Probability Foundations for Electrical engineers
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Lecture - 15
Examples: Independence

Welcome to this lecture on examples in a probability, today we are going to see examples of independent events and dependent events.

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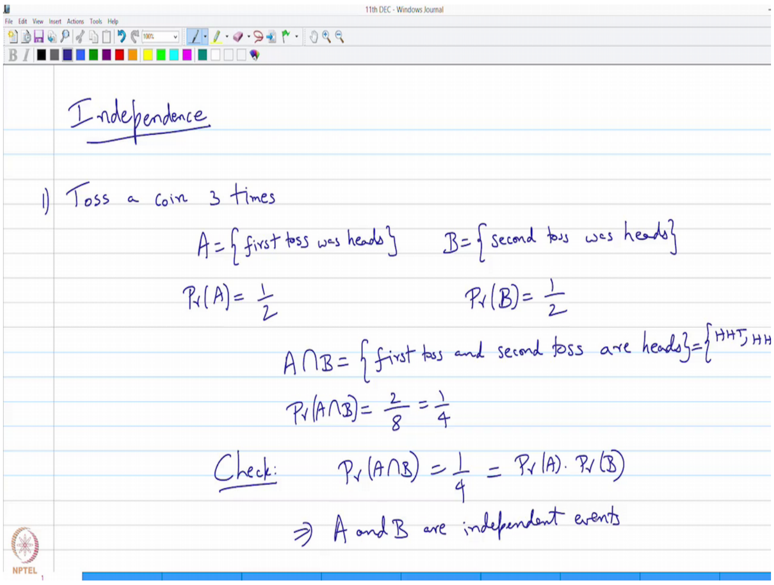

So, independence is the main theme of the example of these examples, you must have seen professor Aravind's lectures, where he defines clearly what independence its gives the definition gives a lot of examples himself. So, I would like to push those examples a little bit further and get started. So, the first experiment we will see is as usual toss a coin let us say 3 times. So, we toss a coin 3 times we all know what the sample space is I do not have to write this once again.

Let me consider 2 events and then then argue about their dependence and independence. So, the first event A is that first toss was heads see the second event B is that the second toss was heads. So, if you were to compute the probability even though the probability of A is actually 1 by 2 right first toss of heads is 1 by 2 similarly probability of B is 1 by 2. So, these 2 are not enough to conclude about the independence of A and B you have to compute one more probability which is the probability of A intersect B.

What is A intersect B first toss and second toss heads, what is the probability of A intersect B? See you want the first and second toss to be heads if you want you can write it down write down the event explicitly you have HHT, HHH right. So, these are the only 2 possibilities. So, probability of A intersect B is 2 by 8 which is 1 by 4.

Now, what is the definition of independence if 2 events A and B are independent, if probability of A intersect B equals probability of A times probability of B and we can check the probability of A intersect B is 1 by 4 is actually the same as probability of A times probability of B. So, A and B are independent that implies A and B are independent events. So, this is a very very simple example.

Now, from this example one tends to generalize a little bit. So, if you see what is the meaning of 2 events being independent intuitively what it means is whether the first event happened or not does not affect the probability of the second event in any way. So, if you think about it the way I am tossing 3 coins repeatedly, whatever happens in the first toss does not impact what is going to happen in the second toss. So, the probabilities will end up in this fashion. So, this is something important to understand from an intuitive point of view also ok.

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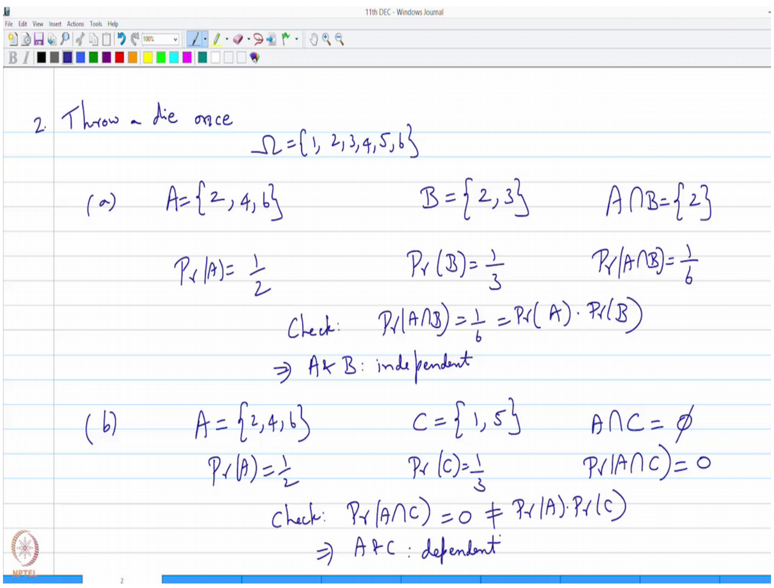

So, let us look at another example with the throw or die a 2 times or let us say throw a die one sorry. So, let us look at another example where we throw a die once. So, so we know what the sample space is you have 6 possibilities all of them are equally likely, I am going to define 2 events here event A is the event 2 comma 4 comma 6.

If you want in English you can think of this event as being the even numbers right you get an even number, and then I am going define another event B which is let us say 2 comma 3. So, do not ask me what this event is let us just some event I am defining it is a valid event 2 comma 3. So, I am going to try and see if these 2 events are independent. So, let us do probability of A its just 1 by 2 probability of B which is 1 by 3 right. So, probability of a is actually the number of elements in A which is 3 divided by the total number of possibilities 6, 3 by 6 is 1 by 2 probability of B is 2 by 6 which is 1 by 2.

So, what about A intersect B, if you look at the intersection its actually 2 and then what would be probability of A intersect B is actually 1 by 6. So, let us do a check probability of A intersect B is 1 by 6 equals probability of A times probability of B. So, in this case A and B are independent right its quite easy to see that its called the part A. Now part B is another example where I am going to take something very similar for a I will keep a as it is, then I will define an event c which is let us say 1 comma 5. So, this is a is the event with the minutes it is said to have happened if you have got 2 or 4 or 6 c is an event its said to happen if you got 1 or 5 ok.

So, that is the definition. So, if you actually compute probability of A; you have a half here probability of C will be 1 by 3, but what happens to A intersect B it is actually the null set there is no intersection. So, what is the probability of A intersect B it is actually 0. So, in your check probability of A intersect B is 0 is actually not equal to probability of a times probability of C I am sorry I [FL] ok.

So, let us do probability of A intersect C it turns out A intersect C is actually the null set there is no intersection here right it says 2 4 6 that is 1 5. So, probability of A intersect C is actually equal to 0. So, in your check you are going to get probability of A intersect C equals 0 which is clearly not equal to probability of A times probability of C. So, A and C are dependent ok.

So, I think these 2 examples a little bit important to illustrate one of the finer concepts in independent events. In fact, the intersection of A and B is very very important; so if then A and B do not intersect then they will not be independent. In fact, if they have to intersect for independence look at look at what happened in the second example A and c did not have any intersection ok.

So, probability of A was nonzero probability of c was nonzero and A intersect C became empty set. So, which means probability is 0. So, if you have no intersection they are going to be dependent. Now if you think about it makes a lot of sense. So, let us look at part B for instance the event A is 2 4 6 and even C is 1 comma 5. Now if I told you that the event a happened right suppose I told you the event A happened then you know for sure that the even C did not happen ok.

So, whether or not event A happened directly affects the probability of event C. So, that is directly affects not the let me not say probability here the conditional probability of event c. So, you see given a is going to be equal to 0 for instance. So, so the fact that these 2 guys do not have an intersection actually leads to a lot of dependence between these 2 events. So, when people say independence it is not that the sets representing the events do not intersect it is not that at all they have to intersect and just intersection alone is not enough to make them independent.

If you want independence the fractions should work out you know the number of elements in the intersection should be exactly, such that the probability is multiplied in that fashion. So, maybe I should give you one more example to illustrate what I mean by that.

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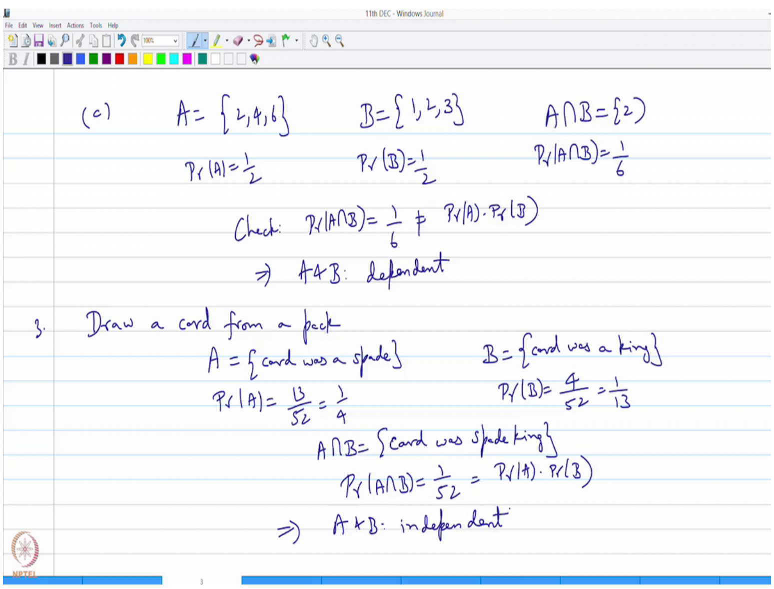

If you take an example here where I keep A as again 2 4 6 and then I keep B as let us say 1 2 3 or maybe; so 1 2 3 that is ok. So, A is 2 4 6 B is 1 2 3 and again if you do probability of A now you get 1 by 2 probability of B you again get 1 by 2 right. So, both these events are probability 2.

So, if you do A intersect B you actually get 2 and probability of A intersect B is ending up being 1 by 6. So, now, once again if you check probability of A intersect B is 1 by 6 and its not equal to probability of A times probability of B. So, this implies A and B are dependent events. So, here again there was an intersection between A and B, but the intersection was in a way it was no too small or something like that right.

So, you are not going to get independence with this kind of intersection. So, hopefully these 3 examples give you an illustration of what is needed for dependence and independence, the 2 events have to intersect for independence to be possible, but just because they intersect it does not mean that they will be independent, the probabilities have to work out correctly the probability check probability of A intersect B has to be equal to probability of a into probability of B only then they will be independent. And also just because the 2 events are disjoint as in they do not intersect you will. In fact, these joint events will never be independent. So, disjoint events will always be dependent.

Because whether or not event a happen tells you something about event B right. So, that is what is very important to understand here. So, independence is a little bit more subtle in these fashions and we better watch out when you define independence. So, let me give one more example in the same spirit here, let us say you draw a card from a pack. So, note again I am drawing only one card case. So, it is not 2 cards I am drawing one card from one pack ok.

So, I am going to define the event as card is A spade card was a spade, and the event B is card was a king. So, what is the probability of a then we know this is 13 by 52 is 1 by 4 what s the probability of event B it is actually 4 by 52 which is 1 by 13. So, what is A intersect B card was spade king right. So, you draw one card one event is that the card was spade the other event is the card was king the question is are these 2 events independent. So, we see that this event a can be written down card versus spade, there are 13 possibilities here even B can be written down card was the king a 4 possibilities here probability of a is 13 by 52, 1 by 4 that is probability of B is 4 by 52 1 by 13 that is ok.

Now, A intersect B is not a null set. So, that is promising. So, card was spade king and they could be independent, but we have to verify whether the probabilities were called correctly or not probability of A intersects B is there is only one card here. So, that is 1 by 52 and that is really equal to probability of a times probability of b. So, that implies A and B are independent. So, this is next way of checking this.

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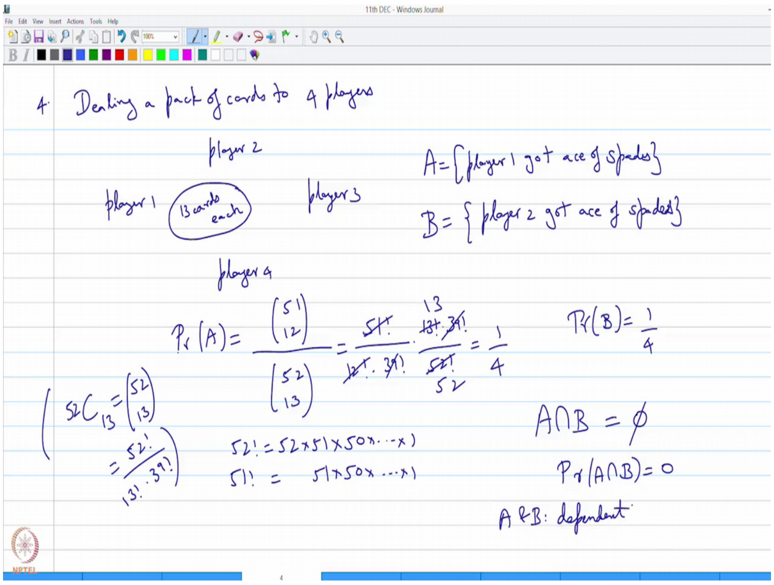

So, I am going to give you a couple of slightly non trivial examples slowly the next example we will see is let us say dealing a pack of cards to 4 players. So, there are 4 players let us say player 1, player 2, player 3, player 4 this is how we play cards is not it I mean you 4 players and you deal the cards or distribute the cards to all 4 people and each person after distribution will end up with 13 cards each right ok.

So, they will end up with 13 cards each. So, one can think of when you can imagine how this is being dealt. So, I will assume it is a very very fair way of distributing the cards as an every possibility, is equally likely in the distribution you shuffled the cards very much. So, that is what happens. So, let me define 2 events here a is player 1 got case of spades even B is player 2 got ace of spades ok.

So, these are the 2 events its mean if you probably already know whether or not a or A and B are going to be independent, let us look at the possibility now its simple example, but before that I want to illustrate some of these tricks and probability to calculate probability. So, for instance what is the probability of a let us say you want to compute the probability of a. So, you have 52 cards player one is equally likely to get any one of the 13 possibilities.

So, you can start computing in this that fashion. So, I am going to do it that way first and then I will show you a nice little simple way to all give what the final answer has to be and you will see it will be working out correctly. So, if you look at what cards a player one can ge, there are 13 different cards out of the 52 he will get. So, all of them are equally likely. So, you will have a 52 choose 13 in the denominator.

Remember once again what is my notation this is some of some of you write it as 52 c thirteen. So, combinations of 13 from 52 I will also write it like this, what is this this is 52 factorial by 13 factorial times 39 factorial. So, this is number of ways in which you can take 13 things out of a group of a 52 things.

So, now this is in the denominator this is all the possible things he could have got. Now in how many of these cases would he have definitely got the ace of spades. So, that is one card already chosen for you right ace of spades is chosen for you there are only 12 other cards left out of the remaining 51 all these 12 possibilities are equally likely right. So, you will have 51 choose 12 is that ok.

The ace of cards was chosen for you already and then you have 51 choose 12. So, if you do this calculation you will have a 51 factorial by 12 factorial times 39 factorial 13 factorial times 39 factorial by 52 factorial, things will cancel and you will get remember this will cancel this will cancel 51 factorial will cancel with 52 factorial, remember what is 52 factorial that is 52 into 51 into 50 so on till 1 and what is 51 factorial, its 51 into 50 into so on till one.

So, if you write 51 factorial by 52 factorial, this 51 factorial is going to cancel and it will leave you just 52. Same there with 13 factorial and 12 factorial it will cancel and will only leave thirteen. So, 13 by 52 that is 1 by 4. So, that is the probability that the player one got ace of spades now one might actually quickly arrive at the sense of 1 by 4.

If you think about the 4 events play a one got ace of spades player, 2 got ace of spades player ,3 got ace of spades player, 4 got ace of spades right there are 4 events like this and the ace of spades has to go to one of these players right. So, all of them already know together they make up the entire set of possibilities and each of these events is going to be equally likely by some sort of a symmetry in this situation right you would be distributing the cards very fair and in a very fair manner.

So, it should be equally likely that in any player any one player has the ace of spades. So, probability that the player one got ace of spades should be 1 by 4 similarly probability of B that player 2 got ace of spades is also 1 by 4. Now if you think of the intersection what is the intersection play a one got ace of spades and player 2 got ace of spades that is actually the null set right there is no such event I mean you cannot have distribution of the cards where both the player one and the player 2 has the ace of spades. So, that is not a valid pack of cards because in the pack of cards there is only one ace of spade. So, this is actually 0 ok.

So, clearly A and B are dependent events; so anyway, I was just trying to give you another illustration of an example where you have to argue very carefully. So, now, in many real life experiments that you want to model and you know real life scenarios you want a model in probability, such kind of things will happen you will you will have one event and you will have another event using symmetry you will have to argue that they will have to maybe be equal probability and also you have you have to argue, that they are disjoint in some sense all these things you have to notice very carefully. So, hopefully this was an example that gave you the idea ok.

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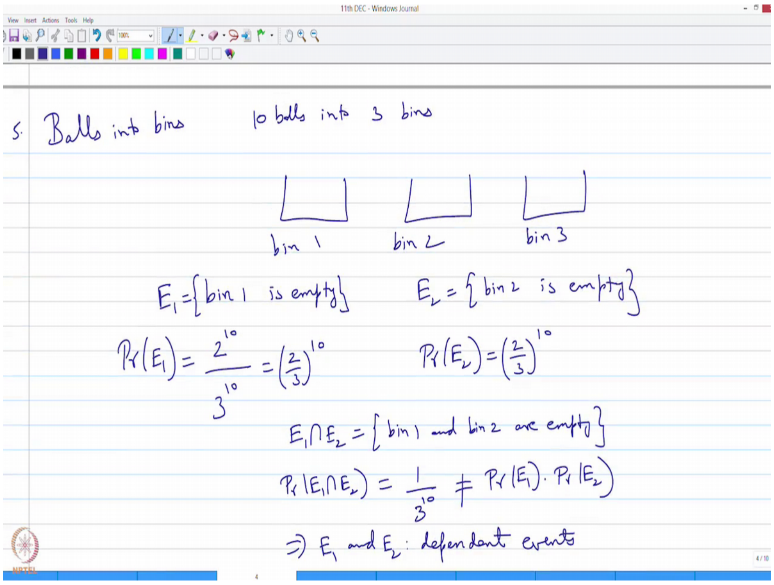

So, we are going to see one more example which is just the fifth one balls into bins. So, let me take 3 bins for simplicity bin 1, bin 2, bin 3. So, you remember this experiment what happens is there are balls that you close your eyes, and throw in and it is equally likely that they will end up in each of these bins and at the end of the day you want to count and see how many balls are in how many bins ok.

So, let us say we throw I do not know let us say 10 balls in to 3 bits. So, I am going to define an event E 1 which is bin 1 is empty. So, this is E 1, and then you have the event E 2 which is bin 2 is empty. So, things are going to get a little bit subtle here you have to pay a attention. So, event E 1 is the event that bin 1 is empty event E 2 is the event that bin 2 is empty.

Now, are these 2 events independent or not that is the question that I am going to ask here. So, this actually needs the calculation. So, if you think about it, there is an even intersect E 2 there are various possibilities under which, there is a possibility and there is bin 1 and bin 2 can be empty what happens in that case all the balls go into bin 3. So, that can happen. So, the 2 events are not disjoint. So, they may be dependent or may not be depend it may be independent or may not be independent you have to actually calculate the probabilities and check.

So, let us try and calculate probabilities here, what is the probability of E 1. So, what is the probability that bin 1 is empty every ball when you throw it is equally likely to go to any one of the 3 bins and I want bin 1 to be empty which means what every ball should either go to bin 2 or go to bin 3 right. So, every ball has 2 power 10 possibilities out of a total of 3 power 10 possibilities. So, each ball could have gone to any one of the 3 bins. So, that is 3 power 10 total possibilities out of which 2 power 10 are favorable to you.

So, you have 2 power 10 divided by 3 power 10 or if you want you can write it as 2 by 3 power 10. So, now, we can quickly argue that probability of E 2 will actually be equal to probability of E 1 why is that because bin 1 and bin 2 are sort of symmetric in some sense probability that they are both empty each of them was empty bin 1 is empty and bin 2 is empty should be the same so. In fact, that will also be 2 by 3 power 10.

So, what about E 1 intersect E 2? Bin 1 and bin 2 are empty. So, now, if you do probability for this E 1 intersect E 2, you notice there is only one possibility for any binned ball to go to right you cannot we cannot put it in bin 1 or you cannot put in bin 2 you should put it only in bin three. So, there is only one possibility maybe if you want you can call it 1 power 10 divided by 3 power 10, I am sorry 3 power 3 power 10. So, that is the probability of E 1 intersect E 2 and you can quickly check this is not equal to probability of E 1 times probability of E 2 it is a.

You agree. So, you can see that the probability of E 1 intersect E 2 is 1 by 3 power 10 its not equal to probability of E 1 times probability of E 2. So, now, there is a small calculation that I want to do here. So, supposing. So, this implies let me finish it off this implies E 1 and E 2 are dependent events. So, let me push this example a little bit further not necessarily in independence example.

But I want to show you how to do calculations when the events are dependent. So, when events are independent probability of E 1 intersect E 2 is probability of E 1 into probability of E 2 it kind of simplifies things in some fashion, when they are dependent some calculations become more difficult. So, for instance since E 1 and E 2 are dependent.

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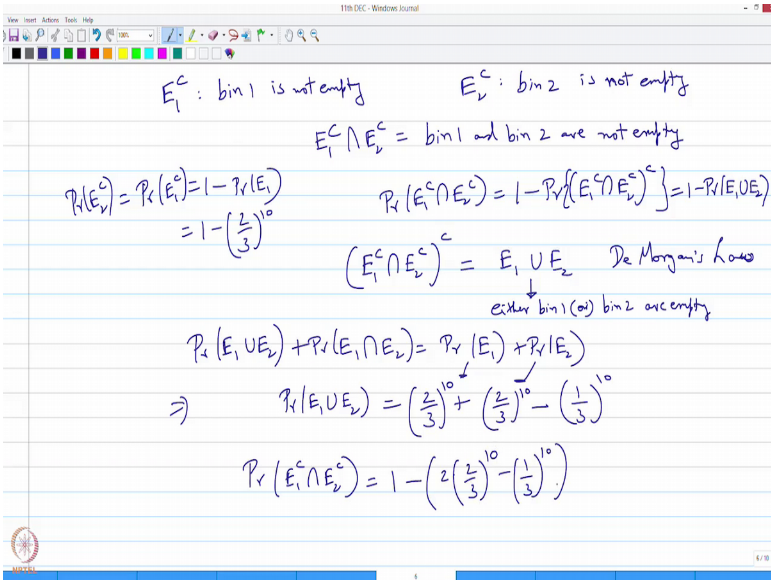

Supposing I asked about E 1 complement what is E 1 complement bin 1 is not empty, and what is E 2 complement then 2 is not empty ok.

Now, there is a very nice result in a probability you can think about proving it if you like; if E 1 ana E 2 are dep are dependent E 1 complement and E 1, E 2 complement are also dependent. So, you can think about it know if one says something about E 2 clearly even say something more one complement also. So, even complement will also say something body to. So, it is there will be dependent.

If E 1 and E 2 are dependent E 1 complement and E 2 complement also a dependent, but this E 1 complement intersect E 2 complement there is an interesting event what is E 1 complement intersect E 1 complement E 2 complement bin 1 and bin 2 are not empty. So, quite often you are interested in problems like this. So, you want bin 1 and bin 2 to be not empty and if you remember from the very very early lectures, I defined this event and I kind of showed you how for this kind of events its difficult to enumerate ok.

Bin one and bin 2 are not empty. So, these kinds of things are slightly more complicated to enumerate. So, supposing you want to find probability of E 1 complement this is not too bad you can find probability of E 1 compliment, how do you find probability one compliment its 1 minus probability of E 1 and you know probability of E 1. So, this is one minus 2 by 3 power 10. In fact, this is the same as probability of E 2 compliment both of these individually I can find the probability.

So, how will I find the probability of this guy? E 1 complement intersect E 2 complement, it looks a little tricky. So, it turns out one can use this very nice little trick here. So, it turns out instead of trying to find this, one needs to observe this interesting little result if you do a complement of this event using de Morgan's law you will get E 1 union E 2. So, this is just de Morgan's law or you saw professor Aravind's lectures on this de Morgan's law of sets. So, E 1 complement intersect E 2 complement the whole complement is actually E 1 union E 2 ok.

So, probability of E 1 complement intersect E 2 complement is one minus probability of E 1 complement intersect E 2 complement the whole complement right and that is one minus probability of E 1 union E 2, because E 1 union E 2 is the same as this big complicated little expression with E 1 union 2. So, its easy to kind of argue this in English also right.

What is E 1 complement intersect E 2 complement? That is bin 1 and bin 2 are not empty what is E 1 union E 2 right what is E 1 union E 2 this is either bin 1 or bin 2 are empty right E 1 is bin 1 is empty E 2 is bin 2 empty if you say E 1 union E 2 it is either bin 1 or bin 2 must be empty.

What is E 1 complement intersect E 2 complement bin 1 and bin 2 are not empty. So, these 2 clearly are complements of each other right. So, this is the essence of the way de Morgan's law works. So, instead of finding probability of E 1 complement intersect E 2 complement I will simply find probability of E 1 union E 2 and then do a subtraction.

Now, how do you find probability of E 1 union E 2. So, here you can use this other nice result probability of E 1 union E 2 plus probability of E 1 intersect E 2, it is actually equal to probability of E 1 plus probability of E 2 right. So, now, why is this nice because I know probability of E 1, I know probability of E 2, I also know the probability of E 1 intersect E 2.

So, once I have these 3 probabilities I can find the probability of E 1 union E union E 2. So, this tells you probability of E 1 union E 2 its probability of E 1 which is 2 by 3 power 10 plus probability of E 2 which is 2 by 3 power 10 minus probability of E 1 intersect E 2 which is 1 by 3 power 10 ok.

So, this tells you; what is the probability of E 1 union E 2. So, what is the probability that of this other guy here E 1 complement intersect E 2 complement, its the probability that bin 1 and bin 2 are not empty its 1 minus this guy. So, maybe you can write it a little bit simpler 2 times 2 by 3 power 10 minus 1 by 3 power 10. So, that is the answer I mean one can simplify this further if you like, but that is fine enough we can write it down.

So, this was just an example to illustrate how you can have dependent events in many real life scenarios, and how you can use this sort of a union rule for computing probabilities in a very clever way, and compute all sorts of interesting events even though they are dependent. So, it is a little bit more complicated than when they are independent, but when they are dependent also hope is not lost you can do computations with events that are dependent. So, of course, independent events are very very nice they are very easy to combine.

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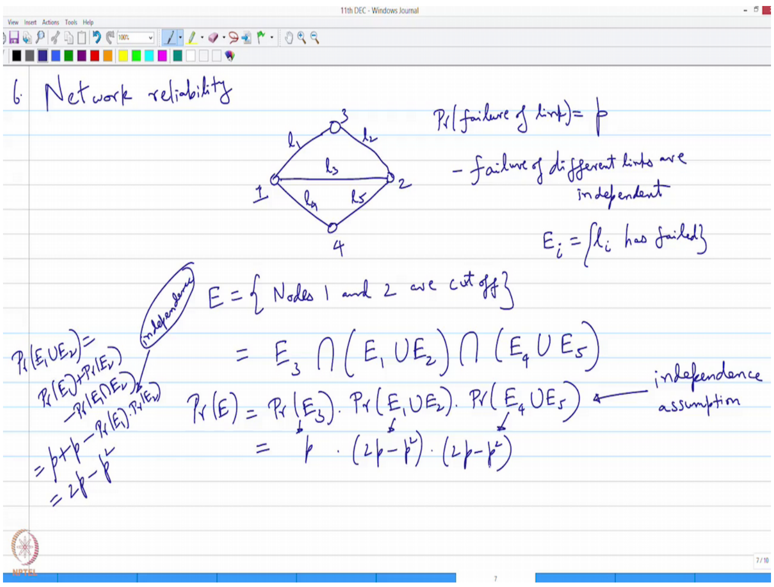

So, you know you know intersecting them is very very easy very nice illustration of this is going to come in the next example and with we will conclude this lecture. It is some sort of a network reliability problem. So, I am going to draw a very simple network here let us say you have one link here, another link here and a third link here. So, this is let us call it point 1 and point 2.

So, these are this is just a network I am just drawing these nodes and connecting them with lines you can think of them in various ways. So, for instance these could be cities in a map and these could be roads that collect connect them together. So, 1 and 2 are connected directly or they connected to 3 or they connected through 4. And there might be probability of failure in any of these links or roads or whatever you want to imagine this might be a probability.

So, maybe I will call equal probability. So, probability of failure of a link equals p and in very many statistical models you can assume that the failures of failure of link of links failure of different links are independent now this is a very very very very crucial assumption. So, quite often when you have networks like this, it is reasonable to assume the different links are not affected by each other. So, if you imagine these as roads connecting cities maybe what happens in one road, it is not connected to what happens in another road whether there is a some agitation in the middle of this road which causes the traffic there to stop or traffic on another road is stopping there.

They are probably independent in some sense you can you can make use of this assumption in your calculation, but if you want to apply results from such a probabilistic model you should be extremely aware of this assumption that you are assuming that is independent. So, for instance if there was a major natural catastrophe in this entire area came a major earthquake or something and it is very likely that all of the roads will be cut off right. So, it can also happen. So, you should be aware under which models under which situations this kind of a model of independence actually holds, and it can be violated if its violated then all your inferences will go wrong ok.

So, it is very very important to know what any probability model. So, now, one of the questions they will ask us any particular link fails with probability p. I am interested in the event E that nodes 1 and 2 are cut off what do we mean by cut off. They cannot communicate to each other you cannot go from one to two. So, what is the probability that nodes 1 and 2 are disconnected in some fashion as a new you cannot start at one and there is no way for you to reach 2 ok.

So, now you have to call these links. So, let me call this l one this is l 2 this is l 3 this is l 4 this is l 5 right and they will call event E 1 E i as l i has failed. So, now, what is this event E I can write it in an interesting fashion you can think of it as various possibility. So, now, E can occur if let us say E c E 3 has to always occur.

If E 3 does not occur right clearly you can go from 1 to 2 always, link 3 is active means you can go from one to 2 and the event E would not occur. So, E 3 must definitely occur link 3 should definitely fail. Now what about the remaining links, you have to argue very carefully out of l one and l 2 at least one link should fail out of l 3 and l 4 and l 5 at least one link should fail ok.

So, this event E if you argue it out carefully you can write it as E 3 intersect E 1 or E 2 intersect E 4 or E 5 is that from or if I am sorry, hopefully you saw how I did this calculation the event E would have occurred nodes 1 and 2 would have been cut off E 3 has to occur. So, because link 3 is active you have a connection from 1 to 2, E 3 has to occur and between E 1 and E 2 I think link one and link 2 at least one should fail which is what either link one or link 2 should fail.

So, E 1 union E 2 because if both are active then clearly you have a route same thing happens with E 4 and E 5 now remember events involving different links are all independent and these are all events involving different links and I am doing an intersection of these things. So, when I do probability when I know these things are independent I can multiply these probabilities probability of E 3 times probability of E 1 union E 2 times probability of E 4 union E 5 is that ok.

So, how did I manage to do this the independence assumption is crucial here independence assumption. So, it is very critical. So, if you did not have the independence assumptions I cannot multiply the fact that I multiply tells me something about this even e. So, right computation when it is valid under what situation is this valid when you translate it into reality you have to be aware of the assumptions you made in the model ok.

So, once you do this it is not too bad. So, this probability is just p what is this probability what is the probability of E 1 union E 2 let me again do that carefully here; probability of E 1 union E 2 its actually again I have to use this union rule probability of E 1 plus probability of E 2 minus probability of E 1 intersect E 2 ok.

Now, what is probability of E 1 that is p what is probability of E 2 that is another p what is probability of E 1 intersect E 2? Remember E 1 and E 2 are again independent whether the link one fails or link 2 fails its independent. So, this becomes simply probability of E 1 times probability of E 2. Again we use independence here very crucial. So, you see here its 2 times p minus p squared. So, that is the probability of E 1 intersect union E 2. So, its p times 2 p minus p squared times what about this guy probability of E 4 union E 5 you do not have to do repeat the same calculation again, it has to be the same right it has to be the same as E 1 union E 2. So, you will get 2 p minus p squared. So, that is the final answer for probability that nodes one and node 2 will be cut off.

So, hopefully this illustrated to you that in your model, if you assume things are independent any composite event can be easily decomposed in terms of individual events if they are all independent, and you can easily multiply them with each other compute probabilities of unions intersections etcetera in this nice fashion using both the union rule, and the prop fact that you can multiply probabilities for independent events ok.

So, hopefully this these example showed you various situations in which we are doing calculations, we will slowly complicate things a little bit more we will look at more complicated examples of independence, and all that as we go along in the next examples lectures.

Thank you very much.