylindrical wave equation in terms of the TE to z and TM to z modes. So, similar to the rectangular coordinate system where **A** is equal to $\hat{u}_z \psi$ the psi gives rise to the TM to z mode. And **F** equal to $\hat{u}_z \psi$ psi gives rise to the TE to z modes.

In this case, the same substitution made in the cylindrical coordinate systems will yield the same kind of modes namely the substitution **A** equal to $\hat{u}_z \psi$ in the cylindrical coordinate system will yield the TM to z cylindrical modes and the substitution **F** equal to $\hat{u}_z \psi$ is going to yield the TE to z modes.

However, the cylindrical interface problem is more involved than the rectangular problem because in a rectangular dielectric interface if this is an epsilon1 this is mu1, epsilon2, mu2 and incident TE wave in this direction is going to be reflected as our TE wave and transmitted as a TE wave. Similarly, an incident TM mode in this direction is going to be reflected as reflected TM mode and a transmitted TM mode.

So, this is the nature of the rectangular isotropic dielectric to dielectric boundary. So, this is a boundary between two dielectric medium. Let us continue this boundary. However the same is not true unfortunately for the cylindrical dielectric boundary making the cylindrical boundary far more complex than the rectangular boundary which means that if I have a cylindrical boundary here, so, if this is epsilon1 mu1 and this is epsilon2 mu2. The incident TE mode is going to give rise to a reflected TE mode as well as reflected TM mode and also a transmitted TE or TM mode and the similar situation will occur if this is TM mode. So, an incident TE mode incident at a dielectric to dielectric cylindrical boundary is going to yield reflected TE and TM types of modes giving rise to hybrid mode propagation in cylindrical waves or cylindrical dielectric structures.

However, we say that the modes TE and TM are decoupled at the planar or rectangular boundary whereas the modes TE and TM they are coupled at a cylindrical boundary. So, as a result of this if we want to characterize the cylindrical boundary in terms of reflection coefficients, we have to go towards the reflection matrix instead of the reflection coefficient here because the reflected field is described only in terms of the reflected TE/TM mode.

If you talk of the reflection coefficient here the question immediately arises are you talking of the reflected TE mode or the reflected TM mode. So, therefore, in order to describe this complex behavior for the cylindrical wave system or the cylindrical guided wave structures particularly cylindrical dielectric guided structure we have to take recourse to the reflection matrix instead of the reflection coefficient which characterize the planar medium.

So, these are the like the distinctions, inherent distinctions between the planar or the rectangular and the cylindrical like the dielectric to dielectric interface. So, now, before we do that we must also remember that in this case if this is our PEC boundary as we are interested in a PEC boundary in this course, if this boundary is a PEC and this is going to plus infinity and this is minus infinity the modes TE and TM are decoupled. They are decoupled. Because this boundary is our PEC boundary or a perfectly electric conductor boundary as in the case of the circular waveguide.

Therefore, in the circular waveguide the TE and TM modes are decoupled because this boundary this boundary this boundary and this boundary the whole thing that mean is the single boundary. The whole thing is metallic. In such a case, for the cylindrical coordinate system the TE and TM cases will be decoupled.

But, if the same circular waveguide is coated by a dielectric layer, so, if this is epsilon1 mu1 and if this is ε_o, μ_o in the dielectric layer inside the dielectric layer the modes will be coupled.

So, this will be a very practical problem. If I want to have, for instance a conformal patch here and fed by a coaxial probe from here. So, the inner conductor of the coaxial probe will be touching the patch to a hole and the outer conductor will be shorted to this ground plane. So, these situations are going to arise in the design and rigorous analysis of all kinds of conformal antennas on cylindrical surfaces where regardless mode coupling has to be considered at the dielectric regions.

Otherwise, your analysis is going to be extremely inaccurate, it cannot describe the wave behavior in these regions. So, this is important. However, we are not going to go to such advanced treatments. We will be limited in our analysis to the circular waveguide. Before we go to the circular waveguide however, we have to as we said we have to describe the solutions the electric and magnetic fields for **A** equal to $\hat{u}_z\psi$ and **F** equal to $\hat{u}_z\psi$. So, which we will do in the upcoming future lecture. So, let us stop here. Thank you.