Advanced Microwave Guided-Structures and Analysis

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Lecture 61

Cylindrical Wave Functions (Contd.)

So, let us continue in this lecture for the cylindrical wave functions. And in this part we are going to see how the appropriate choice of the magnetic vector potential and the electric vector potential in terms of the z directed ψ function or the substitutions **A** equals to $\hat{u}_z\psi$ leads to TM to z mode in the cylindrical coordinate system. And what are the field distributions for those modes? And how the substitution **F** equal to $\hat{u}_z\psi$ leads to the TE to z modes? And what is the field distribution, electrical and magnetic field distributions, mathematical expressions for such fields?

This is going to directly lead to the solution of the circular waveguide where we will find that the modes inside the circular waveguide are indeed the TE to z and the TM to z types because of the very simple reason that those fields which we are going to find now, they are going to satisfy the boundary conditions at the conducting walls of the circular waveguide. So, therefore conceptually the treatment is not different from the rectangular waveguide. It is only the mathematical language that changes.

So, the whole cylindrical coordinate circular waveguide system proceeds in exactly the same way except the difference in the mathematical language. So, we are just now going to start our exploration of what are the fields of **A** equal to $\hat{u}_z\psi$ and **F** equal to $\hat{u}_z\psi$. Let us go to the lecture.

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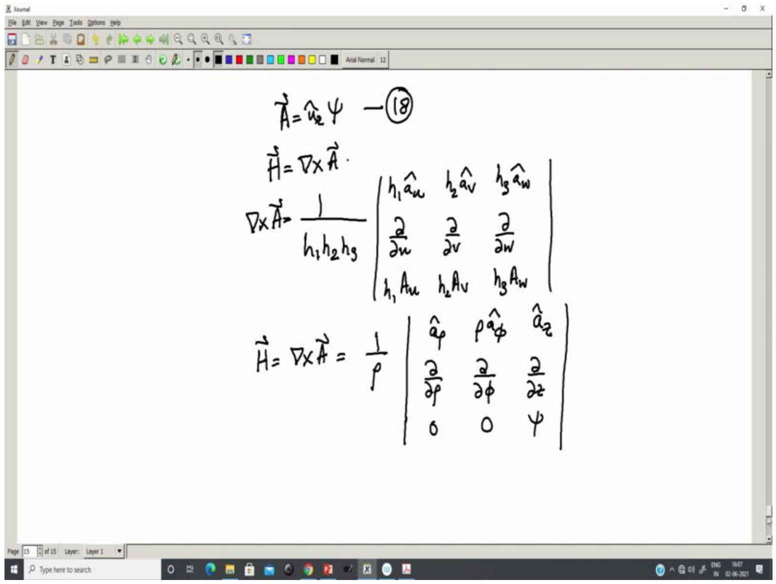

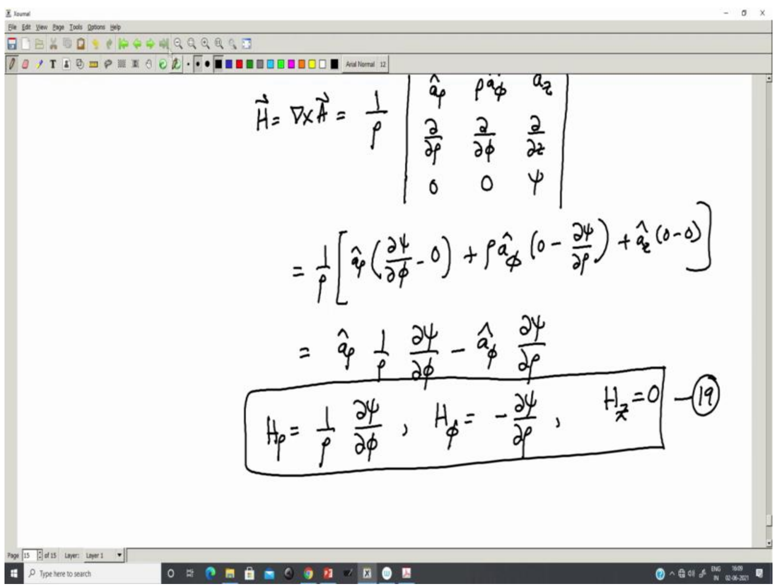

So, let us suppose that, first of all that **A** equal $\hat{u}_z \psi$. We call this equation 18. We know $\vec{H} = \nabla \times \vec{A}$. And $\nabla \times \vec{A}$ in the cylindrical coordinate system is $\frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{a}_u & h_2\hat{a}_v & h_3\hat{a}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 A_u & h_2 A_v & h_3 A_w \end{vmatrix}$. So therefore, $\vec{H}$ given by $\nabla \times \vec{A}$ becomes $\frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$.

Because ψ has only z component, not new to us; similar treatment we had done in the rectangular coordinate system. And this is equal to $\frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$. So, that becomes $\frac{1}{\rho}\left[\hat{a}_\rho\left(\frac{\partial\psi}{\partial\phi}\right) - \rho\hat{a}_\phi\left(\frac{\partial\psi}{\partial\rho}\right)\right]$. So, H_ρ is given by $\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}$. H_ϕ is given by $-\frac{\partial\psi}{\partial\rho}$ and H_z is given by 0. That we can easily deduce from the above equation. So, we call this equation or sets of equation for the magnetic fields, equation number 19.

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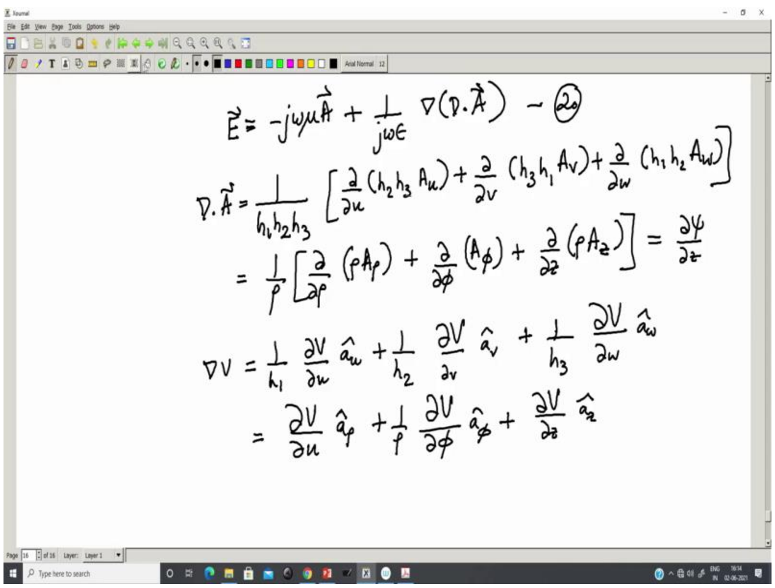

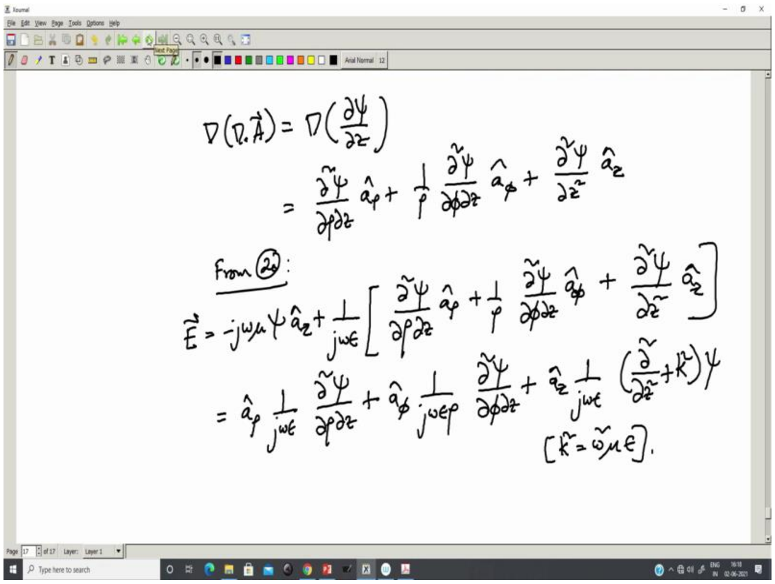

Similarly, for the electric field we encountered this equation in the rectangular coordinate system, $\vec{E} = -j\omega\mu\vec{A} + \frac{1}{j\omega\varepsilon}\nabla\left(\nabla \bullet \vec{A}\right)$. And now we are going to express the same thing in the cylindrical wave system, or a cylindrical coordinate system. So, we call this equation 20.

So now, $\nabla \bullet \vec{A}$ in the cylindrical coordinate is $\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(h_2 h_3 A_u\right) + \frac{\partial}{\partial v}\left(h_3 h_1 A_v\right) + \frac{\partial}{\partial w}\left(h_1 h_2 A_w\right)\right]$.

That is equal to $\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho A_\rho\right) + \frac{\partial}{\partial \phi}\left(A_\phi\right) + \frac{\partial}{\partial z}\left(\rho A_z\right)\right]$. That is equal to $\frac{\partial \psi}{\partial z}$, as **A** has only Az component.

Now grad of V in the cylindrical coordinate system where V is the scalar is given by $\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u}\widehat{a}_u + \frac{1}{h_2}\frac{\partial V}{\partial v}\widehat{a}_v + \frac{1}{h_3}\frac{\partial v}{\partial w}\widehat{a}_w$. And grad of divergence **A** can therefore be written as $\nabla\left(\frac{\partial \psi}{\partial z}\right)$, because we already found that divergence of **A** is $\frac{\partial \psi}{\partial z}$. And that is equal to $\frac{\partial^2 \psi}{\partial \rho \partial z}\widehat{a}_\rho + \frac{1}{\rho}\frac{\partial^2 \psi}{\partial \phi \partial z}\widehat{a}_\phi + \frac{\partial^2 \psi}{\partial z^2}\widehat{a}_z$. So now, from equation 20, we can write the electric field as $-j\omega\mu\psi\widehat{a}_z + \frac{1}{j\omega\varepsilon}\left[\frac{\partial^2 \psi}{\partial \rho \partial z}\widehat{a}_\rho + \frac{1}{\rho}\frac{\partial^2 \psi}{\partial \phi \partial z}\widehat{a}_\phi + \frac{\partial^2 \psi}{\partial z^2}\widehat{a}_z\right]$. So, that can be written as

$\hat{a}_\rho \frac{1}{j\omega\varepsilon}\frac{\partial^2\psi}{\partial\rho\partial z} + \hat{a}_\phi \frac{1}{j\omega\varepsilon}\frac{1}{\rho}\frac{\partial^2\psi}{\partial\phi\partial z} + \hat{a}_z \frac{1}{j\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$, where k square is given by $\omega^2\mu\varepsilon$ which is well-known.

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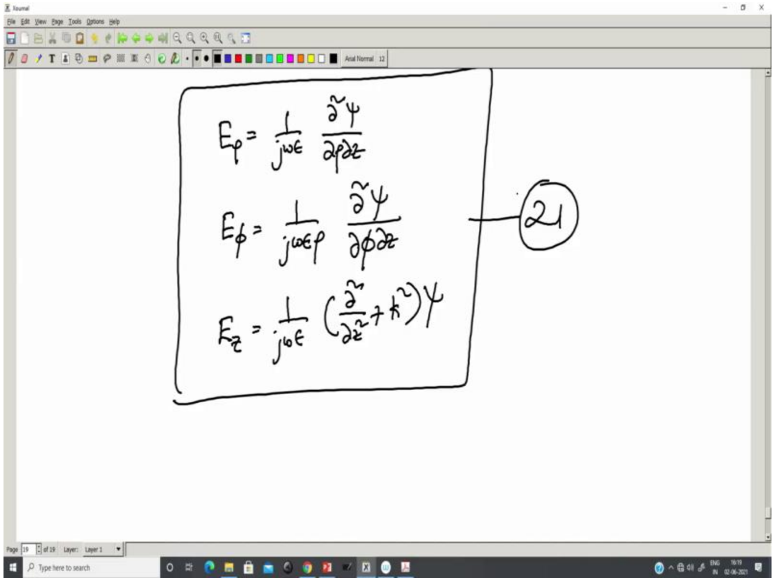

Therefore, we can collect together the terms for the electric field from this expression. So, E_ρ will be equal to $\frac{1}{j\omega\varepsilon}\frac{\partial^2\psi}{\partial\rho\partial z}$. E_ϕ will be $\frac{1}{j\omega\varepsilon}\frac{1}{\rho}\frac{\partial^2\psi}{\partial\phi\partial z}$. and E_z will be equal to $\frac{1}{j\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi$.

So, these constitute electric fields for the TM to z mode.

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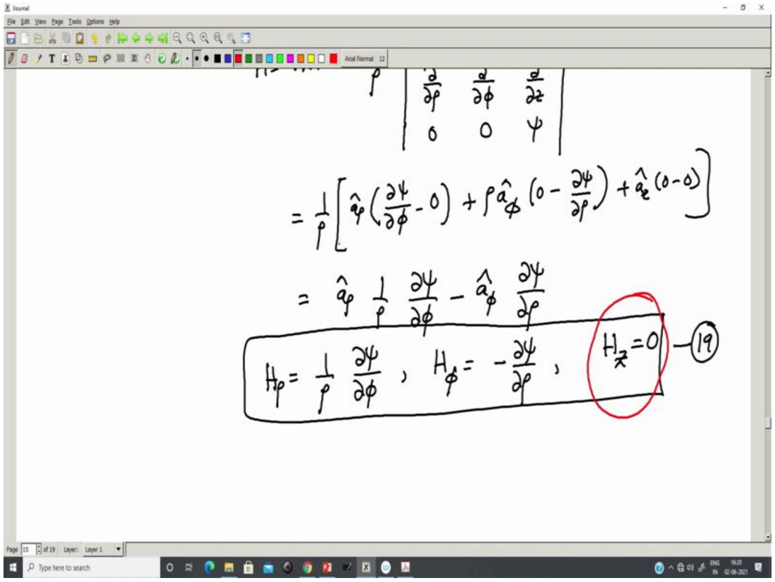

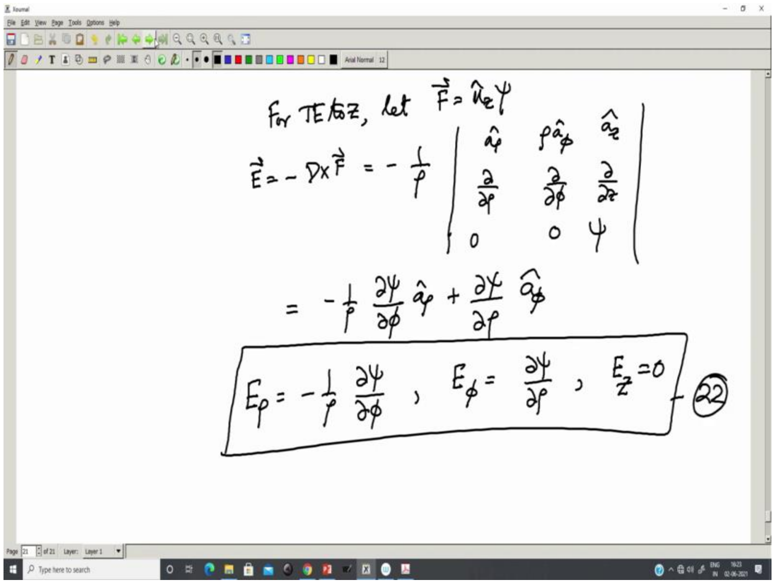

Why is the mode TM to z now? Because back from equation number 19, we see that the z directed magnetic field is 0. So, the mode is TM to z. So, similar to the rectangular waveguide or the rectangular coordinate system, z- directed magnetic vector potential will give rise to the TM to z mode in the cylindrical coordinate system as well.

So, let us explore now what will happen for the z- directed electric vector potential, whether it gives rise to the TE to z mode as it does for the rectangular coordinate system. So, for the TE to z, let **F** equal to uz psi. So, **E** is given by $-\nabla \times \vec{F}$, and that is given by $\frac{-1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$. And that ultimately yields $\frac{-1}{\rho}\frac{\partial \psi}{\partial \phi}\hat{a}_\rho + \frac{\partial \psi}{\partial \rho}\hat{a}_\phi$.

So therefore, E_ρ is given by $\frac{-1}{\rho}\frac{\partial \psi}{\partial \phi}$. E_ϕ is given by $\frac{\partial \psi}{\partial \rho}$ and Ez is given by 0. So, we clearly see from here itself that the mode is TE because the Ez, the z component of electric field is 0. Call this equation number 22.

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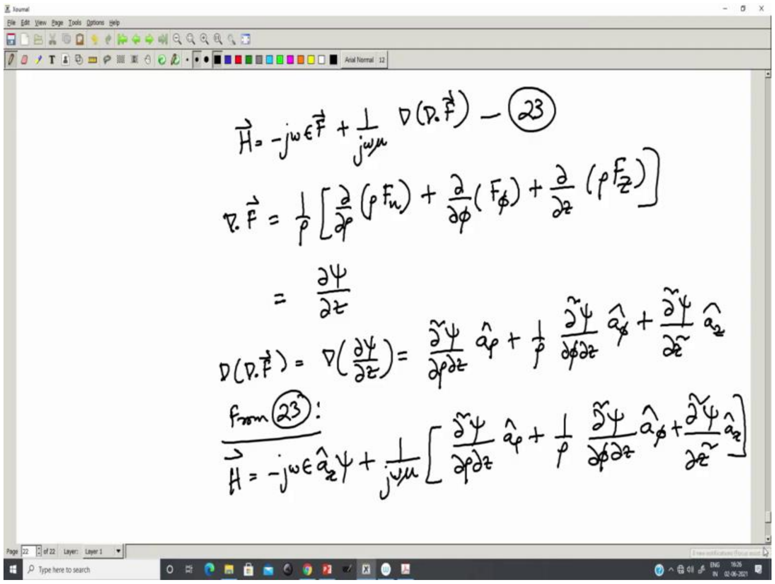

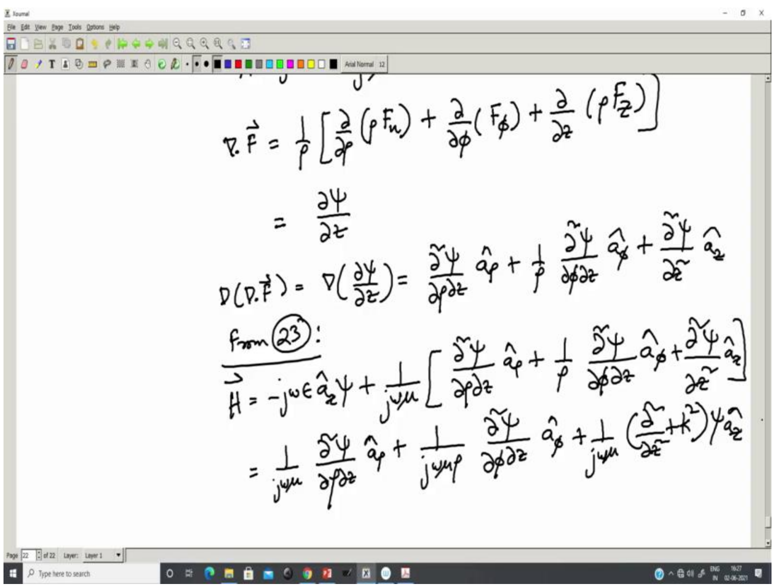

Then we go ahead with the magnetic field. **H** is equal to $-j\omega\varepsilon\vec{F}+\frac{1}{j\omega\mu}\nabla\left(\nabla\bullet\vec{F}\right)$. Call this 23. Divergence **F** is given by $\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho F_\rho\right)+\frac{\partial}{\partial\phi}\left(F_\phi\right)+\frac{\partial}{\partial z}\left(\rho F_z\right)\right]$. And because **F** has only a z component this boils down to $\frac{\partial\psi}{\partial z}$. Now grad of divergence **F** becomes $\nabla\left(\frac{\partial\psi}{\partial z}\right)$. That is equal to $\frac{\partial^2\psi}{\partial\rho\partial z}\hat{a}_\rho+\frac{1}{\rho}\frac{\partial^2\psi}{\partial\phi\partial z}\hat{a}_\phi+\frac{\partial^2\psi}{\partial z^2}\hat{a}_z$.

So now, from 23 we can compute **H** as $-j\omega\varepsilon\psi\hat{a}_z + \frac{1}{j\omega\mu}\left[\frac{\partial^2\psi}{\partial\rho\partial z}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial^2\psi}{\partial\phi\partial z}\hat{a}_\phi + \frac{\partial^2\psi}{\partial z^2}\hat{a}_z\right]$. That is $\hat{a}_\rho \frac{1}{j\omega\mu}\frac{\partial^2\psi}{\partial\rho\partial z} + \hat{a}_\phi \frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial^2\psi}{\partial\phi\partial z} + \hat{a}_z \frac{1}{j\omega\mu}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$.

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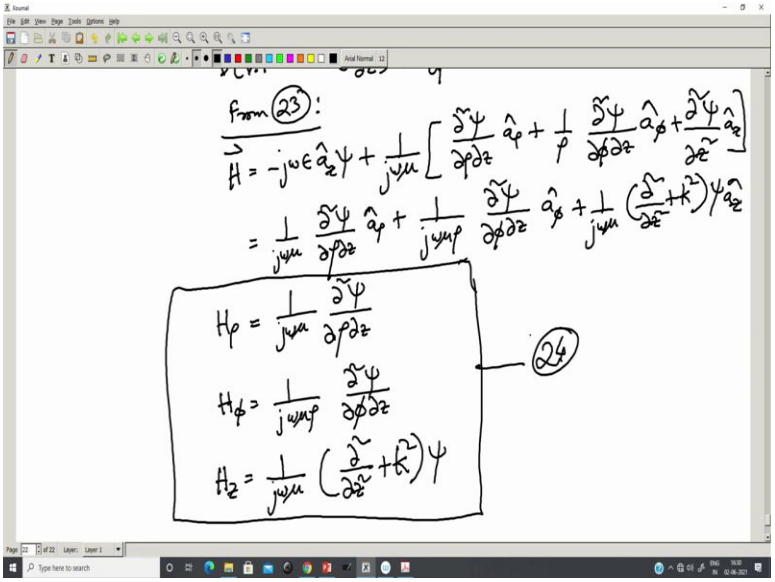

So, from here we can find out all the magnetic field components which is H_ρ equal to $\frac{1}{j\omega\mu}\frac{\partial^2\psi}{\partial\rho\partial z}$. H_ϕ is $\frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial^2\psi}{\partial\phi\partial z}$. And Hz as $\frac{1}{j\omega\mu}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi$. So, these are the three magnetic field components in the cylindrical wave system or the cylindrical coordinate system for **F** is equal to $\hat{u}_z\psi$. That is all, this equation 24.

And an arbitrary field, one which has both Ez and an Hz can be expressed as a superimposition of the TE to z and the TM to z fields just like in the rectangular coordinate system. So, any arbitrary field can be expressed as a summation over TE to z and TM to z fields.

So, this completes the derivation of the TE to z and the TM to z electric and magnetic fields in the cylindrical coordinate system. And the simple conclusion is that **A** equal to $\hat{u}_z\psi$ leads to the TM to z modes and as in the rectangular waveguide and the substitution **F** equal to $\hat{u}_z\psi$ gives rise to or yields the TE to z modes. Also, similar to the rectangular coordinate system. So, let us stop here.