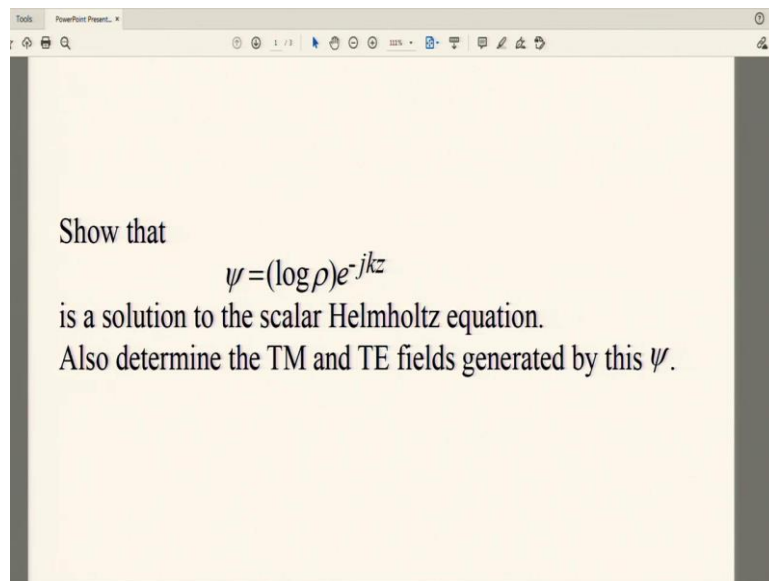


Advanced Microwave Guided – Structures and Analysis
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Lecture 64
Cylindrical Wave Functions Tutorials

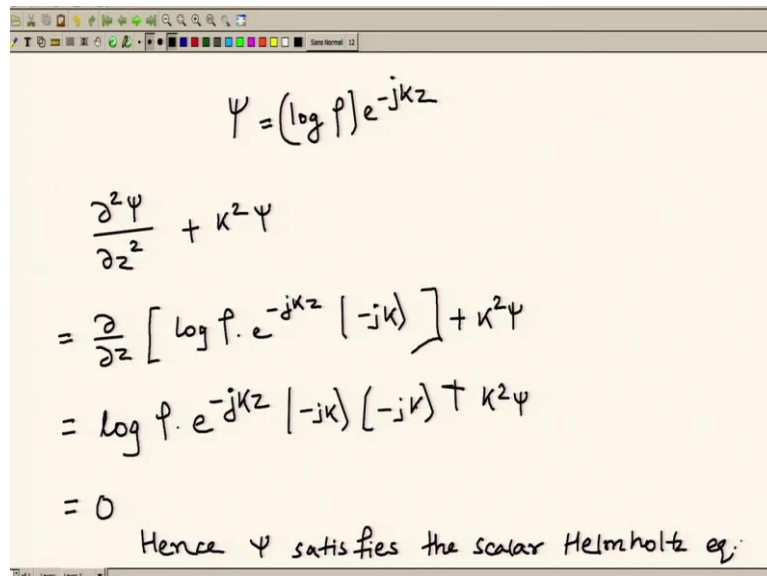
Hello everyone. So, in today's session we will solve some numerical problems based on cylindrical wave functions.

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So, the first question is that we need to show $\psi = \log(\rho)e^{-jkz}$ is a solution to the scalar Helmholtz equation. Also we need to determine the TM and TE fields generated by this ψ .

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$$\begin{aligned}\psi &= (\log \rho) e^{-jkz} \\ \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi &= \frac{\partial}{\partial z} \left[\log \rho \cdot e^{-jkz} (-jk) \right] + k^2 \psi \\ &= \log \rho \cdot e^{-jkz} (-jk) (-jk) + k^2 \psi \\ &= 0\end{aligned}$$

Hence ψ satisfies the scalar Helmholtz eq.

So, to start with we have $\psi = \log(\rho) e^{-jkz}$, so this is the psi function that has been provided in the question. So, at first what we need, we need to show that this ψ is a solution to the scalar Helmholtz equation. So, we will find out $\frac{\partial^2 \psi}{\partial z^2} + k^2 \psi$, so we will perform the derivation with respect to z .

So, it will be $\frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial z} \right)$, first is $\frac{\partial \psi}{\partial z}$, so we will differentiate this function with respect to z first. So, on differentiating, we get $\log(\rho) e^{-jkz} (-jk)$. Again, we need to perform differentiation with respect to z , so it will be, $\log(\rho) e^{-jkz} (-jk) (-jk)$ plus $k^2 \psi$, so this is 0, and hence ψ satisfies the scalar Helmholtz equation. Now what? Now the question says like we need to determine the TM and TE fields generated by this ψ .

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For TM case
we can obtain the fields TM to z by letting
 $A = \hat{u}_z \psi$
In cylindrical co-ordinates :
 $H = \nabla \times A$
$$\nabla \times A = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$$

So, to start with will start for the TM case first. We can obtain the fields TM to z by letting \mathbf{A} as $u_z \psi$. Now in cylindrical coordinates, we know \mathbf{H} will be curl of \mathbf{A} .

We will find out curl of \mathbf{A} in cylindrical coordinates, so we will have curl of \mathbf{A} equals to

$$\frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$$

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$$\nabla \times A = \frac{1}{\rho} \left[\hat{a}_\rho \left(\frac{\partial \psi}{\partial \phi} \right) + \rho \hat{a}_\phi \left(-\frac{\partial \psi}{\partial \rho} \right) + \hat{a}_z (0) \right]$$

$$= \frac{1}{\rho} \left[\hat{a}_\rho \left(\frac{\partial \psi}{\partial \phi} \right) + \rho \hat{a}_\phi \left(-\frac{\partial \psi}{\partial \rho} \right) \right]$$

$$H_\rho = \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \quad \text{--- (1)}$$

$$H_\phi = -\frac{\partial \psi}{\partial \rho} \quad \text{--- (2)}$$

$$H_z = 0 \quad \text{--- (3)}$$

$$H_\phi = -\frac{\partial \psi}{\partial \rho} \quad \text{--- (2)}$$

$$H_z = 0 \quad \text{--- (3)}$$

$$\mathbf{E} = \frac{1}{\epsilon_0} \nabla (\nabla \cdot \mathbf{A}) - \hat{z} A_z$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial \psi}{\partial z}$$

Since \mathbf{H} is curl of \mathbf{A} , so now we can write the fields as $H_\rho = \frac{1}{\rho} \frac{\partial \psi}{\partial \phi}$, $H_\phi = -\frac{\partial \psi}{\partial \rho}$, $H_z = 0$. So, let us denote these equations as equation number 1, this is 2 and this is 3.

Now for the \mathbf{E} fields we will write \mathbf{E} as $\frac{1}{\epsilon_0} \nabla (\nabla \cdot \mathbf{A}) - \hat{z} A_z$. Now, we can write $\nabla \cdot \mathbf{A}$ as in cylindrical coordinates $\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \phi} (A_\phi) + \frac{\partial}{\partial z} (A_z)$. So, for this case we are left with, this will be 0 and so we are left with $\frac{\partial}{\partial z} (\psi)$.

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$$\nabla (\nabla \cdot \mathbf{A}) = \left(\hat{a}_\rho \frac{\partial}{\partial \rho} + \hat{a}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{a}_z \frac{\partial}{\partial z} \right) \left(\frac{\partial \psi}{\partial z} \right)$$

$$= \hat{a}_\rho \frac{\partial^2 \psi}{\partial \rho \partial z} + \hat{a}_\phi \frac{1}{\rho} \frac{\partial^2 \psi}{\partial \phi \partial z} + \hat{a}_z \frac{\partial^2 \psi}{\partial z^2}$$

$$E_\rho = \frac{1}{\epsilon_0} \frac{\partial^2 \psi}{\partial \rho \partial z} \quad \text{--- (4)}$$

$$E_\phi = \frac{1}{\epsilon_0} \cdot \frac{1}{\rho} \frac{\partial^2 \psi}{\partial \phi \partial z} \quad \text{--- (5)}$$

$$E_z = \frac{1}{\epsilon_0} \frac{\partial^2 \psi}{\partial z^2} - \hat{z} \psi = \frac{1}{\epsilon_0} \left[\frac{\partial^2}{\partial z^2} + k^2 \right] \psi \quad \text{--- (6)}$$

Therefore, we can write $\hat{a}_\rho \frac{\partial^2 \psi}{\partial \rho \partial z} + \hat{a}_\phi \frac{1}{\rho} \frac{\partial^2 \psi}{\partial \phi \partial z} + \hat{a}_z \frac{\partial^2 \psi}{\partial z^2}$.

Now, we will substitute this and then find out the **E** fields. So, we will

$$E_\rho = \frac{1}{\hat{y}} \frac{\partial^2 \psi}{\partial \rho \partial z}$$

get $E_\phi = \frac{1}{\hat{y}} \frac{1}{\rho} \frac{\partial^2 \psi}{\partial \phi \partial z}$, so we will denote this as 4, this as 5 and this as equation 6.

$$E_z = \frac{1}{\hat{y}} \left[\frac{\partial^2}{\partial z^2} + k^2 \right] \psi$$

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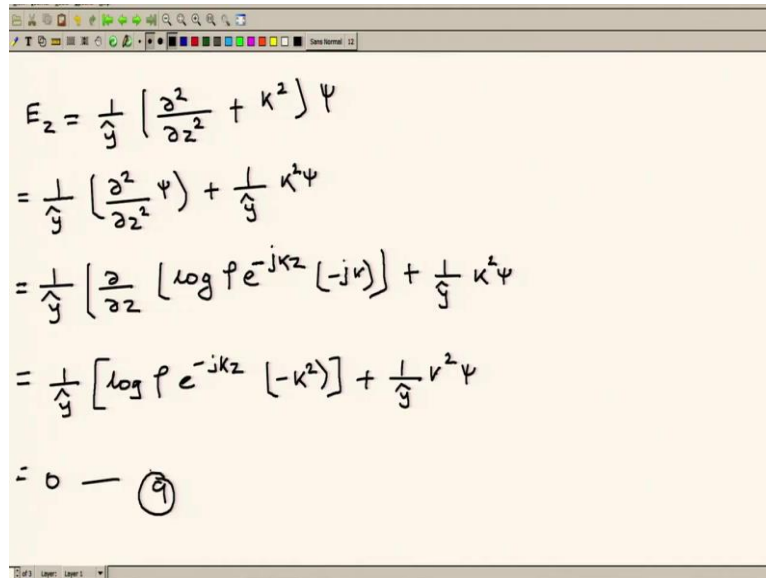
The image shows two screenshots of a digital whiteboard with handwritten mathematical derivations. The top screenshot shows the derivation for the radial component of the electric field, E_ρ . It starts with the potential function $\psi = (\log \rho) e^{-jkz}$. Then, it calculates $E_\rho = \frac{1}{\hat{y}} \frac{\partial^2 \psi}{\partial \rho \partial z}$. This is expanded to $\frac{1}{\hat{y}} \frac{\partial}{\partial \rho} [(\log \rho) e^{-jkz} (-jk)]$, which simplifies to $\frac{1}{\hat{y}} \cdot \frac{1}{\rho} (-jk) e^{-jkz}$, labeled as equation 7. The bottom screenshot shows the derivation for the azimuthal component, E_ϕ . It starts with $E_\phi = \frac{1}{\hat{y}} \cdot \frac{1}{\rho} \frac{\partial^2 \psi}{\partial \phi \partial z}$. This is expanded to $\frac{1}{\hat{y}} \cdot \frac{1}{\rho} \cdot \frac{\partial}{\partial \phi} [(\log \rho) e^{-jkz} (-jk)]$, which simplifies to 0, labeled as equation 8.

Now in the question we have $\psi = \log(\rho)e^{-jkz}$. So, we will have $E_\rho = \frac{1}{\hat{y}} \frac{\partial^2 \psi}{\partial \rho \partial z}$ that we derived

previously. Now we will substitute this ψ in this place, so this will be $\frac{1}{\hat{y}} \frac{1}{\rho} e^{-jkz} (-jk)$. So this

is equation 7. Now we will have E_ϕ , we have E_ϕ as 0. This is equation number 8.

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The image shows a handwritten derivation for E_z on a digital whiteboard. The steps are as follows:

$$E_z = \frac{1}{\hat{y}} \left[\frac{\partial^2}{\partial z^2} + k^2 \right] \psi$$

$$= \frac{1}{\hat{y}} \left(\frac{\partial^2}{\partial z^2} \psi \right) + \frac{1}{\hat{y}} k^2 \psi$$

$$= \frac{1}{\hat{y}} \left[\frac{\partial}{\partial z} \left[\log \rho e^{-jkz} (-jk) \right] \right] + \frac{1}{\hat{y}} k^2 \psi$$

$$= \frac{1}{\hat{y}} \left[\log \rho e^{-jkz} (-k^2) \right] + \frac{1}{\hat{y}} k^2 \psi$$

$$= 0 \quad \text{--- (9)}$$

Now for E_z , we can write E_z as $\frac{1}{\hat{y}} \left[\frac{\partial^2}{\partial z^2} + k^2 \right] \psi$.

Now, again differentiating with respect to z , we will have 0, so basically this term gets as 0.

E_z is also 0, so give this equation number as equation number 9.

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$$= \frac{1}{\rho} \left[\frac{\partial}{\partial z} [\log \rho e^{-jkz}] (-j^k) \right] + \frac{1}{\rho} k^2 \psi$$

$$= \frac{1}{\rho} [\log \rho e^{-jkz} (-k^2)] + \frac{1}{\rho} k^2 \psi$$

$$= 0 \quad \text{--- (9)}$$

$$H_\rho = \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} = 0 \quad \text{--- (10)}$$

$$H_\phi = -\frac{\partial \psi}{\partial \rho} = -\frac{\partial}{\partial \rho} [\log \rho e^{-jkz}]$$

$$H_\phi = -\frac{1}{\rho} e^{-jkz} \quad \text{--- (11)}$$

$$H_z = 0 \quad \text{--- (12)}$$

For TE case

By letting $F = \hat{y} \psi$

$$E = -\nabla \times F$$

$$H = \frac{1}{\omega \mu} \nabla (\nabla \cdot F) - \hat{y} F$$

Now, H_ρ , H_ϕ and, H_z . So, H_ρ we derived it as $H_\rho = \frac{1}{\rho} \frac{\partial \psi}{\partial \phi}$, so differentiating ψ with respect to ϕ will give us 0, so this is equation number 10. Now H_ϕ , so H_ϕ we derived as $-\frac{\partial \psi}{\partial \rho}$. So, it comes as minus $\frac{1}{\rho} e^{-jkz}$, so this is equation number 11 and H_z was 0, so these are the fields for TM case.

Now similarly for TE case we can obtain the field TE to z by considering \mathbf{F} as $u_z \psi$. So, we can write \mathbf{E} as minus curl of \mathbf{F} and \mathbf{H} as $\frac{1}{\omega \mu} \nabla (\nabla \cdot \mathbf{F}) - \hat{y} \mathbf{F}$. So, in the similar pattern we will find out the fields for the TE case.

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$$H = \frac{1}{z} \nabla (\nabla \cdot F) - \hat{\psi} F$$

$$\nabla \times F = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$$

$$= \frac{1}{\rho} \left[\hat{a}_\rho \left(\frac{\partial \psi}{\partial \phi} \right) + \rho \hat{a}_\phi \left(-\frac{\partial \psi}{\partial \rho} \right) + \hat{a}_z (0) \right]$$

$$= \frac{1}{\rho} \left[\hat{a}_\rho \left(\frac{\partial \psi}{\partial \phi} \right) + \rho \hat{a}_\phi \left(-\frac{\partial \psi}{\partial \rho} \right) + \hat{a}_z (0) \right]$$

$$\therefore E_\rho = -\frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \quad \text{--- (13)}$$

$$E_\phi = \frac{\partial \psi}{\partial \rho} \quad \text{--- (14)}$$

$$E_z = 0 \quad \text{--- (15)}$$

So, we will start from finding out $\nabla \times \vec{F}$. We can write $\nabla \times \vec{F}$ as $\frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$. This will

give us $\frac{1}{\rho} \hat{a}_\rho \frac{\partial}{\partial \phi} (\psi) + \hat{a}_\phi \frac{\partial}{\partial \rho} (\psi) + \hat{a}_z (0)$.

Therefore, since \mathbf{E} was minus of curl of \mathbf{F} , so we can write the fields as $E_\rho = -\frac{1}{\rho} \frac{\partial \psi}{\partial \phi}$, $E_\phi = \frac{\partial \psi}{\partial \rho}$, $E_z = 0$. So, let us denote this number as 13, this has equation number 14, and this is equation number 15.

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$$E_\phi = \frac{\partial \psi}{\partial \phi} \quad \text{--- (14)}$$

$$E_z = 0 \quad \text{--- (15)}$$

$$H = \frac{1}{z} \nabla(\nabla \cdot \vec{F}) - \hat{y} F$$

$$(\nabla \cdot \vec{F}) = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{\partial \psi}{\partial z}$$

$$\nabla(\nabla \cdot \vec{F}) = \left[\hat{a}_\rho \frac{\partial}{\partial \rho} + \hat{a}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{a}_z \frac{\partial}{\partial z} \right] \frac{\partial \psi}{\partial z}$$

$$= \hat{a}_\rho \left(\frac{\partial^2 \psi}{\partial \rho \partial z} \right) + \hat{a}_\phi \frac{1}{\rho} \frac{\partial^2 \psi}{\partial \phi \partial z} + \hat{a}_z \frac{\partial^2 \psi}{\partial z^2}$$

$$H_\rho = \frac{1}{z} \frac{\partial^2 \psi}{\partial \rho \partial z} \quad \text{--- (16)}$$

$$H_\phi = \frac{1}{z} \frac{1}{\rho} \frac{\partial^2 \psi}{\partial \phi \partial z} \quad \text{--- (17)}$$

$$H_z = \frac{1}{z} \frac{\partial^2 \psi}{\partial z^2} - \hat{y} \psi$$

$$= \frac{1}{z} \left[\frac{\partial^2}{\partial z^2} + k^2 \right] \psi \quad \text{--- (18)}$$

Next for \mathbf{H} we will find $\frac{1}{z} \nabla(\nabla \cdot \vec{F}) - \hat{y} \vec{F}$. So, we will find $H_\rho = \frac{1}{z} \frac{\partial^2 \psi}{\partial \rho \partial z}$, H_ϕ we will get

$H_\phi = \frac{1}{z} \frac{1}{\rho} \frac{\partial^2 \psi}{\partial \phi \partial z}$ and H_z we will have $H_z = \frac{1}{z} \left[\frac{\partial^2}{\partial z^2} + k^2 \right] \psi$. So, let us denote these equations

as equation number 16, this as 17 and this as equation 18.

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$$\Psi = (\log \rho) e^{-jkz}$$
$$E_{\rho} = -\frac{1}{\rho} \frac{\partial \Psi}{\partial \phi} = 0 \quad \text{--- (19)}$$
$$E_{\phi} = \frac{\partial \Psi}{\partial \phi} = \frac{\partial}{\partial \phi} [(\log \rho) e^{-jkz}] = \frac{1}{\rho} e^{-jkz} \quad \text{--- (20)}$$
$$E_z = 0 \quad \text{--- (21)}$$

$$E_z = 0 \quad \text{--- (21)}$$
$$H_{\phi} = \frac{1}{2} \frac{\partial^2 \Psi}{\partial \rho \partial z}$$
$$= \frac{1}{2} \frac{\partial}{\partial \rho} [\log \rho e^{-jkz} (-jk)]$$
$$= \frac{1}{2} \left[\frac{1}{\rho} e^{-jkz} (-jk) \right]$$
$$= \frac{1}{2} \left[-\frac{jk}{\rho} e^{-jkz} \right] \quad \text{--- (22)}$$

$$H_{\phi} = \frac{1}{2} \frac{\partial^2 \Psi}{\partial \rho \partial z}$$
$$= \frac{1}{2} \frac{\partial}{\partial \rho} [\log \rho e^{-jkz} (-jk)]$$
$$= \frac{1}{2} \left[\frac{1}{\rho} e^{-jkz} (-jk) \right]$$
$$= \frac{1}{2} \left[-\frac{jk}{\rho} e^{-jkz} \right] \quad \text{--- (22)}$$
$$H_{\phi} = \frac{1}{2} \frac{1}{\rho} \frac{\partial^2 \Psi}{\partial \phi \partial z}$$

Now in the given question ψ was $\log(\rho)e^{-jkz}$, so we will have E_ρ as 0. Then E_ϕ we have $E_\phi = \frac{\partial \psi}{\partial \rho}$. So, we will get $\frac{1}{\rho}e^{-jkz}$ and then we had E_z as 0. So, let us denote this as an equation 19, 20 and 21.

Now the **H** fields. So, we have H_ρ as $\frac{1}{z} \frac{\partial^2 \psi}{\partial \rho \partial z}$, so it will have $\frac{1}{z} \left[\frac{-jk}{\rho} e^{-jkz} \right]$. We will give it as equation number 22. Then we will find out H_ϕ , so H_ϕ is 1 upon z cap 1 upon ρ del square psi upon del phi del z .

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The image shows a digital whiteboard with the following handwritten mathematical steps:

$$\vec{H}_\phi = \frac{1}{2} \frac{1}{\rho} \frac{\partial}{\partial \phi} [\log \rho e^{-jkz} (-jk)]$$
$$H_\phi = 0 \quad \text{--- (23)}$$
$$H_z = \frac{1}{2} \left[\frac{\partial^2}{\partial z^2} + k^2 \right] \Psi$$
$$= \frac{1}{2} \left[\frac{\partial^2 \Psi}{\partial z^2} \right] + \frac{1}{2} k^2 \Psi$$
$$= \frac{1}{2} \left[\frac{\partial}{\partial z} (\log \rho e^{-jkz} (-jk)) \right] + \frac{1}{2} k^2 \Psi$$
$$= 0 \quad \text{--- (24)}$$

So, H_ϕ is also 0, so this is equation 23. Now H_z , so H_z is $\frac{1}{y} \left[\frac{\partial^2}{\partial z^2} + k^2 \right] \psi$, we will substitute

ψ , so we will get that this is also coming as 0, so this is equation number 24. So, these are the TE and TM fields that are generated by this psi function. So, this is all for now, thank you so much.