Advanced Microwave Guided-Structures and Analysis
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Lecture 62
Cylindrical Wave Functions (Contd.)

Welcome to this part of the lecture on the circular waveguides. So, here we are going to utilize our previous treatment of the cylindrical wave functions particularly the TE to z and TM to z decompositions of the cylindrical solution to the cylindrical Helmholtz equation to the treatment of the circular waveguide and see that they yield the propagation constant of the circular waveguide in terms of the TE to z and TM to Z modes. We are going to compute the propagation constants. And also, we are going to find out from these expressions the cutoff frequencies for the TE to z and TM to z modes and the characteristic impedances of TE to z and TM to z modes.

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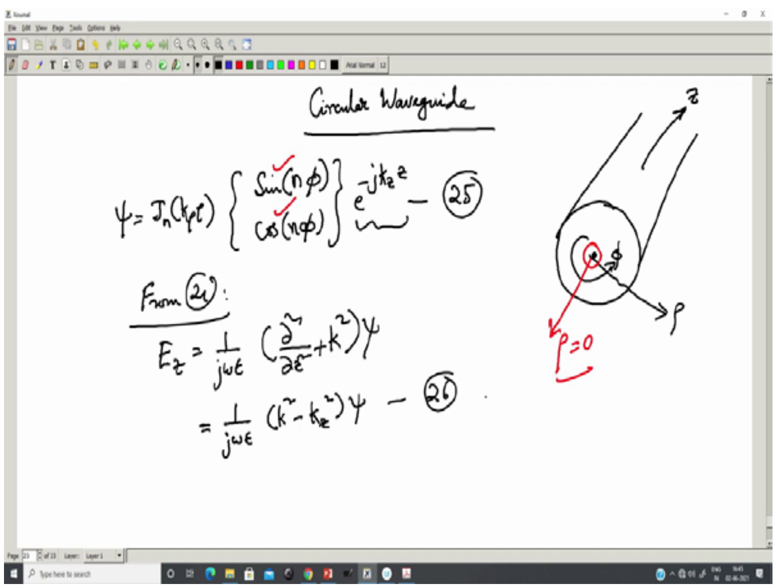

So, for the modes which are TM to z we may express the field, as we said, in terms of the decomposition **A** equal to $u_z\psi$. The solutions we obtained for the electrical and magnetic fields for this corresponding to this substitution **A** is equal to $u_z\psi$.

Now the field is finite at rho equal to 0. So, the wave functions must be of the form of 16. So, we wrote down in 16 that the circular waveguide, this is my circular waveguide. This is the centre. This is my rho. This is my phi. So, because the point of interest includes this region, which is rho equal to 0, the field decomposition inside the circular waveguide must be of the form of 16 where we use the nth order Bessel function of the first kind for describing the fields along the rho direction.

So therefore, we can write the psi function in terms of $j_n\left(k_\rho \rho\right)$. Instead of $e^{jn\phi}$, we can write that as $\sin\left(n\phi\right)$, $\cos\left(n\phi\right)$, and for the forward propagating wave along the plus z direction, the variation of the fields or the appropriate eigenfunction given into the form of $e^{-jk_z z}$. We call this equation 25.

So, either $\sin\left(n\phi\right)$ or $\cos\left(n\phi\right)$ may be chosen. So, we have a mode degeneracy. That means two kind of field distribution corresponding to the single eigennumber n; the two distributions are equally possible. They are given by Helmholtz equation. they are equally possible solutions corresponding to a particular value of n. This is called mode degeneracy except for the case n equal to 0 when sin n phi vanishes and cos n phi become 1. So, that means cos n phi becomes 1, means the field is phi invariant, that is, it does not depend on phi. So, except for the case n equals to 0, we have the mode degeneracy along the phi direction.

Now if we go to the electric field expressions for the TM to z mode from equation 21, so from 21 we can write Ez as $\frac{1}{j\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi$. That is, $\frac{1}{j\omega\varepsilon}\left(k^2-k_z^2\right)\psi$. Performing this differentiation corresponding $e^{-jk_z z}$, so we call this equation 26.

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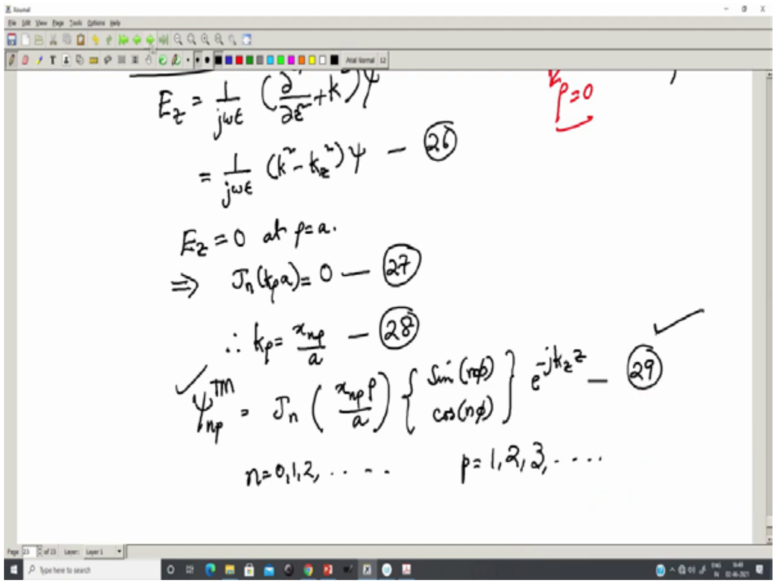

Now, as per the conditions of the circular waveguide because this is a PEC boundary, this is a conducting wall, so Ez must vanish at rho equal to a which is corresponding to the radius of the circular waveguide. So, we call this as a which is the radius of the circular waveguide. So, Ez must be equal to 0 at rho is equal to a. And if Ez must be 0 at rho equal to a that means $j_n\left(k_\rho a\right)$ must be equal to 0, because that is the only way Ez is equal to 0 at rho equal to a from equation number 25. We substitute rho equal to a here. And $j_n\left(k_\rho a\right)$ must be equal to 0 at rho equal to a. I mean that is why we substituted rho is equal to a.

So, corresponding to this the eigenvalues for k_ρ will be determined. So, k_ρ will be equal to X_{np} by a. or $k_\rho a$ is equal to X_{np}. So, these are the roots to this equation 27. So, X_{np} are the roots to equation number 27 corresponding to the zeros of the nth order Bessel functions of the first kind. So, we call this equation number 28.

So, substituting this into 25 we get the TMnp mode functions, so which is ψ_{np}^{TM} will be Jn. Instead of k_ρ we will substitute X_{np}/a. So, $j_n\left(\frac{X_{np}}{a}\rho\right)\begin{Bmatrix}\sin(n\phi)\\ \cos(n\phi)\end{Bmatrix} e^{-jk_z z}$, so which is equation number 29 where n can be 0, 1, 2 etc and p can be 1, 2, 3 etcetera.

For a particular value of n = 0, p can vary between 1, 2, 3, 4, 5, 6, infinite. It can contain infinite number of values. Then corresponding to n equal to 1, again p can take infinite number of values.

So, as we said, so these are the roots to equation number 27 corresponding to the zeros of the nth order function Bessel function of the first kind.

So, the electromagnetic field is then determined from equation number 19 and 21 with this psi function. So, we saw that 19 and 21 gives the electric and magnetic fields of the TM to z modes. So, the electric and magnetic fields for the TM to z modes inside the circular waveguide can be determined from this psi function by substituting this psi function into equation number 19 and 21.

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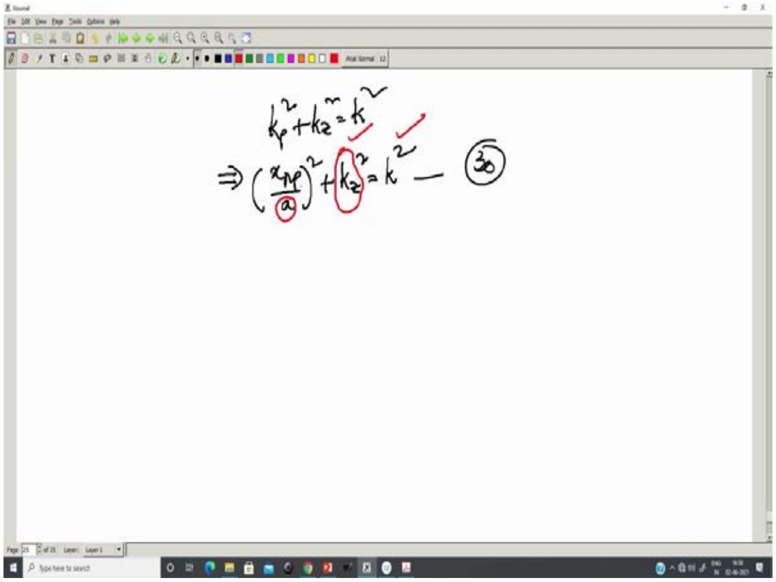

Then, the mode phase constant kz can be determined because $k_\rho^2 + k_z^2 = k^2$. And we substitute instead of k_ρ, $\frac{X_{np}}{a}$ from equation number 28. So, because $k_\rho^2 + k_z^2 = k^2$ square, this will imply $\left(\frac{X_{np}}{a}\right)^2 + k_z^2 = k^2$. So, this is equation 30. So, from this, the values, the propagation constant kz can be determined at a given frequency for a given radius of the circular waveguide a for a particular mode corresponding to n and p.

Similarly, for the modes TE to z they are expressed in terms of the electric vector potential F having only z component of psi. So, this wave function must be of the form of 25. So, the wave function must be of the same form as 25. But we have to now substitute into the circular waveguide boundary condition corresponding to the fields of TE to z modes. So, because Ez is 0 we have to work with Ephi which will also vanish over the wall ρ is equal to a.

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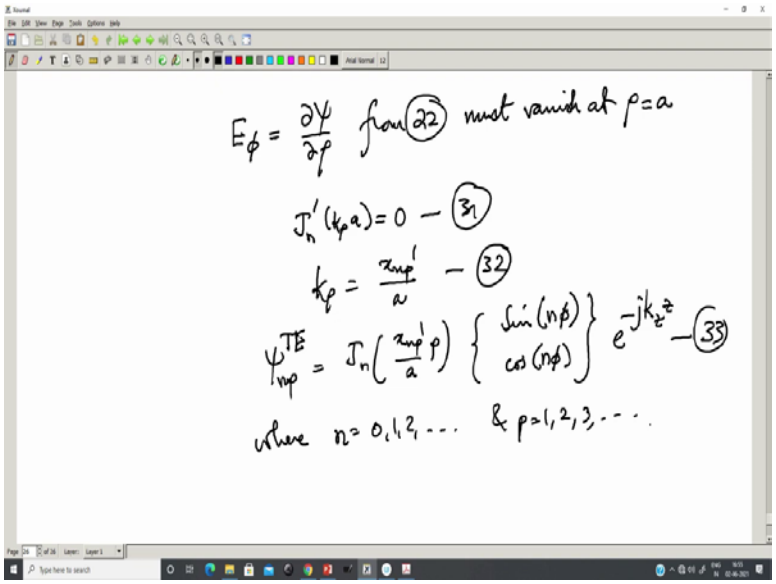

So, Ephi which is $\frac{\partial\psi}{\partial\rho}$ from 22 must be equal to 0 or must vanish at rho equal to a. So, you can easily see that 22 gives the electric fields for TE to z mode. And Ephi is $\frac{\partial\psi}{\partial\rho}$. And therefore if this has to be true we differentiate once with respect to rho. So, therefore Jn dash k rho a must be equal to 0 at, and therefore $j_n'\left(k_\rho a\right)$ must be equal to 0.

Now we know that Jn are oscillatory functions. We have found out or we have discussed before that Jn are analogous to $\cos\left(k\rho\right)$. So, they are oscillatory functions. So, Jn prime or Jn dash k rho a must also be an oscillatory function. So, we can satisfy this equation which is equation 31 by choosing k_ρ as ${X'_{np}}/{a}$, which is equation 32. So, X'_{np} corresponds to the roots of equation number 31, and also corresponding to the ordered zeros of the derivative of the Bessel function.

So from 32, the wave function for the TEnp mode can be written as ψ_{np}^{TE} as Jn, now instead of $k_\rho a$ we write $\frac{X'_{np}}{a}\rho$, the degenerate harmonic functions along the phi directions which is $\begin{cases}\sin\left(n\phi\right)\\ \cos\left(n\phi\right)\end{cases} e^{-jk_z z}$, where n is 0, 1, 2 and p is 1, 2, 3.

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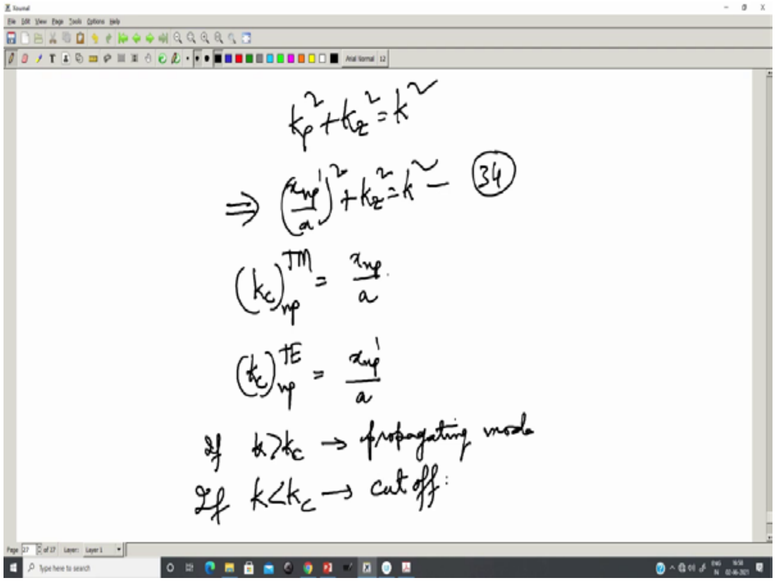

And now the mode propagation constant is again given by the same expression $k_\rho^2 + k_z^2 = k^2$. So, substituting for k_ρ we get, that is $\left(\frac{X'_{np}}{a}\right)^2 + k_z^2 = k^2$, which is equation number 34. Now the cutoff wave number for a mode similar to the rectangular waveguide is that for which kz will vanish, the mode propagation constant will vanish.

So, from equation 30, corresponding to the TM to z mode and from equation 34 corresponding to the TE to z mode, we obtain $(k_c)_{np}^{TM}$ as X_{np}/a. And $(k_c)_{np}^{TE}$ we can write as $\frac{X'_{np}}{a}$. So, kc becomes the cutoff wave number.

So, if k is greater than kc we have a propagating mode and if k<kc, the mode is cutoff.

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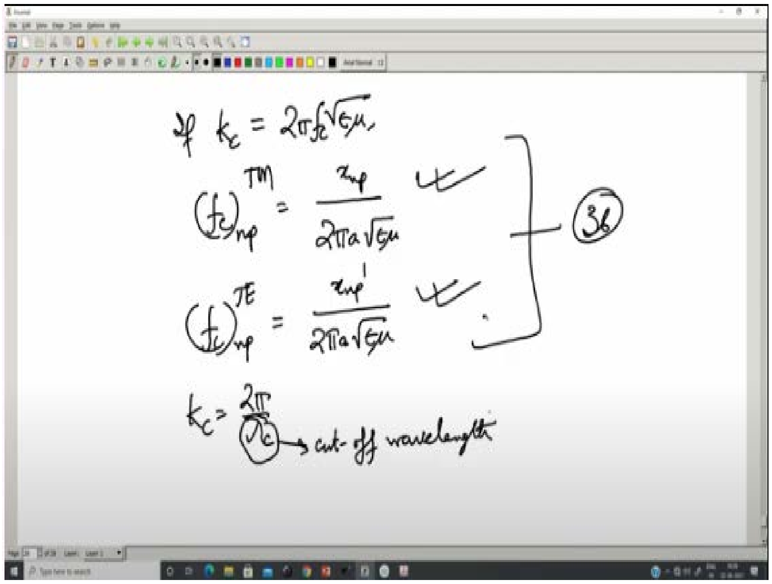

If we write kc as $2\pi f_c\sqrt{\varepsilon\mu}$, we obtain the cutoff frequencies $(f_c)_{np}^{TM}$ for the TM mode as $\frac{X_{np}}{2\pi a\sqrt{\varepsilon\mu}}$. And $(f_c)_{np}^{TE}$, the cutoff frequency for the TE mode as $\frac{X'_{np}}{2\pi a\sqrt{\varepsilon\mu}}$. So, this is the cutoff frequency for the TM mode. This is the cutoff frequency for the TE mode. So, we call the sets of equations 36. Setting kc is equal $\frac{2\pi}{\lambda_c}$ where λ_c is the cutoff wavelength.

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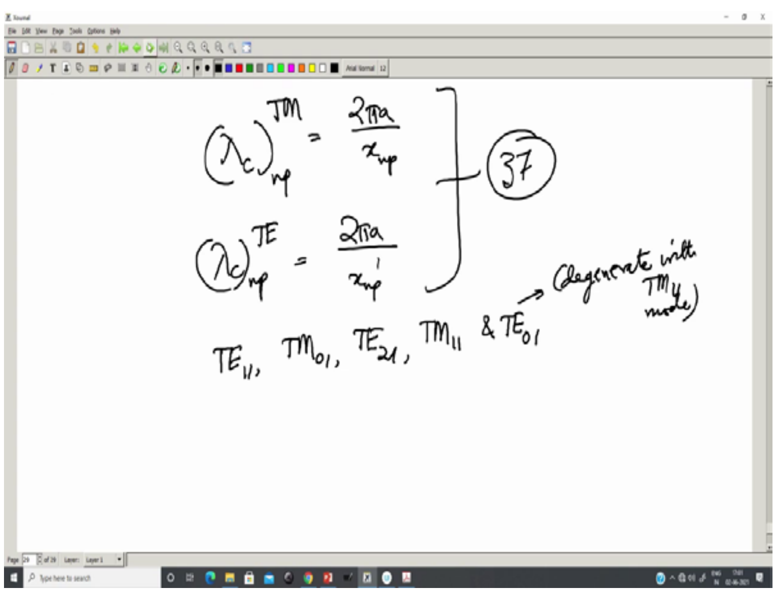

We can obtain similarly and easily the cutoff wavelengths corresponding to the TM and TE modes as $(\lambda_c)_{np}^{TM} = \dfrac{2\pi a}{X_{np}}$. And $(\lambda_c)_{np}^{TE} = \dfrac{2\pi a}{X'_{np}}$. So, we call these equations 37. So, we see that the cutoff frequencies are proportional to X_{np} for the TM modes, and they are proportional to the X'_{np}, for the TE modes from equation number 36.

Now the modes in ascending order of cutoff frequency are TE11, the lowest order mode or the dominant mode in the circular waveguide followed by the TM01, then TE21, then TM11 and TE 01 which is degenerate with the TM 11 mode. That means they have the same cutoff frequencies. So, the modes of the circular waveguide have the z-directed wave impedances as we found in the rectangular waveguide, and we can find them out easily.

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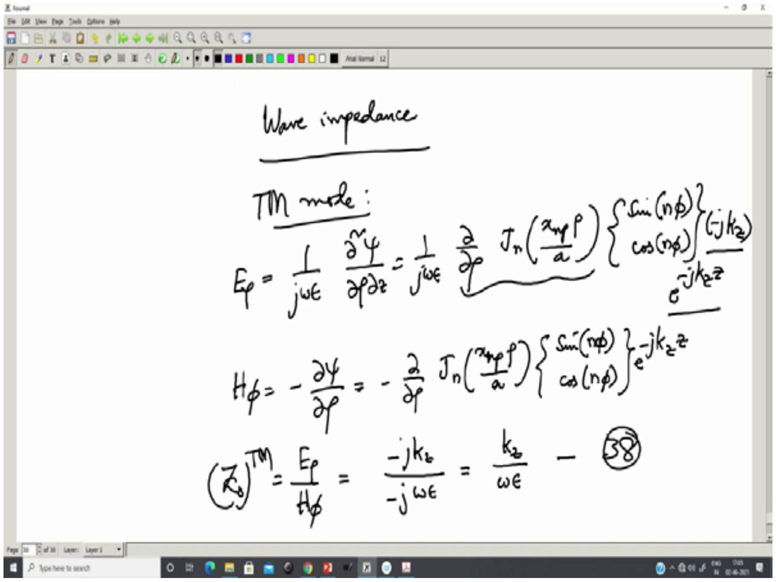

So, for the TM mode we have E_ρ as $\dfrac{1}{j\omega\varepsilon}\dfrac{\partial^2\psi}{\partial\rho\partial z}$. And this we can find substituting the appropriate psi function for the TM mode. And we can also find out H_ϕ as $\dfrac{-\partial\psi}{\partial\rho}$, substituting for the appropriate psi. And then Z_o^{TM} is going to be given by $\dfrac{E_\rho}{H_\phi}$.

So, if we do this, this becomes $\frac{-jk_z}{j\omega\varepsilon}\frac{\partial}{\partial\rho}\left(J_n\left(\frac{X_{np}}{a}\rho\right)\right)\begin{Bmatrix}\sin(n\phi)\\ \cos(n\phi)\end{Bmatrix}e^{-jk_z z}$. So, one differentiation with respect to rho, another differentiation with respect to z.

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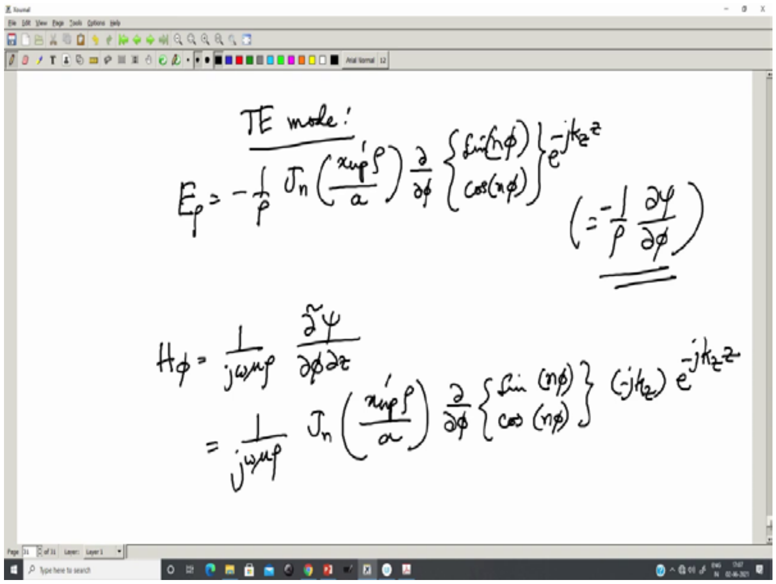

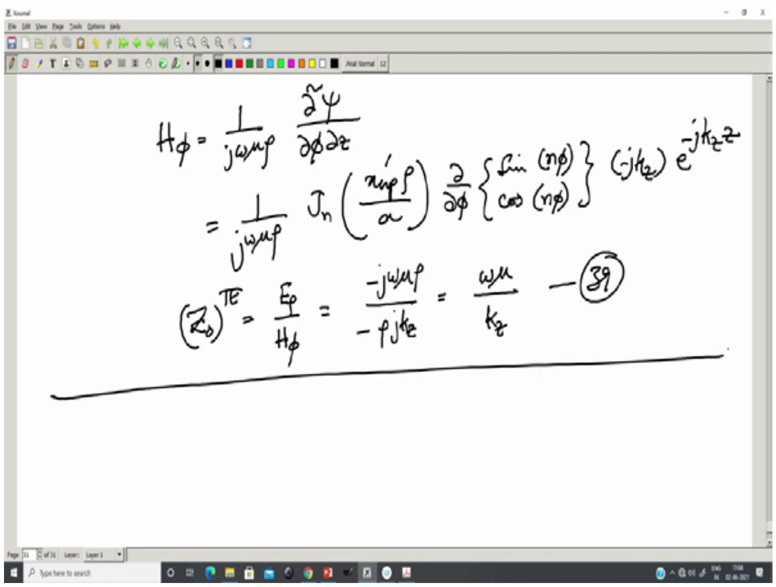

And for the TE mode if you have to compute $\frac{E_\rho}{H_\phi}$, we just find E_ρ as $\frac{-1}{\rho} J_n\left(\frac{X'_{np}}{a}\rho\right)\frac{\partial}{\partial\phi}\begin{cases}\sin(n\phi)\\ \cos(n\phi)\end{cases} e^{-jk_z z}$. So, this expression has been obtained previously for the TE to z mode, this should be minus.

And H_ϕ is $j\omega\mu\rho\frac{\partial^2\psi}{\partial\phi\partial z}$, and that is equal to $j\omega\mu\rho J_n\left(\frac{X'_{np}}{a}\rho\right)\frac{\partial}{\partial\phi}\begin{Bmatrix}\sin(n\phi)\\ \cos(n\phi)\end{Bmatrix}(-jk_z)e^{-jk_z z}$. And therefore, from here we can find the Z_o^{TE} as $\frac{E_\rho}{H_\phi}$ that is $\frac{-j\omega\mu\rho}{-\rho jk_z}$. So, that is $\frac{\omega\mu}{k_z}$. So, this is equation 39.

So, this completes our treatment of the circular waveguide in terms of the propagation constant of the TE to z and TM to z modes, the cutoff frequencies and the characteristic impedances of the TE to z and TM to z modes. Next we are going to utilize the knowledge earned from the circular waveguide to the investigation of the circular cavity. Thank you.